Solvable Lattice Models

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Preface

Here are 20 lectures I gave at Stanford in the fall of 2023. The class of solvable lattice models we consider here are based on a planar graph, the edges of which correspond to elements of a braided category, such as the module category of a quantum group. Associated with such a system is a partition function, which can itself be something important such as a character of a Lie group. In practice, many important instances of the Yang-Baxter equation can be organized into families, parametrized by a group or groupoid, so I also emphasized parametrized Yang-Baxter equations.

I try to emphasize connections with representation theory. This includes the representation theory of Lie algebras, quantum groups and Hecke algebras. Still I do not assume too much background in Lie theory. Some knowledge of finite-dimensional simple Lie algebras, particularly root systems, Weyl groups, dominant weights, etc. will be useful. But no knowledge is assumed about quantum groups or affine Lie algebras.

I wanted to explain that some important modules in this theory will be Verma modules, including Kac modules for Lie superalgebras. However I could not go deeply into this topic. So the approach I took was to show why we need these modules for quantized enveloping algebras, then explaining the required mathematics only for the unquantized enveloping algebra. This has the advantage of conveying some intuition without getting too involved in technicalities. It has the disadvantage that to actually do research on this topic, the technicalities will be needed.

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LECTURE 1

Origin in Statistical Mechanics

Lattice models were introduced in statistical mechanics in order to study realistic systems. Statistical mechanics will *not* be a focus of this course. Indeed, it turns out that solvable lattice models have important connections with representation theory, for example of quantum groups regarding the underlying mechanism, and other areas such as representation theory of p-adic groups, algebraic combinatorics, algebraic geometry, and conformal field theory. We will review the statistical mechanical origins of the theory, referring to Baxter [**5**] for much more information, before turning away to other subjects.

1. Thermodynamics

The purpose of this section is to give a quick impressionistic treatment of statistical mechanics. Since we will soon migrate away from this subject, we will not try for any depth.

Statistical mechanics is a development from thermodynamics. Thermodynamics was an empirical discovery which started with the theory of gases, motivated by considerations related to engines and refrigeration.

Thermodynamics can be *axiomatized* in the form of several laws, most importantly the *second law of thermodynamics* which contains a subtle and important concept, entropy. The laws of Thermodynamics are sometimes stated thus:

(1) Energy is conserved in a closed system.
(2) Entropy is increasing.
(3) If the temperature is decreased to zero, entropy approaches a fixed value, called the *residual entropy*.

The concept of entropy is of great importance, and universal in its surprising applicability to different areas, such as information theory and black holes. It has important philosophical implications, since it gives a direction to the arrow of time. This is paradoxical since the laws of physics are invariant under time reversal (CPT symmetry).

We take for granted the concept of energy, and its conservation. In thermodynamics and statistical mechanics, it is important to take into account both closed systems, that do not interact with their environment, and systems that do interact. Thus we imagine that energy can be put into a system, or extracted from it. *Work* can be described as energy that is extracted from a system, for example by operating a piston or generating electricity.

Heat is a form of energy that we now understand to be due to the kinetic energy of molecules in a substance. Carnot, whose investigations of the steam engine led to the concepts of thermodynamics, thought of heat as a fluid like water, that can flow from higher levels to lower, and in the process can be made to do work. The first law of thermodynamics can be expressed in the formula

$$dU = dQ + dW,$$

where U is a variable expressing the total amount of energy in the system, Q is the amount of heat, and W is a variable expressing work, energy that is put into a system, or extracted from it.

As Carnot realized, certain processes are reversible. We may imagine a perfectly efficient engine, with frictionless parts, where energy is put in in the form of fire or electricity, and mechanical work is extracted. But other processes, such as friction, are irreversible. In friction, work is transformed into heat, and this is energy that can never be extracted from the system. A processes involving friction is irreversible.

Again, if a system consists of two bodies of different temperatures, energy can be extracted as work by a mechanism such as a dipping bird. But if heat flows from one body to the other, until they reach the same temperature, the energy still exists, but can no longer be extracted as work. Thus the cooling of a hot object is an irreversible process.

The second law of thermodynamics regulates such irreversible processes. The second law postulates that there is a quantity S, called *entropy* that can only increase. Irreversible processes are precisely those that increase the entropy. Conversely, a process is reversible if it does not increase entropy. A system is at *maximal entropy* the entropy can no longer increase. An example would be a system in thermal equalibrium, where all parts are at the same temperature.

Also related to the second law is the notion of *free energy*. This is the amount of energy that can be extracted from a system as useful work. Thus the entropy of the system is maximal if the free energy is zero.

2. Statistical Mechanics and the Partition Function

The physical basis for thermodynamics is *statistical mechanics*. Thus heat is understood as being the kinetic energy of atoms and molecules, and the laws of thermodynamics can be derived from statistical considerations.

We will consider a system with many possible states, which is not strictly subject to the first law, in that not all states have the same energy. The source of this uncertainty is usually interaction with the environment. For example, one considers a system that is in contact with a heat bath at a constant temperature. The system itself is assigned a temperture that may be constant, or could vary within the medium. The system may also depend on other parameters, such as pressure or the strength of an applied electromagnetic field.

An important question that is investigated in Statistical Mechanics is the behavior of a system at a phase transition point. We may consider the melting or boiling of a substance as an example. In an idealized form, we may imagine the process as follow. In a "frozen" state, there are correlations between the local structure of the system at locations that are separated in distance, but in the "melted" form, there are no such correlations. The phase transition point or critical temperature is the value where the structure changes from frozen to melted.

A statistical mechanical system $\mathfrak{S}$ is an ensemble of *states*. Each state $\mathfrak{s}$ has an energy $e(\mathfrak{s})$, and there is a probability measure on $\mathfrak{S}$, with high energy states being less probable. The system may depend on some external parameters, notably the *temperature* of the system. The probability of the state $\mathfrak{s}$ with energy $E(\mathfrak{s})$ is proportional to $\beta(\mathfrak{s}) = e^{-E(\mathfrak{s})/kT}$, where k is *Boltzmann's constant*. Since the sum of the probabilities must be 1, the actual probability is

$$\frac{1}{Z}\beta(\mathfrak{s}), \qquad Z = Z(\mathfrak{S}, T) := \sum_{\mathfrak{s}} \beta(\mathfrak{s}).$$

The quantity $\beta(\mathfrak{s})$ is called the *Boltzmann weight* of the state, and the quantity Z is called the *partition function*. Note that as the temperature increases, energetic states become more probable.

The partition function is a powerhouse in statistical mechanics. For example the mean energy is

$$\langle E \rangle := \frac{1}{Z}\sum_{\mathfrak{s}} \beta(\mathfrak{s})\, E(\mathfrak{s}) = kT^2 \frac{\partial}{\partial T} \log(Z).$$

The free energy, which we recall is the amount of energy that can be extracted from the system as work, equals

$$F = -kT\,\log(Z),$$

and the entropy is

$$S = k\,\log(Z) + \frac{1}{kT}\langle E \rangle.$$

If the partition function depends on other parameters such as a magnetic field strength, differentiating with respect to those will yield other values of significance.

The partition function also occurs in other areas of physics, such as quantum field theory. For us, the partition function will be a main object of study, even though we will soon leave its origins in statistical mechanics behind.

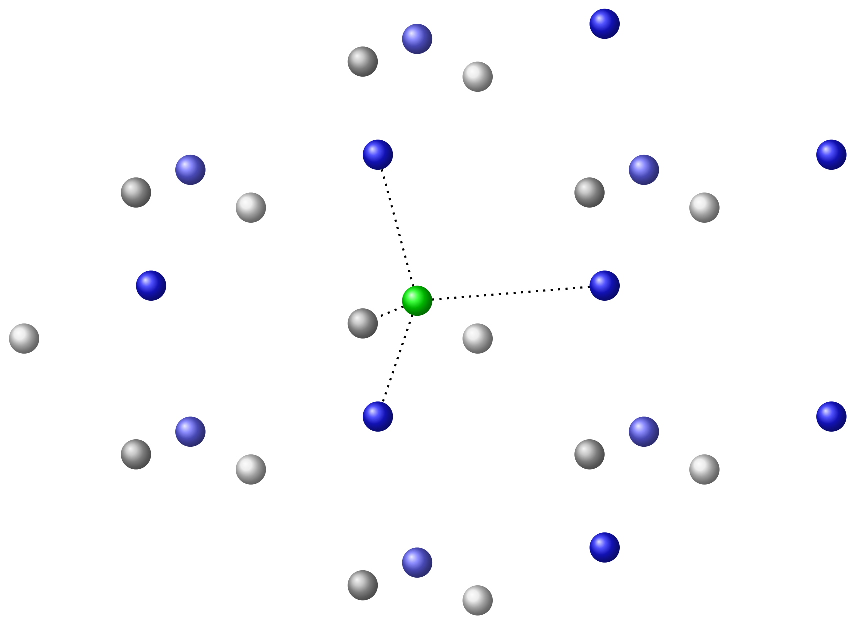
3. Ice

We may consider ice (frozen H_2O), where the larger oxygen atoms have fixed locations at the vertices of a grid. In its usual form (called Ice I_h) these oxygen atoms are arranged in a three-dimensional hexagonal lattice. We can envision the oxygen atoms as lying on the vertices of a three-dimensional hexagonal crystal lattice. Each oxygen atom will have four neighbors, lying at the vertices of a tetrahedron. We may consider the 4-regular graph Γ whose vertices are the oxygen atoms and whose edges are the segments joining them to the four nearby atoms.

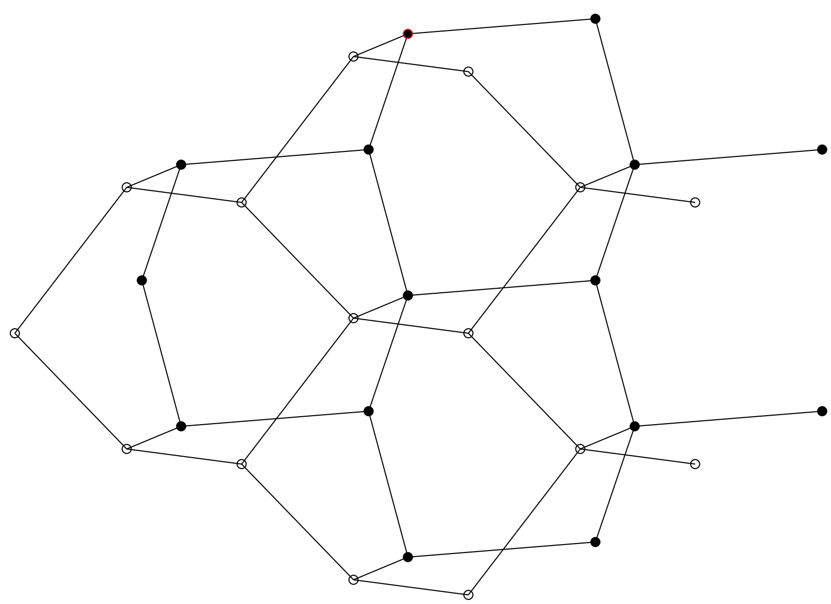
Linus Pauling computed the entropy and free energy of ice by means of a three-dimensional lattice model. Let us describe a grid whose vertices are the oxygen atoms in a crystal. We consider two oxygen atoms *adjacent* if they share a hydrogen bond. They then form a graph Γ that is nearly 4-regular in that each oxygen atom, except those at the boundary of the crystal, have 4 neighbors. (Here we are ignoring a detail about boundary edges, and we will give a proper discussion of Γ below in Section 4.)

Ice has many possible crystalline structures. Under normal conditions, Ice I_h is the usual one. This crystal occurs in sheets or layers. The graph is bivalent. Each layer is a tesselated by hexagons, with oxygen atoms at their vertices. Furthermore, each atom has a bond with one in either the layer above or below, depending on its valence.

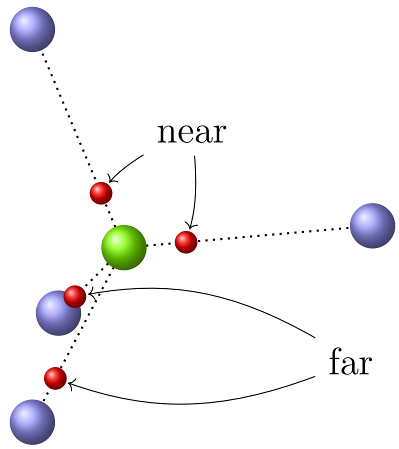
Here is the hexagonal Ice I_h lattice, showing the segments joining a sample Oxygen atom (green) to its four neighbors.



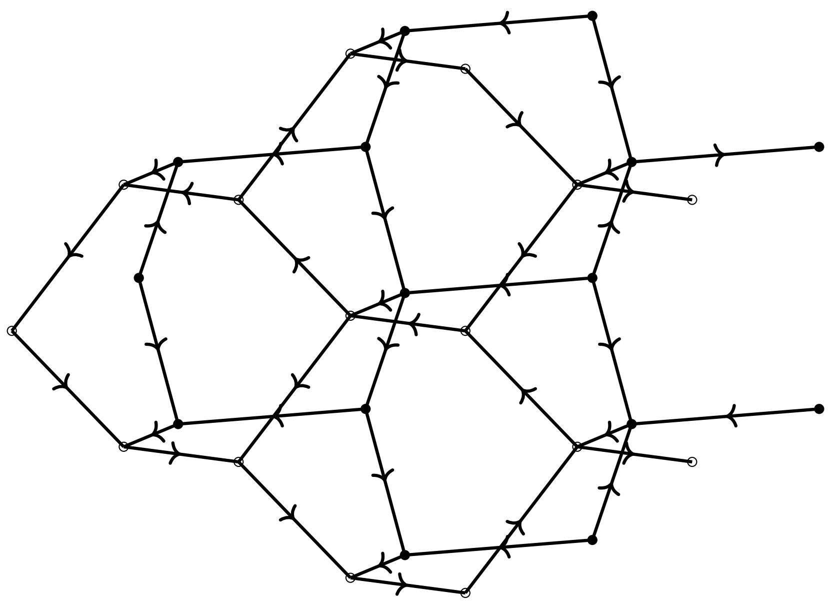
Here is the graph Γ showing two adjacent layers.



While the location of the oxygen atoms is fixed, and forced into a crystalline pattern, the location of the hydrogen nuclei (protons) is another matter. Due to its position in the periodic table, oxygen is allowed two covalent bonds. The oxygen atom will therefore borrow electrons from two hydrogen atoms. This causes the protons to lie on the segments between two adjacent oxygen atoms, but each proton will be closer to one or the other of the two oxygen atoms. There are many possible configurations, which are subject to quantum superposition.



We may represent this graphically by making the graph Γ into a *directed* graph. We decorate the edges with arrows, each pointing towards the hydrogen atom on the edge.

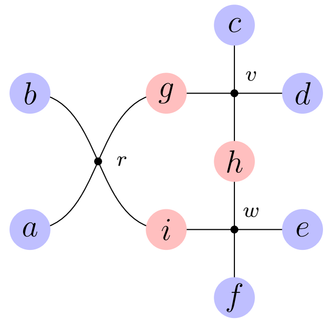


Then we obtain the following model: we have a 4-regular graph, based on the three-dimensional hexagonal lattice. A state of the system is a refinement of the graph to a directed graph, with every vertex having two incoming and two outgoing arrows.

4. A class of lattice models

4.1. Graphs. We have formalized the ice crystal into a system based on a *graph*, which is almost but not the same as a graph in the usual combinatorial definition. Let us define a graph to be a set of *vertices* and a set of *edges* with an incident relation, that some edges are *through* or *adjacent to* certain vertices. We will assume that every edges is through either exactly two vertices, or a single vertex. The edges that are through a single vertex will be called *boundary edges*. The edges that connect two vertices are *interior edges*.

As an example, let us consider this graph:

(1) 

Here we have a graph with three vertices, labeled v, w and r. There are nine edges, labeled a, b, c, d, e, f and g, h, i. The edges a, b, c, d, e, f are boundary edges.

The graph is *planar* if it can be embedded in the plane. **We will consider mainly planar 4-regular graphs.** The Ice I_h graph is 4-regular, but not planar. On the other hand, the graph (1) is planar.

4.2. Spins. In the class of models we will consider, every edge e will be assigned a set Σ_e of possible states, called *spins*. In the Ice models, the spins are the two possible orientations of the arrow that points towards the hydrogen atom. We will require that the spins of the boundary edges are fixed, and are part of the data describing the system. On the other hand, the spins of the interior edges are variable.

4.3. States. A *state* of the model is an assignment of an element of its spinset to every edge of the model. We will assume that the boundary spins have fixed assignments. Indeed, this will be part of the data describing the model.

Almost always there will be local constraints at each vertex on the possible configurations of spins adjacent to a particular vertex. We will call a state in which these constraints are satisfied at every vertex *admissible*.

For example in the Ice I_h model that we have described, the spins are directions or orientations of the edges, which we can represent by arrows, and the constraint is that there there are two "in" arrows and two "out" arrows. This means that there are $\binom{4}{2} = 6$ possible configurations of local spins at the vertex.

4.4. Boltzmann weights. Every admissible configuration $\mathfrak{s}$ is therefore a state of the system $\mathfrak{S}$, which is the ensemble of all states. It is to be assigned a *Boltzmann weight* $\beta(\mathfrak{s})$. We will assume that this is a product of *local Boltzmann weights* $\beta_v(\mathfrak{s})$ at each vertex v:

$$\beta(\mathfrak{s}) = \prod_v \beta_v(\mathfrak{s}),$$

where $\beta_v(\mathfrak{s})$ depends only on the configuration of spins at the edges adjacent to the vertex. We can extend this definition to states that are not admissible by defining $\beta_v(\mathfrak{s}) = 0$ if the local configuration is not admissible. Then $\beta(\mathfrak{s}) = 0$ for inadmissible states $\mathfrak{s}$.

The partition function is

$$Z(\mathfrak{S}) = \sum_{\mathfrak{s}} \beta(\mathfrak{s}).$$

We may sum over all states, or over admissible states.

5. The Six Vertex Model

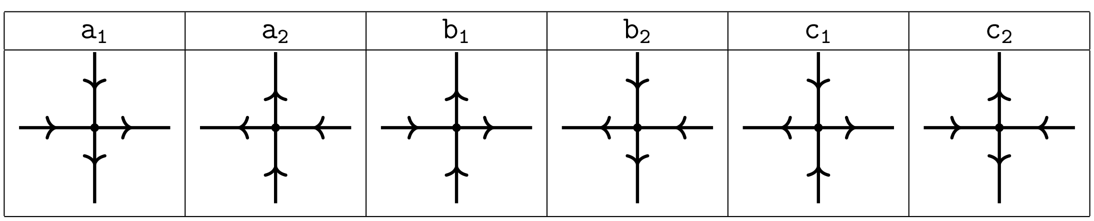
Certain lattice models are called *solvable* since algebraic methods based on the Yang-Baxter equation, which will be a major focus of this course, allow the partition function to be computed exactly. Historically the first example was Onsager's 1944 study of the 2-dimensional Ising model. However we will start with an even simpler model, the *six-vertex model*, which is also related to ice.

Solvable lattice models are almost exclusively 2-dimensional. This means that the underlying graph is planar. The Ice I_h model that we considered is not solvable as far as we know, and its graph is not planar.

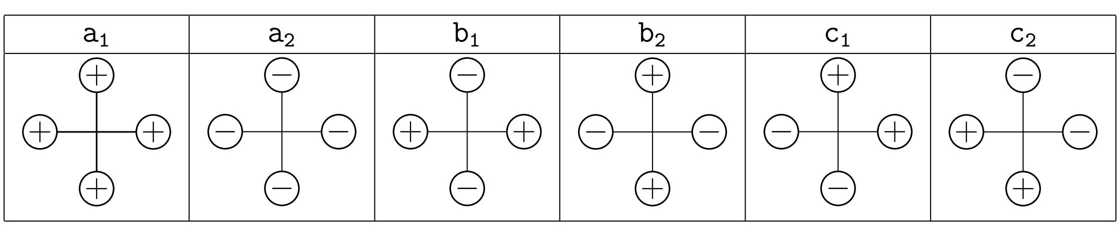
While Pauling had considered the realistic problem of 3-dimensional Ice and heuristically computed the number of states, one can also consider 2-dimensional Ice, in which the oxygen atoms are restricted to a plane, and form a crystal with the oxygen atoms at the vertices of a square lattice. This was investigated by Nagle [**81**], after which Lieb [**72, 70, 71**] and Sutherland [**89**] found exact solutions for the entropy problem. For 2-dimensional Ice, Lieb found that the residual entropy was $kN\,\log(W)$ with $W = (4/3)^{3/2}$.

The mathematical model of 2-dimensional ice is the famous 6-vertex model, which is the archetype of a large class of important solvable lattice models. It is realistic enough to have a phase transition, which was of great interest to the early investigators. We will therefore discuss it at length.

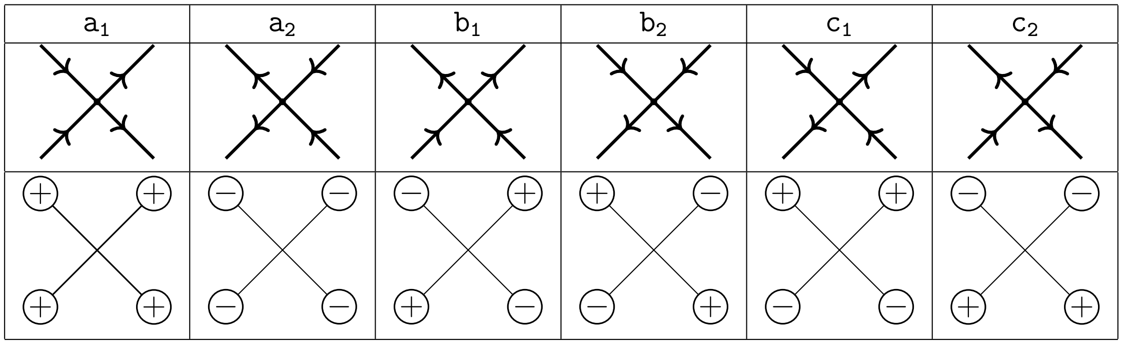
The six-vertex model is nearly identical to the I_h models we have discussed, except that the underlying crystal is 2-dimensional, based on a square lattice. We will give two versions of the Boltzmann weights. Recall that the spinset of an edge is a set of possible states. For the six-vertex model, the spinset has cardinality two. In one version of the six vertex model, the spinset of an edge is an orientation. The Boltzmann weights depend on six parameters, $\mathtt{a}_1$, $\mathtt{a}_2$, $\mathtt{b}_1$, $\mathtt{b}_2$, $\mathtt{c}_1$ and $\mathtt{c}_2$, which may depend on the vertex v, so we may write $\mathtt{a}_1(v)$, etc. We label the possible states as follows:



On the other hand, it will also be convenient to dispense with the orientation and take the spinset to be the 2-element set $\{+,-\}$. Then the labeling of the states is as follows:



Although the lattice model will be based on a rectangular grid, we will also encounter vertices that are in a rotated orientation, and we will use the following labels for these.

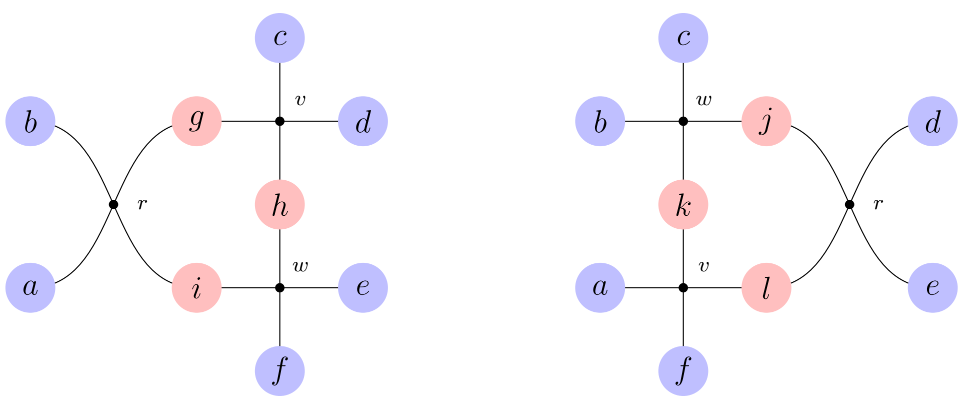


6. Solvability

Baxter introduced an important method of studying certain vertex models, and he used it to solve not just the six-vertex model, but the more difficult eight-vertex model, and with it the XYZ Heisenberg spin chain, a related quantum mechanical problem. This method is based on the *Yang-Baxter equation*, so named by Faddeev. The study of the Yang-Baxter equation leads to interesting mathematics, namely braided categories and quantum groups. Indeed, historically, the six-vertex model was a key example.

We will consider vertices in a model (as described in Section 4) as being associated with a set of Boltzmann weights. We will say that a class of models is *solvable* if when v and w are vertices that can occur adjacent to each other in the class, there is another type of vertex that we will denote r such that the two following systems are equivalent:

(2)



This means that for every possible assignment of spins to the six boundary edges a, b, c, d, e, f, the partition functions of the two systems are equivalent. Thus we sum over all possible assignments of spins to the interior edges, g, h, i on the left-hand side, or j, k, l on the right-hand side. If this is so, we say the *Yang-Baxter equation* is satisfied for these vertices v, w, r.

We will consider two families of solutions to the Yang-Baxter equation within the six-vertex model.

6.1. Field-Free Six-Vertex Model. The vertex v with Boltzmann weights $a_i(v)$, $b_i(v)$ and $c_i(v)$ will be called *field-free* if $a_1(v) = a_2(v)$, $b_1(v) = b_2(v)$ and $c_1(v) = c_2(v)$. We will suppress the subscript in the field free case and write just $a(v) = a_1(v) = a_2(v)$.

We will make the following assumptions:

ASSUMPTION 6.1. *in addition to the field-free assumption that $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, we will assume that $a, b \neq 0$ and that $b \neq \pm c$.*

These assumptions should not be taken seriously and will be eliminated in Theorem 4.2, which shows precisely what the solutions are. So the following result is provisional, and is included as an illustration of how to approach an unknown Yang-Baxter equation. The assumptions that $a, b \neq 0$ and $b \neq \pm c$ are made for convenience, and need to be dispensed result.

Define

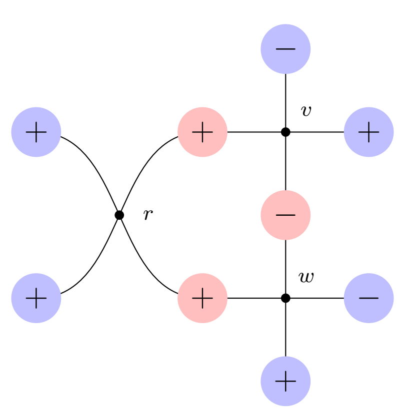
$$\Delta(v) = \frac{a^2(v) + b^2(v) - c^2(v)}{2a(v)b(v)}. \tag{3}$$

The special case $\Delta = 0$ is the free-fermionic Yang-Baxter theorem. The cases $\Delta = \pm 1$ are also special.

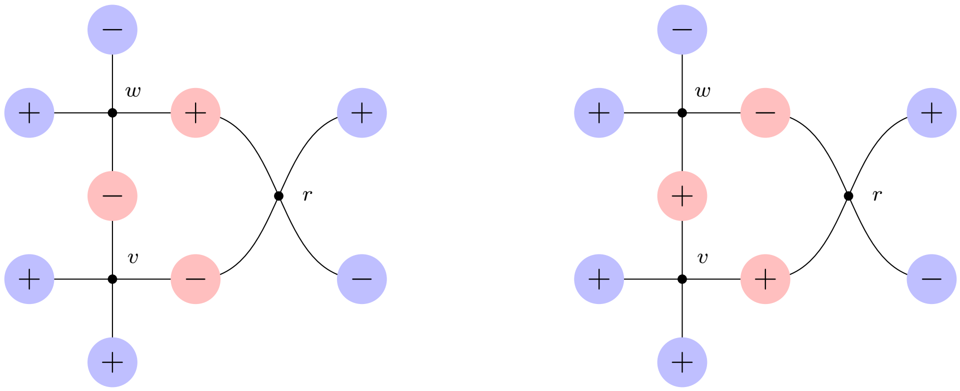
For an interesting and important generalization, see Baxter's analysis of the field-free eight vertex model in Section 10.4 of [**5**]

THEOREM 6.1 (Baxter). *Let $\Delta \in \mathbb{C}$, and let v and w be two field-free six-vertex model vertices with $\Delta(v) = \Delta(w) = \Delta$. In addition to the requirement in Assumption 6.1 that $a(v)$ and $a(w)$ are nonzero, we are assuming that $b(v)$ and $b(w)$ are nonzero. Then there is another field-free six-vertex model vertex r such that the Yang-Baxter equation (2) is satisfied. We have $a(r)^2 + b(r)^2 - c(r)^2 = 2a(r)\, b(r)\, \Delta$, so if $b(r) \neq 0$ then $\Delta(r) = \Delta$.*

PROOF. We claim that there are three equations that must be satisfied for the Yang-Baxter equation to be satisfied. First take $(a, b, c, d, e, f) = (+, +, -, +, -, +)$. The left-hand side of the Yang-Baxter equation has one admissible state:



This has Boltzmann weight $b(v)c(w)a(r)$. On the other hand, there are two admissible states on the right-hand side:



These have weights $c(v)b(w)c(r)$ and $a(v)c(w)b(r)$. So we obtain the equation

$$b(v)c(w)a(r) = c(v)b(w)c(r) + a(v)c(w)b(r). \tag{4}$$

Taking $(a, b, c, d, e, f) = (+, +, -, -, +, +)$ gives

$$c(v)a(w)a(r) = c(v)b(w)b(r) + a(v)c(w)c(r), \tag{5}$$

and taking $(a, b, c, d, e, f) = (+, -, +, -, +, +)$ gives

$$b(v)a(w)c(r) = c(v)c(w)b(r) + a(v)b(w)c(r). \tag{6}$$

Taking other combinations of a, b, c, d, e, f give a total of 12 equations altogether, but they turn out to be these same three equations, repeated. So we need to show that we can construct the vertex r to satisfy (4–6).

Since $\Delta(v) = \Delta(w)$ we have

$$(a(v)^2 + b(v)^2 - c(v)^2)a(w)b(w) = (a(w)^2 + b(w)^2 - c(w)^2)a(v)b(v).$$

This identity implies that

$$\frac{b(v)a(w)b(w) - a(v)b(w)^2 + a(v)c(w)^2}{a(w)} = \frac{a(v)b(v)a(w) - a(v)^2b(w) + c(v)^2b(w)}{b(v)},$$

and we define this to be $a(r)$. Then we define

$$b(r) = b(v)a(w) - a(v)b(w), \qquad c(r) = c(v)c(w).$$

Now it may be checked that the identities (4), (5), (6) are satisfied. For example, to prove (4), the right-hand side equals

$$c(w)(a(v)b(v)a(w) - a(v)^2b(w) + c(v)^2b(w))$$

and using the second expression for $a(r)$ this equals $a(r)c(w)b(v)$. We leave the other two cases to the reader. Checking that $\Delta(r) = \Delta$ is an easy calculation. $\square$

We will explain later how this Yang-Baxter equation can be applied to study the partition functions for the field-free two-dimensional ice models, and what some of the applications are.

6.2. The Free-Fermionic Six Vertex Model. The second case where there is solvability is the *free-fermionic case*. Here the relevant Yang-Baxter equation was found (partly) by Korepin around 1981. See [**62**], page 126, with referencces to earlier literature. Later Brubaker, Bump and Friedberg rediscovered this in a slightly more general form and gave applications. See [**19**]. Within the six-vertex model, this is a very interesting example with important generalizations.

We will call the six-vertex model vertex v *free-fermionic* if

$$a_1(v)a_2(v) + b_1(v)b_2(v) = c_1(v)c_2(v).$$

We are of course dropping the field free-condition.

THEOREM 6.2 (Korepin-Izergin; Brubaker-Bump-Friedberg). *Let v and w be free-fermionic vertices. Then there is a free-fermionic vertex r such that the Yang-Baxter equation is satisfied.*

LECTURE 2

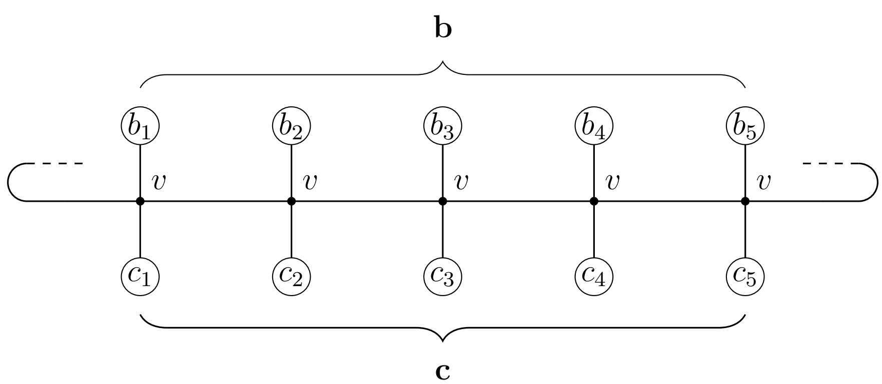
Commuting Transfer Matrices

We will consider systems $\mathfrak{S}$ built up from graphs Γ as in Lecture 1. Recall that a *graph* for us consists of vertices and edges, with an incidence relation between them. Every edge is adjacent to one or two vertices. An edge that is adjacent to two vertices is called *interior*, and an edge that is adjacent to only one vertex is called a *boundary* edge. Every edge $\mathcal{E}$ is assigned a *spinset* $\Sigma_{\mathcal{E}}$ of possible states called *spins*. The spins of boundary edges are fixed, and are part of the data defining the system. A state of the system consists of an assignment of spins to the interior edges.

Also required for the specification of the system $\mathfrak{S}$ is, for every vertex $v \in \Gamma$ a rule β that assigns to a state $\mathfrak{s}$ and a vertex v a weight $\beta(v, \mathfrak{s})$. This should only depend on the spins of the edges adjacent to v. The Boltzmann weight $\beta(\mathfrak{s})$ is the product of the $\beta(v, \mathfrak{s})$ over all vertices, and the partition function is

$$Z(\mathfrak{S}) = \sum_{\text{states } \mathfrak{s}} \beta(\mathfrak{s}).$$

We wish to discuss the concatenation of two systems. To have an example in mind, consider a system consisting of a single row of vertices:



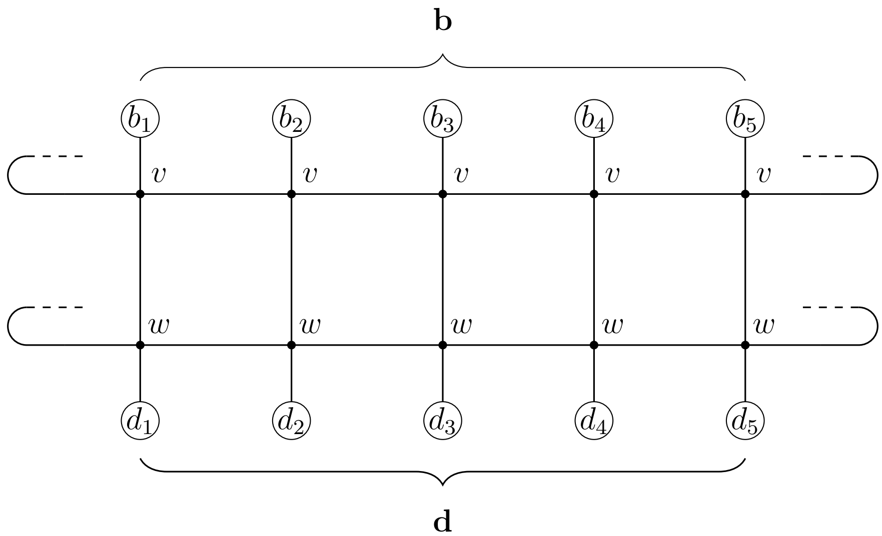
We are imagining that every vertex has the same Boltzmann weight v, so we are giving every vertex the same label. We are imagining that one edge wraps around the back, so the system is periodic. We will refer to this as *cyclindric boundary conditions*.

We have partitioned the boundary edges into two sets, $\mathbf{b} = (b_1, \cdots, b_n)$ where in the picture $n = 5$, and $\mathbf{c} = (c_1, \cdots, c_n)$. The partition function thus depends on $\mathbf{b}$ and $\mathbf{c}$, and we think of $\mathbf{b}$ (somewhat arbitrarily) as *inputs* and $\mathbf{c}$ as outputs, and write

$$Z(\mathfrak{S}) = \langle \mathbf{c} | T_v | \mathbf{b} \rangle$$

where we are using Dirac notation to indicate T_v as a matrix with row entries $\mathbf{b}$ and column entries $\mathbf{c}$. We can think of it as an operator on the free vector space on the set of possible input spins $b_1, \cdots, b_n$, assuming that the spinsets match, so $\Sigma_{b_i} = \Sigma_{c_i}$. We call T_v the *row transfer matrix*.

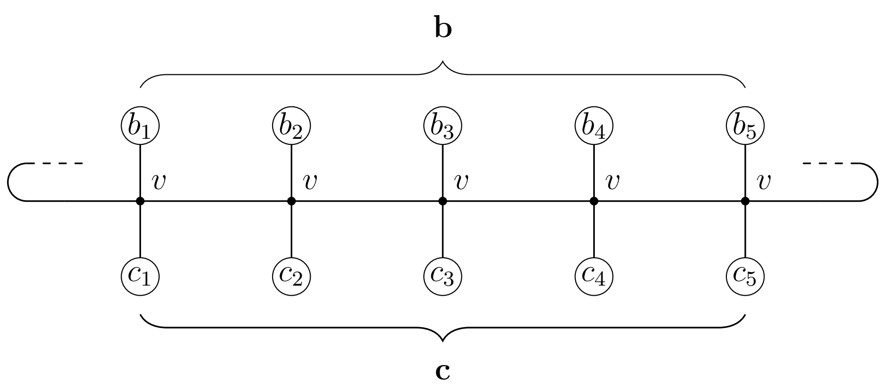
Now let w be another vertex type, and let us consider a system with two layers:

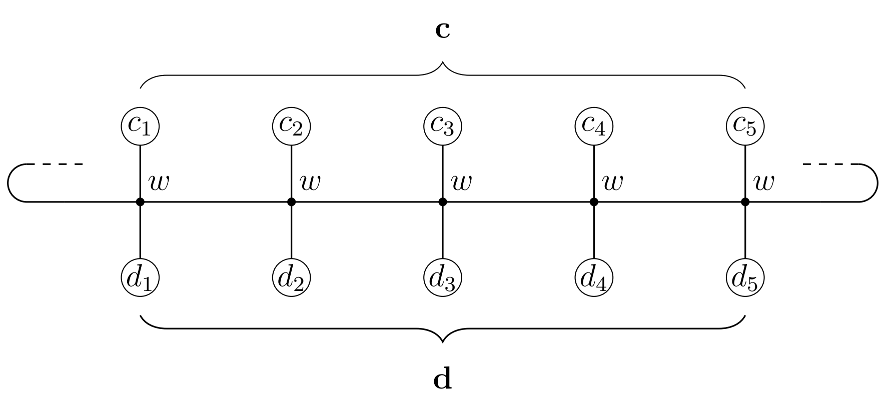


We can express this in terms of the product of two row transfer matrices:

$$\langle \mathbf{d}|T_w T_v|\mathbf{b}\rangle$$

Indeed, we may concatenate the two smaller systems:



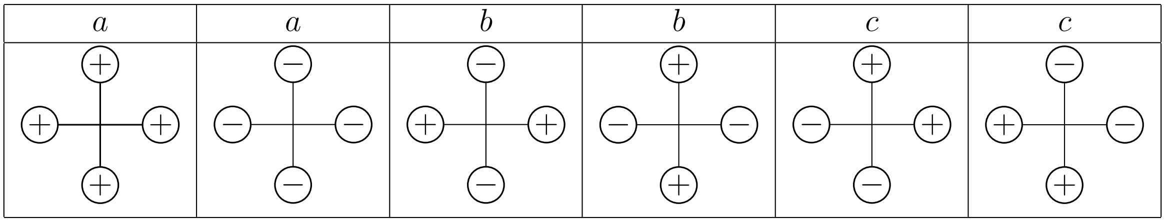


Now the common edges, labeled $\mathbf{c}$ in both cases have become interior edges, so by our rules, we have to sum over the possible states, to obtain:

$$\sum_{\mathbf{c}} \langle \mathbf{d}|T_w|\mathbf{c}\rangle\langle \mathbf{c}|T_v|\mathbf{b}\rangle = \langle \mathbf{d}|T_w T_v|\mathbf{b}\rangle$$

by the usual rule for matrix multiplication.

In preparation for applying the Yang-Baxter equation, we write $v = v(a, b, c)$, where a, b, c are real or complex parameters and the Boltzmann weights are as in the previous lecture.



Baxter's great insight was the use of the Yang-Baxter equation to prove that under certain conditions, row transfer matrices commute.

THEOREM 0.1 (Baxter). *Let $\Delta \in \mathbb{C}^\times$, and let a, b, c, a', b', c' be such that*

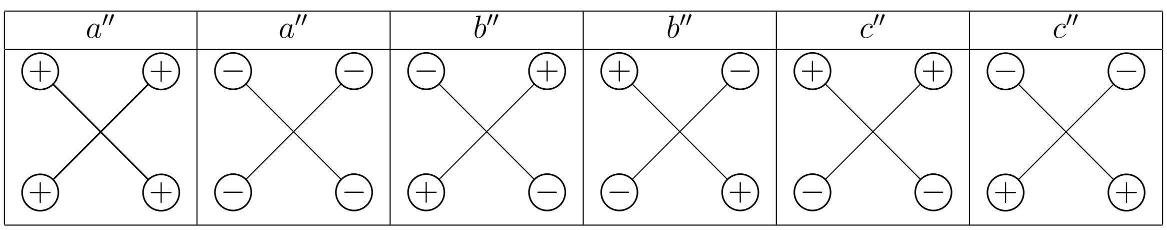
$$\frac{a^2+b^2-c^2}{2ab} = \frac{(a')^2+(b')^2-(c')^2}{2a'b'} = \Delta.$$

Let $v = v(a, b, c)$ and $w = v(a', b', c')$ be the two corresponding vertex types. Then the corresponding row transfer matrices commute:

$$T_w T_v = T_v T_w.$$

We should think of this in the context of "diagonalizing" the matrix T_v, for it is often easier to diagonalize a large family of commuting operators than a single operator.

PROOF. To prove this, we will make use of the Yang-Baxter equation, with the R-matrix r from the last section. We recall from the last lecture that this is the vertex $v(a'', b'', c'')$ which we draw in a rotated orientation, thus:



where

$$a'' = \frac{ba'b' - a(b')^2 + a(c')^2}{a'} = \frac{aba' - a^2b' + c^2b'}{b},$$
$$b'' = ba' - ab', \qquad c'' = cc'.$$

We recall that also

$$\frac{(a'')^2+(b'')^2-(c'')^2}{2a''b''} = \Delta.$$

The matrix r is invertible in the following sense. We think of the two vertices to the right of the matrix as "inputs" and the vertices to the left as "outputs" so that r is represented as a matrix

$$r = \begin{pmatrix} a'' & & & \\ & c'' & b'' & \\ & b'' & c'' & \\ & & & a'' \end{pmatrix}$$

with inverse (as usual matrices):

$$\begin{pmatrix} a''' & & & \\ & c''' & b''' & \\ & b''' & c''' & \\ & & & a''' \end{pmatrix} \begin{pmatrix} a''' & & & \\ & c''' & b''' & \\ & b''' & c''' & \\ & & & a''' \end{pmatrix} =$$

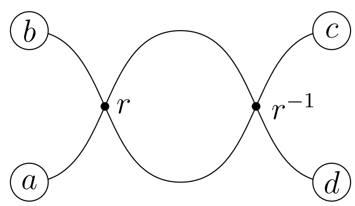
It may be computed that

$$a''' = \frac{1}{a''}, \qquad b''' = \frac{-b''}{(c'')^2 - (b'')^2}, \qquad c''' = \frac{c''}{(c'')^2 - (b'')^2}.$$

Then we compute that also

$$\frac{(a''')^2 + (b''')^2 - (c''')^2}{2a'''b'''} = \Delta.$$

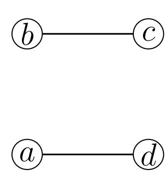
Now we may concatenate the matrices r and r^{-1}, and this is done by ordinary matrix multiplication. In other words, if we compute the partition function of the following system:



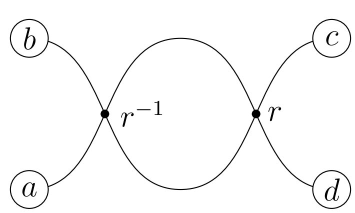
we get 1 if $a = d$ and $b = c$ but 0 otherwise. This is because summing over the middle column (four possibilities) really amounts to just multiplying matrices:

$$\begin{pmatrix} a'' & & & \\ & c'' & b'' & \\ & b'' & c'' & \\ & & & a'' \end{pmatrix}$$

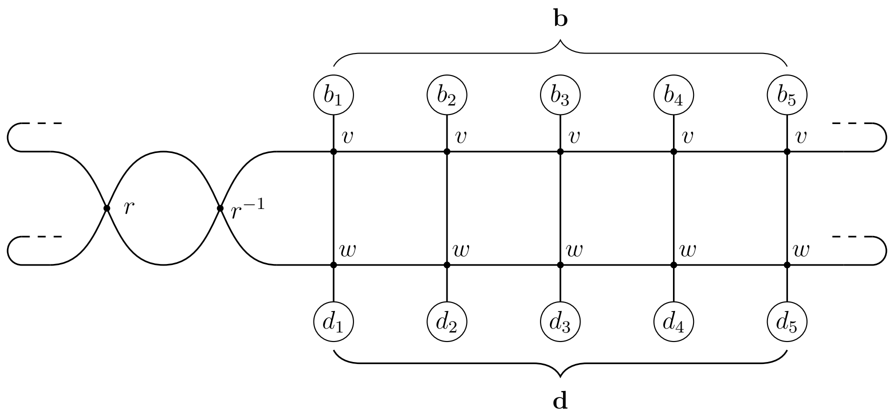
So this concatenation of r and r^{-1} is equivalent to:



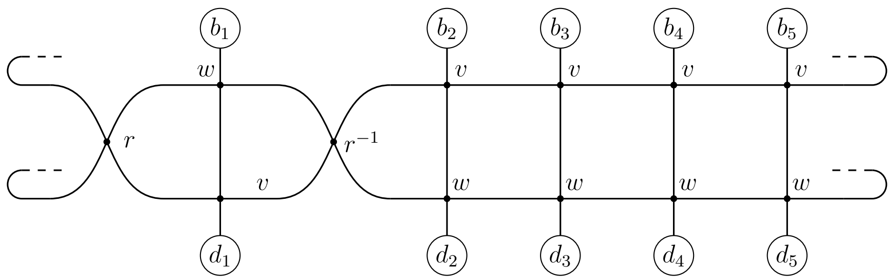
This is also equivalent to



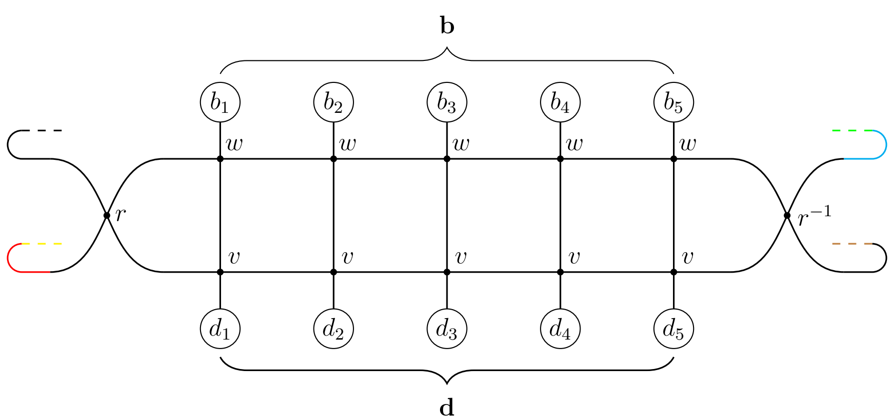
We may insert r and r^{-1} into our system representing $\langle \mathbf{d} T_w T_v | \mathbf{b} \rangle$ to obtain:



Now we use the Yang-Baxter equation to see that this system is equivalent to:



We may repeat this process several times to obtain this system:

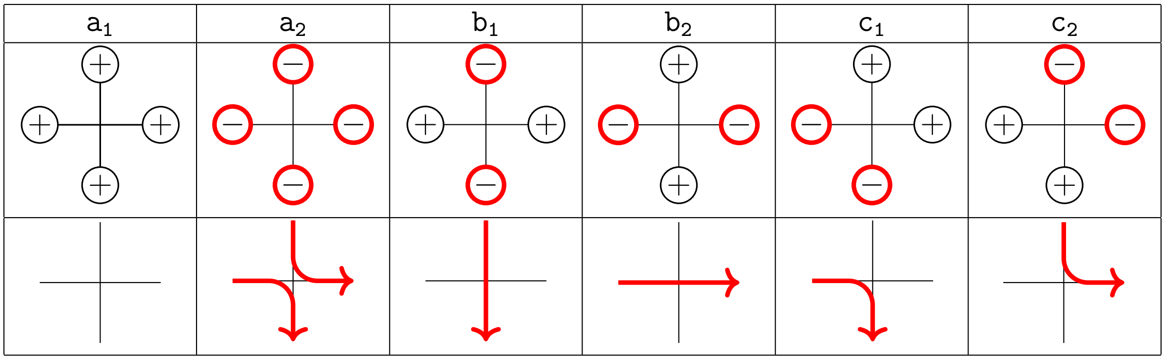


Now due to the cylindric boundary conditions, the r and r^{-1} are again adjacent and may be discarded. But now the system represents $\langle \mathbf{d} T_v T_w | \rangle$. We have proven that the two row transfer matrices commute. $\square$

1. Paths

In many models we may visualize states in terms of paths (or lines) through the lattice. Let us see how this works with the six-vertex model.

We will interpret a $-$ state as the presence of a particle, and $+$ as the absence of a particle. We will visualize the particles as moving from top to bottom, and from left to right.



We have drawn the particles in red, then visualized the paths they must take. In the case of a_2 we have elected not to allow the paths to cross, though in other schemes they *might* cross.

In the last section we considered cylindric boundary conditions, wrapping the grid around into a cylinder. We might also consider *toroidal* boundary conditions, additionally wrapping the top to bottom so that there are no boundary edges. Now, however, we want to do no wrapping, envisioning a rectangular grid with boundary edges on the left, right, top and bottom. To specify the system, we must specify which of these will be $-$ and which will be $+$. We refer to this specification as *domain wall boundary conditions*.

LEMMA 1.1. *Let us consider a grid with domain wall boundary conditions. The number of $-$ on the top and left must equal the number of $-$ on the right and bottom, or else the system will have no admissible states.*

PROOF. Every line must start at the top or left and finish on the right or bottom. This gives a bijection between the $-$ spins on the top or left and those on the right or bottom. $\square$

LECTURE 3

Commuting Transfer Matrices

1. Gelfand-Tsetlin Patterns and States

Remark 1. This section mentions some facts about Lie group representations and Schur polynomials. These are included since they may be helpful to some readers, but may be skipped. Schur polynomials will be properly introduced later and we will prove their principal properties using lattice models.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a sequence of nonnegative integers. We say λ is a *partition* of length $\leqslant n$ if it is weakly decreasing:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

(Note that we say the length is $\leqslant n$. The actual length is the number of nonzero entries.) We say λ is a partition of k, and write $\lambda \vdash k$ if $\sum \lambda_i = k$. The partition is *strict* if

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n \geqslant 0.$$

A strict partition is the same as a partition into unequal parts.

A closely related notion is that of a *dominant weight* for $\mathrm{GL}(n)$. Assuming some Lie theory, we may identify $\mathbb{Z}^n$ with the $\mathrm{GL}(n)$ weight lattice Λ. Then the weight $\lambda = (\lambda_1, \cdots, \lambda_n)$ is *dominant* if

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

If λ is a dominant weight, then by results of Schur and Weyl, there is a unique irreducible representation $\pi_\lambda^{\mathrm{GL}(n)}$ of $\mathrm{GL}(n, \mathbb{C})$ with highest weight λ. Its character χ_λ is essentially the *Schur polynomial* s_λ. This is a symmetric polyomial such that

$$\chi_\lambda(g) = s_\lambda(z_1, \cdots, z_n)$$

where z_i are the eigenvalues of $g \in \mathrm{GL}(n, \mathbb{C})$.

Thus a partition of length $\leqslant n$ is a dominant weight for $\mathrm{GL}(n)$. A dominant weight λ is a partition only if $\lambda_n \geqslant 0$.

Gelfand-Tsetlin patterns are triangular arrays of integers satisfying certain inequalities. We may express this by saying that the rows are weakly decreasing, and adjacent rows *interleave*. This means the following. Suppose that $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\mu = (\mu_1, \cdots, \mu_{n-1})$ are partitions or more generally dominant weights. The condition for λ and μ to interleave is that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \lambda_n.$$

Note: λ and μ interleave if and only if $\pi_\mu^{\mathrm{GL}(n-1)}$ appears in the restriction of $\pi_\lambda^{\mathrm{GL}(n)}$ to $\mathrm{GL}(n-1, \mathbb{C})$. Indeed we will later prove (using lattice models) that

$$s_\lambda(z_1, \cdots, z_{n-1}, 1) = \sum_{\text{dominant } \mu \text{ interleaving } \lambda} s_\mu(z_1, \cdots, z_{n-1}),$$

which is called the $\mathrm{GL}(n) \Rightarrow \mathrm{GL}(n-1)$ *branching rule*.

We may now define a Gelfand-Tsetlin pattern of size n. This is a triangular array

$$A = \left\{ \begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\ & a_{21} & & a_{22} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & & & & & & \\ & & & & a_{n0} & & & & \end{array} \right\}$$

with $n+1$ rows such that each row is a partition, and the rows interleave.

For example, there are 8 Gelfand-Tsetlin patterns with top row $(2,1,0)$. These are:

$$\left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right\}, \quad \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\} \quad \color{red}{\left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\}}, \quad \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{array} \right\}$$

$$\left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array} \right\}, \quad \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{array} \right\} \quad \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \right\}, \quad \left\{ \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\}.$$

These patterns are all strict, except the third one, which we have marked in red.

Strict Gelfand-Tsetlin patterns with top row $(n-1, n-2, \cdots, 0)$ are in bijection with another type of mathematical entity called *alternating sign matrices.* We will not discuss alternating sign matrices much, preferring to work with Gelfand-Tsetlin patterns. But due to the historical importance of alternating sign matrices, here is the rundown.

An *alternating sign matrix* of size n is a square matrix whose entries are all 0, 1 or -1. It is assumed that in every row and column, the nonzero entries alternate between 1 and -1, strating and ending with 1, so there are an odd number of nonzero entries in each row and column. For example, a permutation matrix is an alternating sign matrix.

Lemma 1.1. *There is a bijection between strict Gelfand-Tsetlin patterns with top row* $(n-1, n-2, \cdots, 0)$ *and alternating sign matrices of size* n*.*

Proof. We will explain the bijection with an example. Consider the alternating sign matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We “accumulate” the rows from the bottom up into annother matrix:

$$B = \begin{bmatrix} 1^3 & 1^2 & 1^1 & 1^0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus the bottom row of B is the bottom row of A, the second-from-bottom row of B is the sum of the last two rows of A, and so forth. The little numbers indicate that we have labeled the columns of this $n-1$, $n-2$, $\cdots$, 0 from right to left, with $n=4$ in the example.

Now we read off the columns of B that have nonzero entries (all equal to 1) and these form a strict Gelfand-Tsetlin pattern:

$$\left\{\begin{array}{ccccccc} 3 & & 2 & & 1 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{array}\right\}.$$

We leave it to the reader to convince themselves that this is a bijection. □

Alternating sign matrices originated in a method of computing determinants due to Charles Dodgson (Lewis Carroll). In the 1980's, Mills, Robbins and Rumsey investigated the number of alternating sign matrices of size n and conjectured that the number of ASM of size n

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\,. \tag{7}$$

Here are some values:

n	1	2	3	4	5	6
$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$	1	2	7	42	429	7436

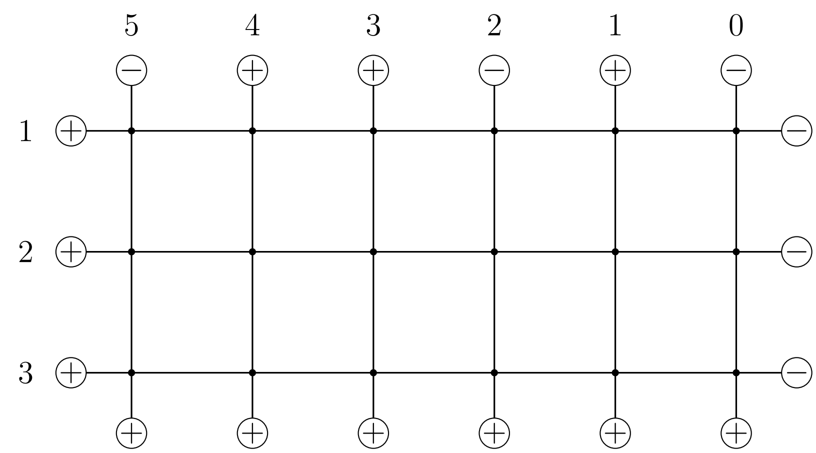
The number 7 when $n = 3$ we have already seen is the number of strict Gelfand-Tsetlin patterns with top row $(2, 1, 0)$.

Robbins and Rumsey consulted with Richard Stanley, who did not know how to prove the conjecture, but told them that the same numbers had appeared in another context in work of Andrews. They equal the number of "totally symmetric self-complementary plane partitions," a seemingly unrelated combinatorial number. The precise number remained only conjectural.

The number (7), conjectured by Mills, Robbins and Rumsey for the number of alternating sign matrices of size n, was proved correct by Zeilberger in difficult work that did not really give insight. Kuperberg then gave another proof using solvable lattice models that gave deep insight. We will eventually cover Kuperberg's proof, and the related Korepin-Izergin determinant formula.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a partition. We consider a model with domain wall boundary conditions as follows. We need a grid with $N + 1$ columns, which we label from 0 to N from right to left, and n rows, labeled $1, 2, \cdots, n$ from top to bottom. We need $N \geqslant \lambda_1$. We have boundary edges on the left, right, top and bottom. On the left and bottom we put $+$ spins, and on the right we put $-$ spins. On the top, the Lemma from Lecture 2 shows that we need n edges with $-$ spins, and the rest must be $+$. We put the $-$ spins at columns labeled $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Thus for example of $\lambda = (5, 2, 0)$ we need six columns and 3 rows, and we arrive at the following boundary conditions:



LEMMA 1.2. *There is a bijection between the states of this system and the strict Gelfand-Tsetlin patterns with top row λ.*

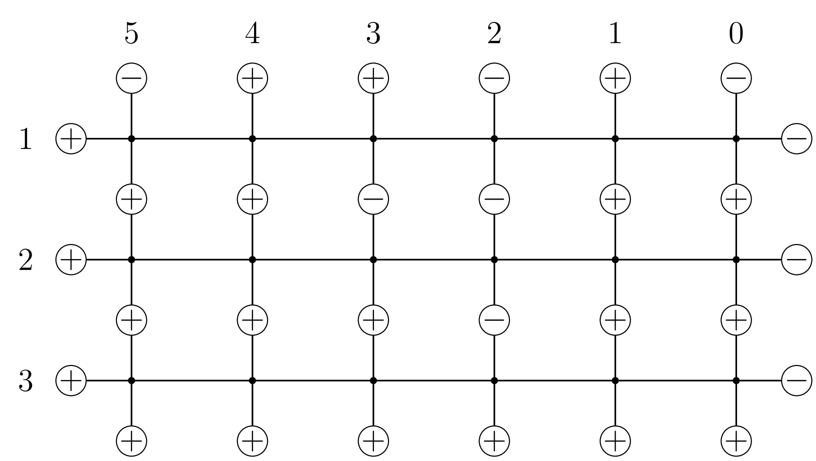
PROOF. We will describe the bijection algorithmically (with an example) and leave it to the reader to convince themselves that the states described are admissible.

First, for each row $a_{i1}, \cdots, a_{i,n+1-i}$ in the given strict Gelfand-Tsetlin pattern, assign a $-$ spin to the vertical edges above the i-th row, and $+$ to the remaining vertical edges. Note that since $\lambda = (a_{11}, \cdots, a_{1n})$, this is consistent with the way we assigned the boundary spins at the top.

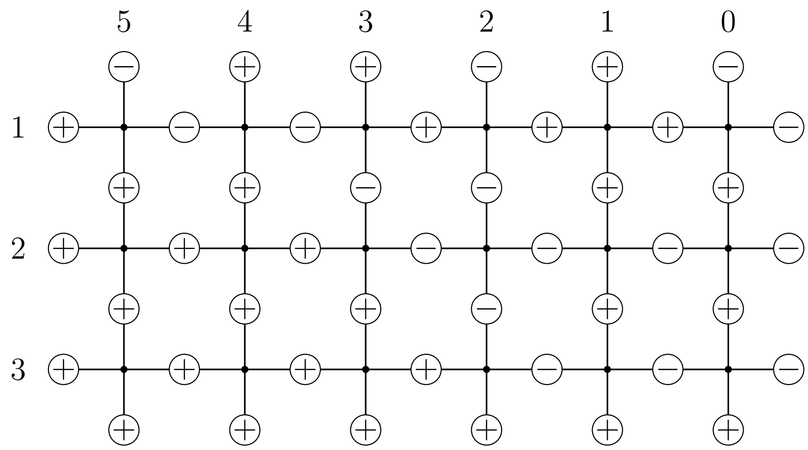
For example suppose that the Gelfand-Tsetlin pattern is:

$$\left\{ \begin{array}{ccccc} 5 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 2 & & \end{array} \right\}$$

This assigns spins as follows:



It remains to be assigned spins to the horizontal edges. We may do this in each row by making use of the fact that the number of $-$ spins adjacent to every vertex is even, to figure out the rows.



This gives the configuration, and it must be checked that it is admissible for the six-vertex model. □

Now let us consider the free-fermionic six-vertex model taking $a = b = c = 1$. Clearly the Boltzmann weight of each state is 1, so the partition function is equal to the number of states. Thus if we can evaluate the partition function we will have counted the number of states. In the special case where $\lambda = (n-1, n-2, \cdots, 0)$, this is equal to the number of alternating sign matrices.

This is the case where the lattice is square, with domain-wall boundary conditions, putting $+$ on the left and bottom boundary edges, and $-$ in the top and right boundary edges. In this case, there is a formula for the partition function as a determinant, due to Korepin and Izergin. This was Kuperberg's approach to the alternating sign matrix conjecture.

The proof of the Korepin-Izergin determinant formula depends again on the Yang-Baxter equation, which we will need to formulate more precisely than before. For the proof of the commutativity of the row-transfer matrices, all we needed to know about the R-matrix was its existence, but for other applications we will need to know its values, a topic that we will come to soon.

2. Vector Yang-Baxter Equation

We will give another notion of the Yang-Baxter equation. Soon we will connect it with the familiar one that we used in the last two lectures.

Let U, V and W be vector spaces. Suppose that we are given three linear transformations:

$$R : U \otimes V \longrightarrow V \otimes U,$$

$$S : U \otimes W \longrightarrow W \otimes U,$$

$$T : V \otimes W \longrightarrow W \otimes V.$$

We will consider two homomorphisms $U \otimes V \otimes W \longrightarrow W \otimes V \otimes U$. The first is the composition

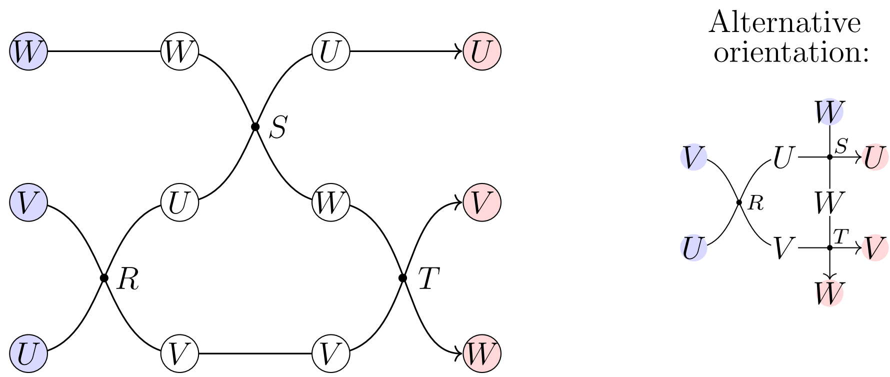
$$U \otimes V \otimes W \xrightarrow{R_{12}} V \otimes U \otimes W \xrightarrow{S_{23}} V \otimes W \otimes U \xrightarrow{T_{12}} W \otimes V \otimes U$$

where the notation is that R_{ij} means R applied to the i and j components of a tensor. Thus $R_{12} = R \otimes I_W$, $S_{23} = I_V \otimes S$ and $T_{12} = T \otimes I_U$. (The subscript notation is popular in Hopf algebra and quantum group literature.) The other homomoprhism is

$$U \otimes V \otimes W \xrightarrow{T_{23}} U \otimes W \otimes V \xrightarrow{S_{12}} W \otimes U \otimes V \xrightarrow{R_{23}} W \otimes V \otimes U.$$

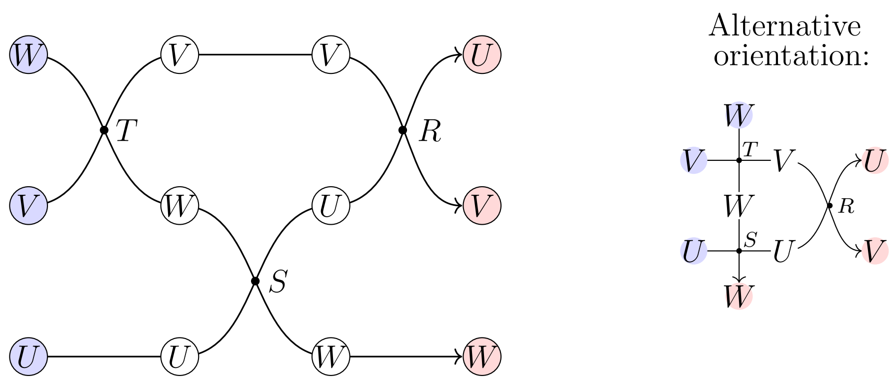
We can diagram the homomorphisms graphically as follows.

$$U \otimes V \otimes W \xrightarrow{R_{12}} V \otimes U \otimes W \xrightarrow{S_{23}} V \otimes W \otimes U \xrightarrow{T_{12}} W \otimes V \otimes U$$



and

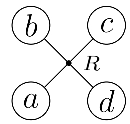
$$U \otimes V \otimes W \xrightarrow{T_{23}} U \otimes W \otimes V \xrightarrow{S_{12}} W \otimes U \otimes V \xrightarrow{R_{23}} W \otimes V \otimes U$$



If these two homomorphisms $U \otimes V \otimes W \longrightarrow W \otimes V \otimes U$ are equal, we will say that R, S, T give an instance of the *Yang-Baxter equation*. We will of course have to explain how this is related to the Yang-Baxter equations we have previously described in terms of Boltzmann weights.

For the six- or eight-vertex models, the vector spaces U, V and W can be taken to be two-dimensional, with bases indexed by the possible spins. Thus U is spanned by u_+, u_- and similarly we have bases v_+, v_- and w_+, w_- for V and W. Let us start with a vertex R with chosen Boltzmann weights $a_1(R), a_2(R)$, etc. and encode these weights in a linear transformation $R : U \otimes V \longrightarrow V \otimes U$ by the following rule. If $a, b, c, d \in \{\pm\}$ then the

Boltzmann weight of the state



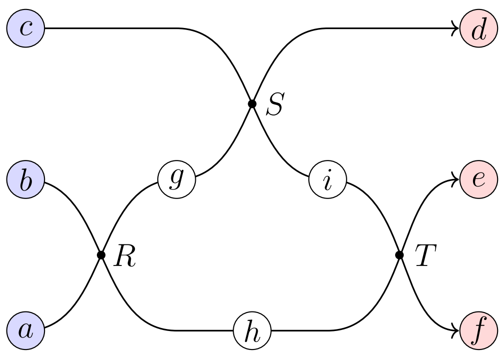
is to be the coefficient of $v_d \otimes u_c$ in $R(u_a \otimes v_b)$. We will write this coefficient in Dirac notation as $\langle v_d \otimes u_c | R | u_a \otimes v_b \rangle$, or if we are thinking of it as a Boltzmann weight as

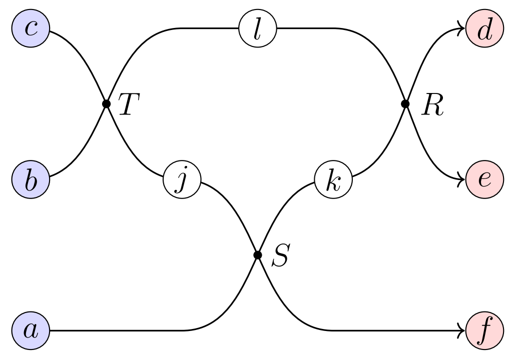
$$\beta_R \begin{pmatrix} b & c \\ a & d \end{pmatrix}.$$

So

$$R|u_a \otimes v_b\rangle := R(u_a \otimes v_b) = \sum_{c,d} \beta_R \begin{pmatrix} b & c \\ a & d \end{pmatrix} |v_d \otimes u_c\rangle.$$

Lemma 2.1. *The partition function of the systems*





equal $\langle w_f \otimes v_e \otimes u_d | T_{12} S_{23} R_{12} | u_a \otimes v_b \otimes w_c \rangle$ *and* $\langle w_f \otimes v_e \otimes u_d | R_{13} S_{23} R_{12} | u_a \otimes v_b \otimes w_c \rangle$.

Proof. We compute

$$T_{12}S_{23}R_{12}|u_a \otimes v_b \otimes w_c\rangle = \sum_{g,h} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} T_{12}S_{23}|v_h \otimes u_g \otimes w_c\rangle$$

$$= \sum_{g,h} \sum_{i,d} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} T_{12}|v_h \otimes w_i \otimes u_d\rangle$$

$$= \sum_{g,h} \sum_{d,i} \sum_{e,f} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} \beta_T \begin{pmatrix} i & e \\ h & f \end{pmatrix} |w_f \otimes v_e \otimes u_d\rangle.$$

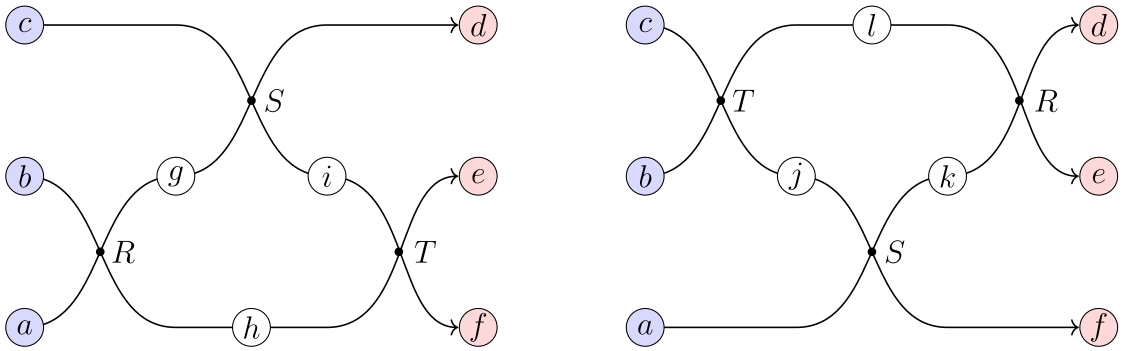
Therefore

$$\langle w_f \otimes v_e \otimes u_d | T_{12}S_{23}R_{12} | u_a \otimes v_b \otimes w_c\rangle = \sum_{g,h} \sum_{d,i} \sum_{e,f} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} \beta_T \begin{pmatrix} i & e \\ h & f \end{pmatrix}.$$

The right hand side is the partition function of the left-side of the Yang-Baxter equation system. As usual, the boundary spins a, b, c, d, e, f are fixed, and the spins of the interior edges g, h, i or j, k, l are summed over in the partition function. We leave the reader to check the other side. □

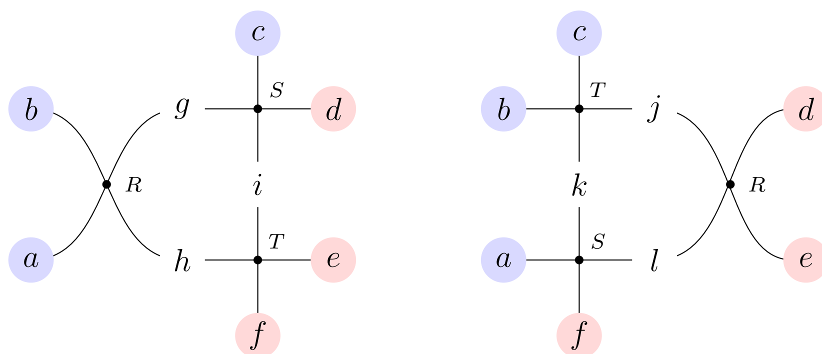
Therefore:

THEOREM 2.2. *Let R, S, T be vertex types, and let U, V, W be as above, and define homomorphisms $R : U \otimes V \longrightarrow V \otimes U$ as above. If for all choices of boindary spins the partition functions of the systems*



agree, then the vector Yang-Baxter equation is satisfied.

One may also reorient the edges and work instead with the systems:



LECTURE 4

Braided Categories and Parametrized Yang-Baxter equations

We will start by surveying the origin of solutions to the Yang-Baxter equation in the notions of *braided monoidal categories* and (sketchily) *quantum groups*. Then we will look again at the notion of a *parametrized Yang-Baxter equation*, in which the vertex types are indexed by a group or groupoid. We will give one example, coming from the field-free Yang-Baxter equation in Lecture 1, obtaining a clearer picture.

1. Braided Monoidal Categories

This section is **optional** and can be skipped or postponed on first reading.

The axioms for a braided monoidal category are due to Joyal and Street [**53**] in the 1980's. It is surprising that such an important concept was not formulated until so late. But there weren't many obvious examples of braided monoidal categories until quantum groups. But it turns out that the modules of a quantum group form a braided category, giving a tremendous fount of examples of the Yang-Baxter equation. We digress to introduce this notion.

Wikipedia link

A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes$ satisfying certain natural axioms. There is a *unit object* I with natural isomorphisms

$$A \otimes I \cong I \otimes A \cong A$$

for A any object in the category, and for three objects A, B, C a natural isomorphism

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

satisfying Maclane's *pentagon axiom*

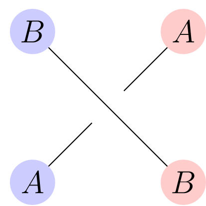
$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur] \arrow[d] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[r] & A \otimes ((B \otimes C) \otimes D) \arrow[ur] &
\end{tikzcd}$$

Maclane's coherence theorem asserts that all similar identities (perhaps involving many tensors) can be deduced from this one.

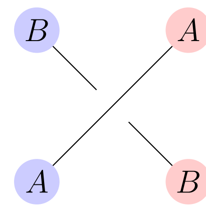
Let $\mathcal{C}$ be a monoidal category. We recall that if A, B, C are objects in $\mathcal{C}$ then there are natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. We will not distinguish between these objects and just denote either as $A \otimes B \otimes C$.

In a braided category there are explicit *braid isomorphisms* $c_{A,B} : A \otimes B \to B \otimes A$ but now we must be careful. For example the composition $c_{B,A}c_{A,B}$ is not assumed to be the identity. So $c_{A,B}$ and $c_{B,A}^{-1}$ are *distinct isomorphisms* $A \to B$.

We will notate the morphism $c_{A,B}$ by an over crossing and $c_{B,A}$ by an under crossing.



$c_{A,B} : A \otimes B \to B \otimes A$



$c_{B,A}^{-1} : A \otimes B \to B \otimes A$

We review the important notion of a natural transformation. We used this implicitly when we defined a monoidal category in Lecture 1, where we said that the isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

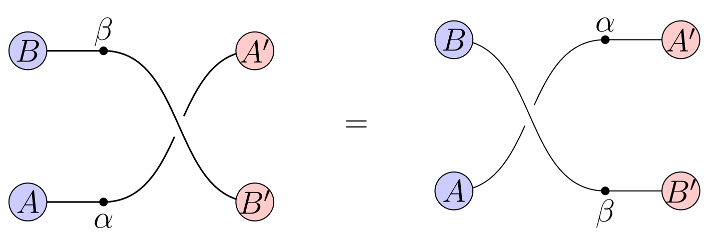
are required to be *natural*.

This means, explicitly, the following. Since $\otimes$ is a bifunctor, if $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ are morphisms then we have on the left and right of the following diagram.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\cong"] \arrow[d, "(\alpha\otimes\beta)\otimes\gamma"] & A \otimes (B \otimes C) \arrow[d, "\alpha\otimes(\beta\otimes\gamma)"] \\
(A' \otimes B') \otimes C' \arrow[r, "\cong"] & A' \otimes (B' \otimes C')
\end{tikzcd}$$

The first axiom of a braided category is that the morphisms $c_{A,B} : A \otimes B \to B \otimes A$ are to be natural. This means that if $\alpha : A \to A'$ and $\beta : B \to B'$ are morphisms, then

$$(\beta \otimes \alpha) \circ c_{A,B} = c_{A',B'} \circ (\alpha \otimes \beta)$$

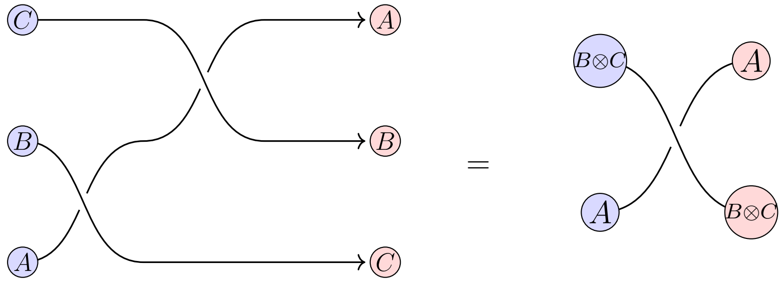


(We are representing the morphisms α, β by dots.)

The braid morphism $c_{A,B} : A \otimes B \to B \otimes A$ is sometimes called an *R-matrix*. It is subject to a couple of axioms. First, it is assumed to satisfy:

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A,B\otimes C}"] \arrow[dr, "c_{A,B}\otimes 1_C"'] & & B \otimes C \otimes A \\
& B \otimes A \otimes C \arrow[ur, "1_B\otimes c_{A,C}"'] &
\end{tikzcd}$$

We can diagram this as follows.

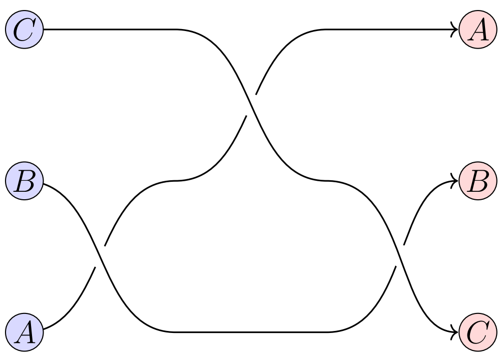


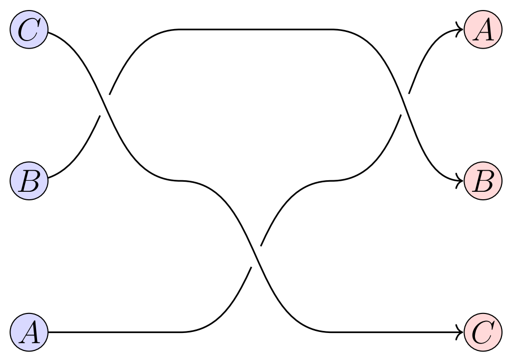
The dual axiom is also needed:

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A\times B,C}"] \arrow[dr, "1_A\otimes c_{B,C}"'] & & C \otimes A \otimes B \\
& A \otimes C \otimes B \arrow[ur, "c_{A,C}\otimes 1_B"'] &
\end{tikzcd}$$

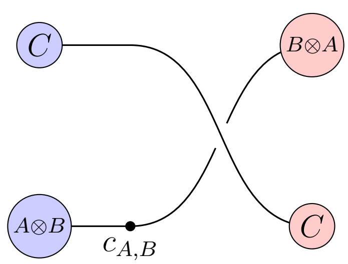
This completes the definition of a braided monoidal category.

THEOREM 1.1. *The Yang-Baxter equation is true in a braided monoidal category. This means we have to show the equivalence of the two following morphisms* $A\otimes B\otimes C \to C\otimes B\otimes A$*:*

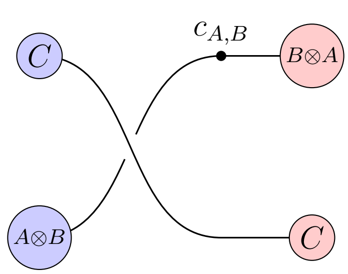




PROOF. Using one of the axioms for the braided category, the first diagram agrees with:



Using naturality, this agrees with



Now using the other axiom, this is equivalent to the morphism in the second diagram. □

2. Quantum Groups

We see that objects in a braided category, particularly if they can be realized as vector spaces, are a potential source of instances of the Yang-Baxter equation. These have applications (as we know) to solvable lattice models, but also to other areas, such as knot invariants (e.g. the Jones polynomial).

Around the same time that Joyal and Street formulated the notion of a braided category, Drinfeld [**35**] invented the notion of a *quasitriangular Hopf algebra*.

If H is an associative algebra, one might hope that the modules form a monoidal category. However if A, B are modules then $A \otimes B$ is not naturally a module for H, but for the tensor product algebra $H \otimes H$. A *Hopf algebra* is an associative algebra H together with an algebra homomorphism $\Delta : H \to H \otimes H$ called the *comultiplication* and some other structure (antipode, counit, various axioms). Using Δ, $A \otimes B$ becomes a module for H, and so the modules become a monoidal category.

A *quasitriangular Hopf algebra* has some further extra structure, a *universal R-matrix* $R \in H \otimes H$ satisfying certain axioms that we will not state here. (See [**35, 77, 59**].) What is important is that using R we may define a braiding $c_{A,B} : A \otimes B \to B \otimes A$, and Drinfeld's axioms for a quasitriangular Hopf algebra are exactly what is needed for the module category to be braided.

Drinfeld then constructed quasitriangular Hopf algebras called *quantum groups* as deformations of more familiar Hopf algebras. If $\mathfrak{g}$ is a Lie algebra, the *universal enveloping algebra* of $\mathfrak{g}$ is an associative algebra $U(\mathfrak{g})$ whose modules are the same as the modules of $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, or more generally a Kac-Moody Lie algebra or superalgebra, then it is possible to deform $U(\mathfrak{g})$ and obtain a family of Hopf algebras $U_q(\mathfrak{g})$ called *quantized enveloping algebras*. If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, it has an *affinization* $\widehat{\mathfrak{g}}$ which is infinite-dimensional. If V is a module for $\mathfrak{g}$, then $\widehat{\mathfrak{g}}$ has a family V_z of modules indexed by $z \in \mathbb{C}^\times$.

There are two choices of $\mathfrak{g}$ that have two-dimensional modules: $\mathfrak{g} = \mathfrak{sl}_2$ (or almost the same thing for this purpose, $\mathfrak{gl}_2$) or the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1|1)$. Both are related to the six-vertex model: $\widehat{\mathfrak{gl}}_2$ is related to the field-free models we have started with, while $\mathfrak{gl}(1|1)$ is related to the free-fermionic models that we will discuss later.

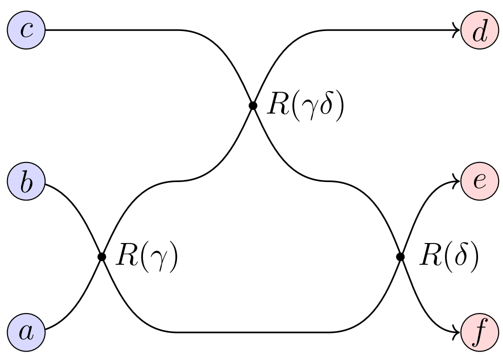
3. Parametrized Yang-Baxter equations

Let Γ be a group, and let V be a vector space. Let $R : \Gamma \longrightarrow \mathrm{GL}(V \otimes V)$ be a map such that for every $\gamma, \delta \in \Gamma$, we have a vector Yang-Baxter equation:

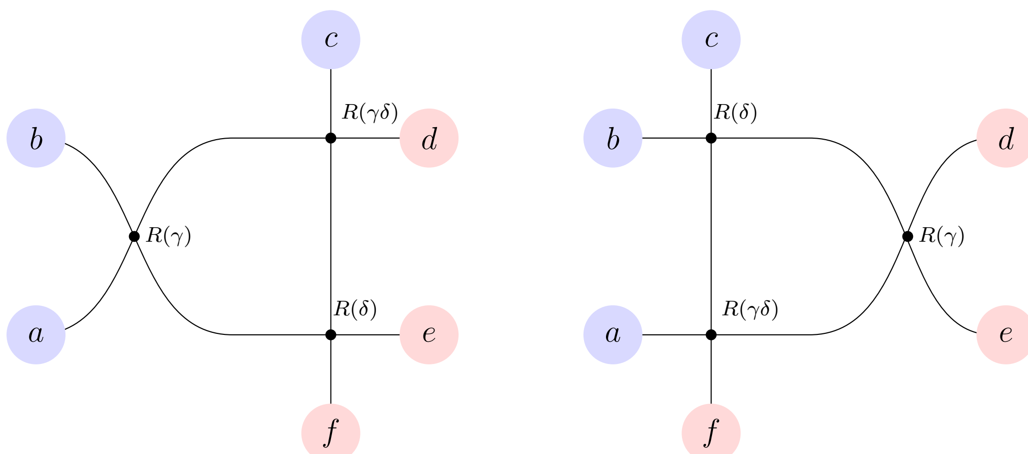
$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "R_{12}(\gamma)"'] \arrow[dr, "R_{23}(\delta)"] & \\
V \otimes V \otimes V \arrow[d, "R_{23}(\gamma\delta)"'] & & V \otimes V \otimes V \arrow[d, "R_{12}(\gamma\delta)"] \\
V \otimes V \otimes V \arrow[dr, "R_{12}(\delta)"'] & & V \otimes V \otimes V \arrow[dl, "R_{23}(\gamma)"] \\
 & V \otimes V \otimes V &
\end{tikzcd}$$

Then we say that we have a *parametrized Yang-Baxter equation* with parameter group Γ.

Alternatively, we could let Σ be a set and for every $\gamma \in \Gamma$ let there be a vertex type, where all edges have the spinset Γ. We will use the notation $R(\gamma)$ for this vertex type. Then we ask that for all a, b, c, d, e, f the two following partition functions are equal:



We can obtain a parametrized Yang-Baxter equation taking V to be the free vector-space on Σ and following the construction of the last section. We could alternatively orient the edges as follows:



In either case, the procedure in Lecture 3 produces a vector Yang-Baxter equation, with V being the free vector space on the spinset Σ.

We will show in the next section that the field-free Yang-Baxter equation of Section 1 gives an example of a parametrized Yang-Baxter equation.

4. Parametrized Field-Free Yang-Baxter equation

Let $\Delta \in \mathbb{C}$ be fixed. Let q be found such that $\frac{1}{2}(q + q^{-1}) = \Delta$. We will use the notation $R(a, b, c)$ for the vertex with Boltzmann weights a, b, c, as before. Let G_Δ be the set of (a, b, c) with $a, b \neq 0$ such that

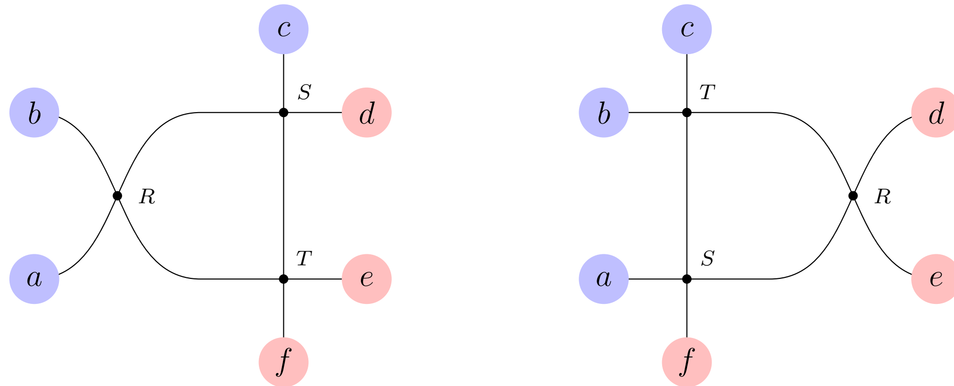
$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta,$$

together with two additional elements $(\pm\Delta, 0, \Delta)$. Eventually we will give G_Δ the structure of a group.

In Lecture 1 we showed that if (a_1, b_1, c_1) and (a_2, b_2, c_2) are in G_Δ then there exists a third $(a_0, b_0, c_0) \in G_\Delta$ such that if (in the notation of Lecture 1) $R = v(a_0, b_0, c_0)$, $S = v(a_1, b_1, c_1)$

and $T = v(a_2, b_2, c_2)$, then we have a Yang-Baxter equation:

(8)



We note that the Yang-Baxter equation is homogeneous in the sense that if any one of (a_i, b_i, c_i) is multiplied by a nonzero constant then the validity of the equation is unchanged. So while R is usually determined by S and T, it is only determined up to constant multiple.

Now we want to start with R and T and compute S. This will give us our first example of a parametrized Yang-Baxter equation. We begin by noting that G_Δ can be parametrized as follows.

Lemma 4.1. *Let $x \in \mathbb{C}^\times$ and let*

$$(a, b, c) = \left(\tfrac{1}{2}(xq - (xq)^{-1}), \tfrac{1}{2}(x - x^{-1}), \tfrac{1}{2}(q - q^{-1})\right). \tag{9}$$

Then $(a, b, c) \in G_\Delta$.

Proof. This is a straightforward calculation. □

Theorem 4.2. *The mapping*

$$R_\Delta : \mathbb{C}^\times \longrightarrow \{\textit{field-free Boltzmann weights } (a, b, c)\}$$

is a parametrized Yang-Baxter equation with parameter group $\mathbb{C}^\times$. Here the Boltzmann weights (a, b, c) of $R_\Delta(x)$ are given by (9).

Proof. The Boltzmann weights are

$$\beta_\Delta(R) = \left(\frac{1}{2}((xq) - (xq)^{-1}), \frac{1}{2}(x - x^{-1}), \frac{1}{2}(q - q^{-1})\right),$$

$$\beta_\Delta(S) = \left(\frac{1}{2}(xyq - (xyq)^{-1}), \frac{1}{2}(xy - (xy)^{-1}), \frac{1}{2}(q - q^{-1})\right),$$

$$\beta_\Delta(T) = \left(\frac{1}{2}(yq - (yq)^{-1}), \frac{1}{2}(y - y^{-1}), \frac{1}{2}(q - q^{-1})\right).$$

Checking the parametrized Yang-Baxter equation is now a matter of computation. There are 12 cases of boundary Boltzmann weights that give nontrivial identities, but acually these are reduandant and there are only 4 distinct indentities that need to be checked. I have posted a computer program called `field-free1.sage` at the class web page that checks this. □

Remark 2. There are three special cases. If $\Delta = 0$, then we are in the free-fermionic case. The parametrized Yang-Baxter equation in Theorem 4.2 can be embeded in a much larger one with parameter group $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$, so in this case Theorem 4.2 is true but it is not the whole story.

Remark 3. On the other hand, if $\Delta = \pm 1$ then $q = \Delta = \pm 1$ is the unique solution to $\Delta = \frac{1}{2}(q + q^{-1})$. We see from (9) that $c = 0$ and $a = \pm b$, so these are very degenerate systems. The values $\Delta = \pm 1$ are phase transition points. See Baxter [**5**], Chapter 8.

Remark 4. Another interesting case is $q = e^{2\pi i/6}$. Then we can take $x = q = -(xq)^{-1}$, and all three Boltzmann weights a, b, c are equal. This fact was exploited by Kuperberg [**66**] in proving the Alternating Sign Matrix Conjecture.

LECTURE 5

Tokuyama Models I

1. Schur Polynomials

Schur polynomials are symmetric polynomials very important in representation theory and combinatorics. Some useful references are [**76, 88, 24, 27**]. They have direct generalizations that are introduced in [**75**]. See [**41, 19, 1, 82**] for treatments using the *free-fermionic* six-vertex model. We recall that the Boltzmann weights are free-fermionic if $a_1(v)a_2(v) + b_1(v)b_2(v) - c_1(v)c_2(v) = 0$ at every vertex. This Lecture and the next are based on [**19**].

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of length $\leqslant n$. If $r < n$ we pad λ with 0's so that $\lambda = (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0)$ has exactly n parts. This is customary in dealing with partitions. We will give two definitions of the Schur polynomial s_λ. It will not be obvious that the two definitions are equivalent. We will use a lattice model to prove this.

1.1. First Definition. Define

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})}. \tag{10}$$

For example, if $n = 3$,

$$s_\lambda(x_1, x_2, x_3) = \frac{\begin{vmatrix} x_1^{\lambda_1+2} & x_2^{\lambda_1+2} & x_3^{\lambda_1+2} \\ x_1^{\lambda_2+1} & x_2^{\lambda_2+1} & x_3^{\lambda_2+1} \\ x_1^{\lambda_3} & x_2^{\lambda_3} & x_2^{\lambda_3} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}}. \tag{11}$$

Wikipedia attributes this definition to Jacobi, who defined Schur functions prior to Schur. The denominator is the Vandermonde determinant:

$$\det(x_j^{n-i}) = \prod_{i<j}(x_i - x_j).$$

It will be useful to introduce the vector $\rho = (n-1, n-2, \cdots, 0)$ so that the exponents are $\lambda_i + \rho_i$ and write the numerator as $\det(x_j^{(\lambda+\rho)_i})$.

Lemma 1.1. *The function s_λ is a symmetric polynomial. It is homogeneous of degree $|\lambda| = \sum \lambda_i$.*

Proof. The polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ is a unique factorization domain. Let us note that the numerator is divisible by every factor $x_i - x_j$ with $i < j$ of the Vandermonde denominator. Indeed, the numerator vanishes when $x_i = x_j$ since two columns of the determinant $\det(x_j^{\lambda_i+n-i})$ are then equal. Thus the numerator is divisible by each factor and therefore by their product since they are coprime. Therefore s_λ is a polynomial. It is symmetric since interchanging x_i and x_j multiplies the numerator and the denominator by -1. The homogeneity is also clear since the numerator and denominator are both homogeneous polynomials. □

Remark 5. The partition λ may be thought of as a dominant weight for the Lie group $\mathrm{GL}(n, \mathbb{C})$. The definition (11) is essentially the Weyl character formula. This formula gives the value character χ_λ of an irreducible representation with highest weight λ for an arbitrary Lie group at a point $\mathbf{z}$ in a fixed maximal torus as

$$\frac{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\rho)}}.$$

In this formula ρ would be usually be the "Weyl vector" which is half the sum of the positive roots, $\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2}\right)$ for $\mathrm{GL}(n, \mathbb{C})$. We've taken $\rho = (n-1, n-2, \cdots, 0)$, but this change just multiplies the numerator and denominator by the same constant. The Weyl group for $\mathrm{GL}(n, \mathbb{C})$ is the symmetric group S_n, so for $\mathrm{GL}(n, \mathbb{C})$, the alternating sum is just the determinant in (11). It follows that s_λ is essentially the character χ_λ of an irreducible representation π_λ of $\mathrm{GL}(n, \mathbb{C})$. More precisely, if $g \in \mathrm{GL}(n, \mathbb{C})$ has eigenvalues $\alpha_1, \cdots, \alpha_n$ then $\chi_\lambda(g) = s_\lambda(\alpha_1, \cdots, \alpha_n)$.

1.2. Second Definition. The *Young diagram* $\mathrm{YD}(\lambda)$ of a partition λ a collection of boxes with λ_1 in the first row, λ_2 in the second row, etc. A *semistandard Young tableau* T (SSYT) of shape λ in the alphabet $\{1, 2, \cdots, n\}$ is a filling of $\mathrm{YD}(\lambda)$ with integers $1, \cdots, n$ such that the rows are weakly increasing, and the columns are strictly decreasing. The *weight* $\mathrm{wt}(T)$ is $(\mu_1, \cdots, \mu_n)$ where μ_i is the number of i's in T.

Example 1.2. Let $\lambda = (5, 2, 2)$ and $n = 5$. Then

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 5 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array}$$

is a SSYT of shape λ. Its weight is $(2, 4, 1, 0, 2)$.

If $\mathbf{z} = (z_1, \cdots, z_n) \in (\mathbb{C}^\times)^n$ and $\mu \in \mathbb{Z}^n$ let $\mathbf{z}^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$. The second definition of the Schur function is due to D.E. Littlewood (1938). It is this formula:

$$s_\lambda(z_1, \cdots, z_n) = \sum_T \mathbf{z}^{\mathrm{wt}(T)} \tag{12}$$

It is not obvious that this is symmetric. On the other hand, the Schur polynomial has another important property, *positivity*, that is not obvious from the first definition. This is the fact that as a polynomial, the coefficients are nonnegative.

We will use a lattice model to show that (12) is symmetric and equivalent to the first definition.

2. Tokuyama models

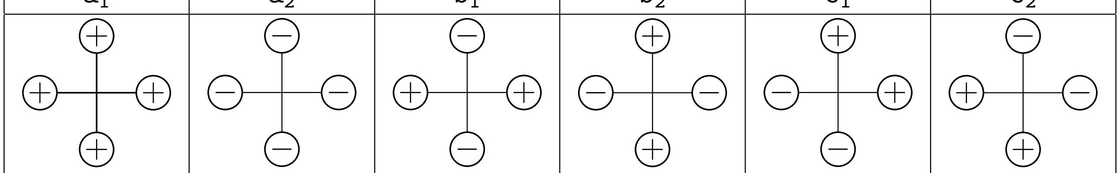
There is a formula due to Tokuyama [**91**] for the Schur function, or more precisely for

$$\left\{\prod_{i<j}(z_i - qz_j)\right\} s_\lambda(z_1, \cdots, z_n)$$

as a sum over strict Gelfand-Tsetlin patterns. If $q = 1$, the product is the Vandermonde determinant in the denominator of the first definition, and Tokuyama's formula reduces to the first definition of the Schur polynomial. On the other hand, if $q = 0$, Tokuyama's formula reduces to the combinatorial definition.

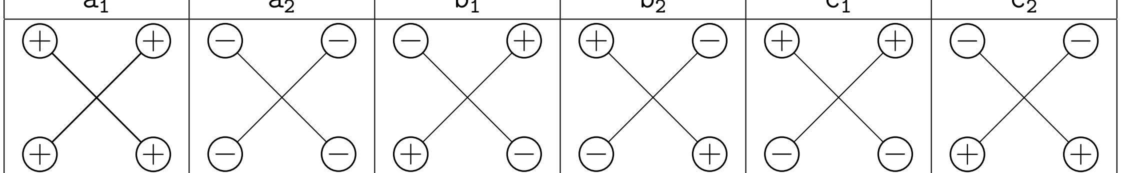
The models we will describe are similar to models in Hamel and King [**41**]. However they did not use the Yang-Baxter equation. The Yang-Baxter equation we need is associated with the quantum group $U_q(\widehat{\mathfrak{gl}}(1|1))$, related to the Lie superalgebra $\mathfrak{gl}(1|1)$. The modules associated with the horizontal edges are related to two-dimensional evaluation modules, but the module associated to the vertical edges are associated to two-dimensional Kac modules. This is not a parametrized Yang-Baxter equation as we formulated it in Lecture 4. However this solution can be embedded in a parametrized Yang-Baxter equation found by Korepin (see [**63**], page 126) and Brubaker, Bump and Friedberg [**19**], with parameter group $\mathrm{GL}(2) \times \mathrm{GL}(1)$ that parametrizes *all* free-fermionic vertices.

We take the following weights, labeled by a complex number z:



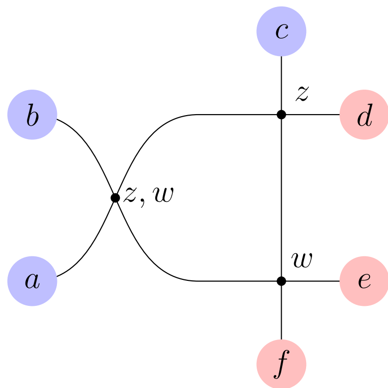
$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	z_i	$-q$	z_i	$z_i(1-q)$	1

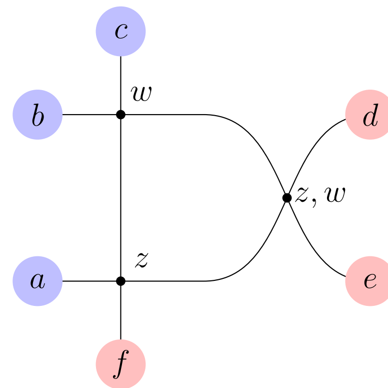
we also take the following R-matrix, labeled by two complex numbers z, w:



$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$z_j - qz_i$	$z_i - qz_j$	$q(z_i - z_j)$	$z_i - z_j$	$(1-q)z_i$	$(1-q)z_j$

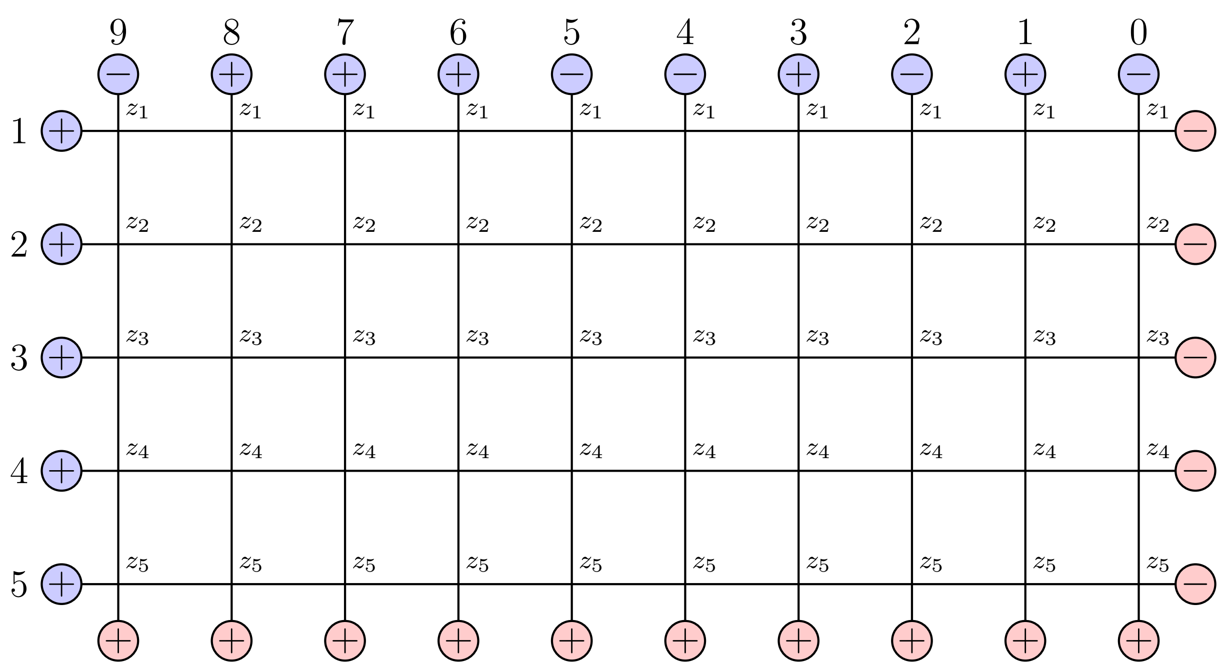
THEOREM 2.1. *The Yang-Baxter equation equation is satisfied in that he following two systems are equivalent for all choices of* $a, b, c, d, e, f \in \{+, -\}$*:*





Now let us explain the models we want to use, called "Gamma Ice" in [**19**].

Fix a partition λ. We begin with grid with rows labeled from top to bottom by $z_1, \cdots, z_n$ and columns labeled $0, \cdots, N$ with $N \geqslant \lambda_1$, ordered *from right to left*. The partition function will turn out to be independent of N. We will use the weights described above, so every vertex in the same row has the same label z_i. We must describe the spins on the boundary edges. On the left and bottom edges we put $+$, on the right we put $-$, and on the vertical j we put $-$ if j is an entry in $\lambda + \rho = (\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_n)$, or $+$ if j is not an element of this vector. For example if $n = 5$ and $\lambda = (5, 2, 2)$, we pad λ with zeros to get $(5, 2, 2, 0, 0)$ and then $\lambda + \rho = (9, 5, 4, 2, 0)$. Therefore we arrive at the following boundary conditions:



We have defined a system that we will denote $\mathfrak{S}_\lambda(\mathbf{z}; q)$. Let $Z_\lambda(z_1, \cdots, z_n; q) = Z_\lambda(\mathbf{z}; q)$ be the corresponding partition function.

THEOREM 2.2. *The partition function*

$$Z_\lambda(\mathbf{z};q) = \prod_{i<j}(z_i - qz_j)S_\lambda(\mathbf{z})$$

where $S_\lambda(\mathbf{z}) = S_\lambda(z_1, \cdots, z_n)$ *is a symmetric polynomial that is independent of* q.

We will give part of the proof in the next section using the train argument and the Yang-Baxter equation. We will then show in Lecture 6 that it implies the equivalence of the two definitions of the Schur function.

3. Proof of the Theorem, part I

We can break the proof into three steps.

PROPOSITION 3.1. *The quotient*

$$S_\lambda(\mathbf{z};q) = \frac{Z_\lambda(\mathbf{z};q)}{\prod_{i<j}(z_i - qz_j)} \tag{13}$$

is symmetric, that is, invariant under permutations of the z_i.

PROOF. We multiply (13) by:

$$\prod_{\substack{1\leqslant i,j\leqslant n \\ i\neq j}} (z_i - qz_j)\ .$$

This is a symmetric polynomial of degree $n(n-1)$ consisting of the $\frac{1}{2}n(n-1)$ factors in the denominator of (13) and $\frac{1}{2}n(n-1)$ others, so we see that it is enough to show that

$$Z_\lambda(\mathbf{z};q)\prod_{i<j}(z_j - qz_i)$$

is symmetric. Let $1 \leqslant k < n$ and let s_k be the "simple reflection" in the symmetric group which interchanges k and $k+1$. These generate the symmetric group, so it is sufficient to show that the last expression is invariant under s_k.

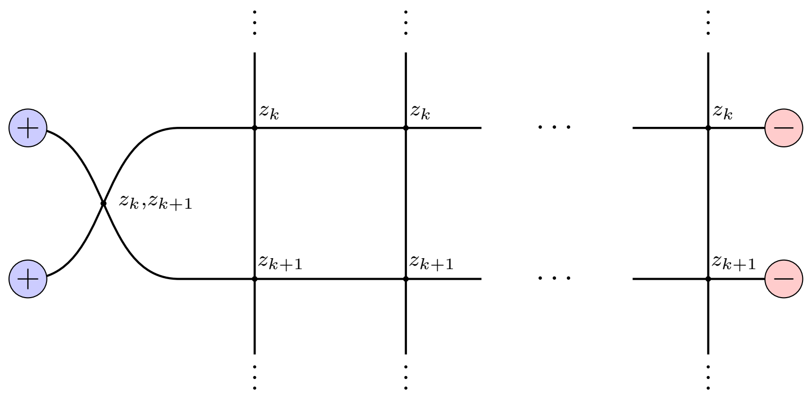
We can pull one factor out and write this as

$$Z_\lambda(\mathbf{z};q)(z_{k+1} - qz_k)\left[\prod_{\substack{i<j \\ (i,j)\neq(k,k+1)}} (z_j - qz_i).\right]$$

The permutation s_k just permutes the $\frac{1}{2}n(n-1) - 1$ factors in brackets. So we may drop these and it is sufficient to show that

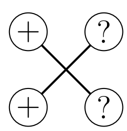
$$Z_\lambda(\mathbf{z};q)(z_{k+1} - qz_k) = Z_\lambda(s_k\mathbf{z};q)(z_k - qz_{k+1}). \tag{14}$$

To see this, let us consider the following system. We attach the R-matrix with coordinates z_k, z_{k+1} to the left at the $k, k+1$ rows:



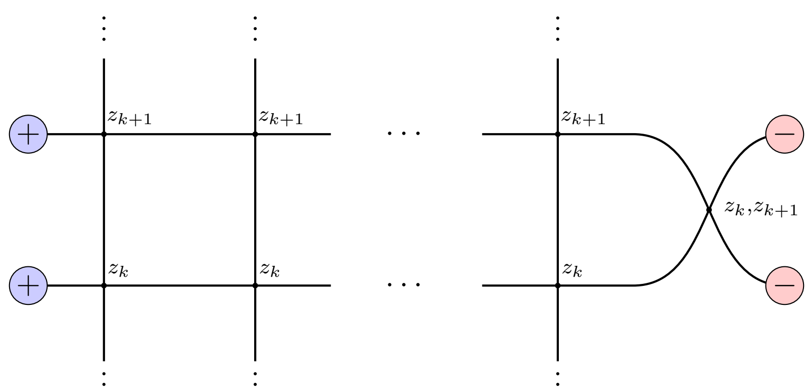
(15)

We note that from the Boltzmann weights if the "input" spins are $+, +$ there is only one possibility for the output spins, which must also be $+, +$:



spins marked ? can only = + !

The Boltzmann weight of the R-matrix is $z_{k+1} - qz_k$, and so the partition function of the system (15) is the left-hand side of (14). Using the train argument, this equals the partition function of



.

and by the same reasoning, this equals the right-hand side of (14). This proves (14) and the symmetry of $S_\lambda(\mathbf{z}; q)$ is established. $\square$

$$Z_\lambda(\mathbf{z}; q) = \prod_{i<j} (z_i - qz_j) S_\lambda(\mathbf{z})$$

Proposition 3.2. *$S_\lambda(\mathbf{z}; q)$ is a polynomial in $z_1, \cdots, z_n$ and q.*

PROOF. It is clear that $Z_\lambda(\mathbf{z};q)$ is a polynomial, since every Boltzmann weight is a polynomial. Rewrite (13) as

$$S_\lambda(\mathbf{z};q) = \frac{\prod_{i>j}(z_i - qz_j) Z_\lambda(\mathbf{z};q)}{\prod_{i\neq j}(z_i - qz_j)}. \tag{16}$$

Both the numerator and the denominator on the right-hand side here are symmetric. In the polynomial ring $\mathbb{C}[z_1, \cdots, z_n, q]$, which is a unique factorization domain, the denominator is a product of coprime polynomials, and it is sufficient to show that it is divisible by each. If $i > j$ then it is obvious that the numerator in (16) is divisible by $z_i - qz_j$ since it is included as a factor in the product defining the numerator. Because it is symmetric, it is divisible by all factors $z_i - qz_j$ because the symmetric group permutes these transitively. Thus the quotient $S_\lambda(\mathbf{z};q)$ is a polynomial. □

LEMMA 3.3. *Let $\mathfrak{s}$ be a state of the model. The total number of patterns of types* $\mathtt{a}_2, \mathtt{b}_1$ *and* $\mathtt{c}_1$ *in the state is* $\frac{1}{2}n(n-1)$.

PROOF. A vertex is of type $\mathtt{a}_2, \mathtt{b}_1$ or $\mathtt{c}_1$ if and only if it has a $-$ in the vertical edge below the vertex. We recall the Gelfand-Tsetlin pattern associated to the state in Lemma 1.3 of Lecture 3. There is a $-$ spin on the vertical edge below the vertex in row i and column j if and only if j is one of the entries in the $(i+1)$-th row of the Gelfand-Tsetlin pattern. There are thus $n-1$ patterns of type $\mathtt{a}_2, \mathtt{b}_1$ or $\mathtt{c}_1$ in the first row, $n-2$ in the second row, and so forth, and $\frac{1}{2}n(n-1)$ altogether. □

PROPOSITION 3.4. *$S_\lambda(\mathbf{z};q)$ is independent of q.*

PROOF. The numerator and denominator in (13) are both polynomials in $z_1, \cdots, z_n, q$ and the denominator has degree $\frac{1}{2}n(n-1)$ in q. We claim that the numerator has a too. Reviewing the Boltzmann weights, only patterns of types $\mathtt{b}_1$ and $\mathtt{c}_1$ can contribute a power of q. The number of such patterns is at most $\frac{1}{2}n(n-1)$ by Lemma 3.3.

Since the degree in q of the numerator of (13) is at most $\frac{1}{2}n(n-1)$, and the degree of the denominator is exactly $\frac{1}{2}n(n-1)$. Since the quotient is known to be a polynomial, it has degree 0 in q, hence is independent of q. □

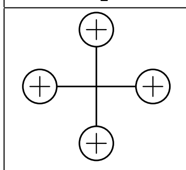
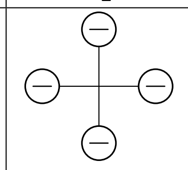
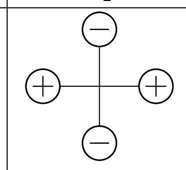
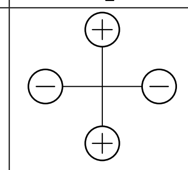
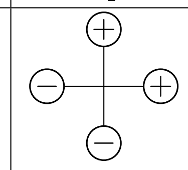
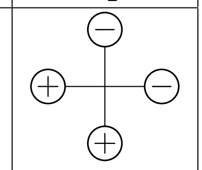
Since $S_\lambda(\mathbf{z};q)$ is independent of q, we may suppress q from the notation and write $S_\lambda(\mathbf{z};q) = S_\lambda(\mathbf{z})$. We have proved that it is a symmetric polynomial. In the next lecture we will show that if $q = 0$, this agrees with the combinatorial definition of $s_\lambda(\mathbf{z})$, and if $q = 1$, it agrees with the Jacobi definition.

LECTURE 6

Tokuyama Models II

1. Our story so far

We continue from Lecture 5, which we briefly review. For reference, here are the Boltzmann weights for Tokuyama ice from Lecture 5:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
					
1	z_i	$-q$	z_i	$z_i(1-q)$	1

We also described boundary conditions depending on a partition λ. The resulting system was denoted $\mathfrak{S}_\lambda(\digamma; q)$. It's partition function was denoted $Z_\lambda(\mathbf{z}; q)$ where $\mathbf{z} = (z_1, \cdots, z_n)$ are the row parameters. We then proved that there is a symmetric polynomial $S_\lambda(\mathbf{z})$, independent of q such that

$$Z_\lambda(\mathbf{z}; q) = \prod_{i<j}(z_i - qz_j)S_\lambda(\mathbf{z}).$$

In this lecture we will prove that S_λ agrees with the Schur function, using either the original Jacobi definition $\det(z_j^{\lambda_i+n-i})/\det(z_j^{n-i})$ when $q = 1$, or the combinatorial definition as a sum over semistandard Young tableaux (SSYT) when $q = 0$.

Our first result does not depend on q.

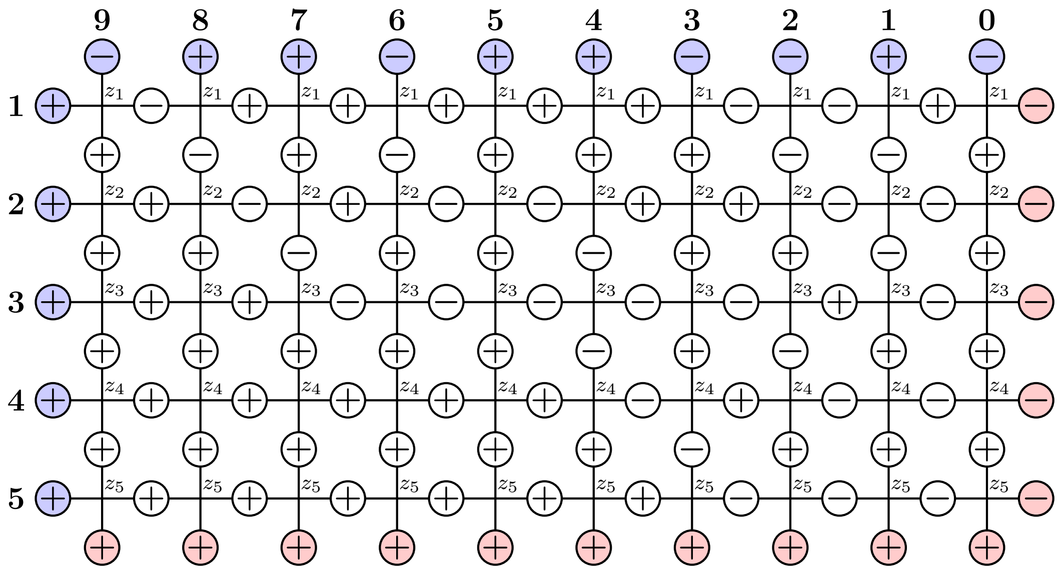
Proposition 1.1. *Let $\mathfrak{s}$ be a state of the system, and let*

$$G = \left\{ \begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{21} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{n1} & & & \end{array} \right\}$$

be the corresponding strict Gelfand-Tsetlin pattern. Let $A_i = \sum_j a_{ij}$ be the row sums. Then the Boltzmann weight $\beta(\mathfrak{s})$ equals a polynomial in q times the monomial $\mathbf{z}^\mu$ where

$$\mu = (A_1 - A_2, A_2 - A_3, \cdots, A_n).$$

Before we prove this, let us work out an example. We will take $n = 5$ and $\lambda = (5, 3, 1, 1) = (5, 3, 1, 1, 0)$. After adding $\rho = (4, 3, 2, 1, 0)$ we get $\lambda + \rho = (9, 6, 3, 2, 0)$, and these are the columns at the top where we put $-$ spins in the boundary conditions. Consider the following state.

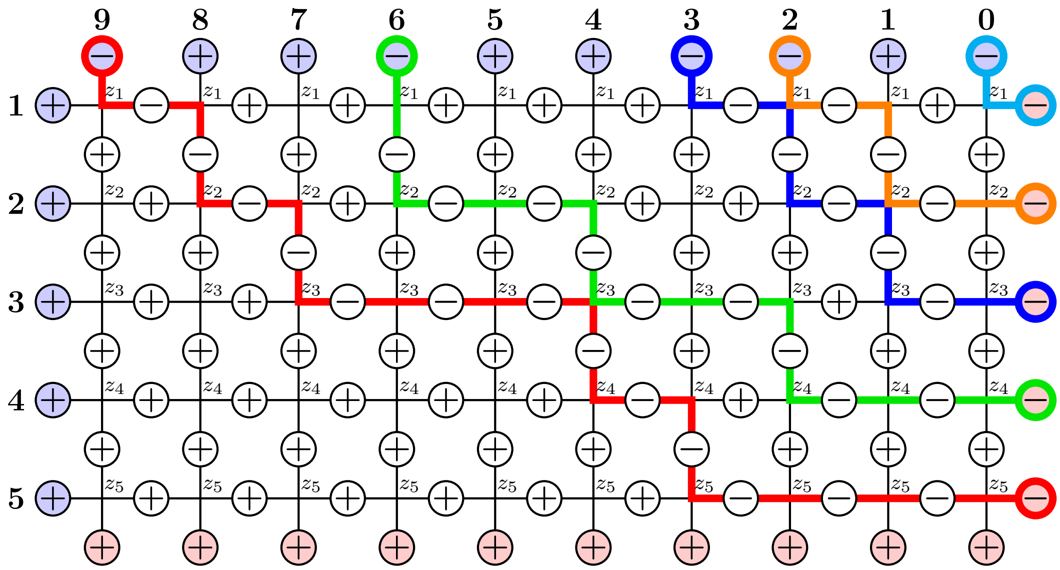


We recall from Lecture 3 that the entries in the corresponding Gelfand-Tsetlin patterns are the columns where a vertical edge with a $-$ spin occurs. Thus:

$$G = \left\{ \begin{array}{ccccccccc} 9 & & 6 & & 3 & & 2 & & 0 \\ & 8 & & 6 & & 2 & & 1 & \\ & & 7 & & 4 & & 1 & & \\ & & & 4 & & 2 & & & \\ & & & & 3 & & & & \end{array} \right\}$$

Conversely, given a strict Gelfand-Tsetlin pattern with top row $\lambda+\rho$, we may put $-$ spins on the vertical edges with entries in the pattern, and $+$ spins in the remaining edges. Then the spins on the horizontal edges are determined by the requirement that the number of $-$ spins adjacent to every vertex must be even, leading to a unique admissible state of the six-vertex model.

Then, we recall from Lecture 2 that we may find paths running through the edges with $-$ spins. Let us see how this works for the above example. There will be six paths, each beginning with an "input" boundary edge (colored blue) and terminating at an "output" edge (colored red). We show the paths as follows, using color to distinguish the six paths.



PROOF OF PROPOSITION 1.1. To prove the Proposition, we note from the Boltzmann weights that $\beta(\mathfrak{s})$ is a polynomial in q times a monomial $\mathbf{z}^\mu$ for some μ. There is a contribution of z_i from every pattern of type $\mathtt{a}_2$, $\mathtt{b}_2$ or $\mathtt{c}_1$. These are precisely the vertices with a $-$ spin to the left of the vertex. Therefore the number of z_i in the product of local Boltzmann weights equals the number μ_i of $-$ spins in the i-th row, not counting the right boundary edge.

Thus in th example, $\mu = (3, 4, 5, 6, 3)$. This is consistent with the statement of the Proposition with where the row sums of the Gelfand-Tsetlin pattern are $(A_1, A_2, A_3, A_4, A_5) = (20, 17, 12, 6, 3)$.

We must show that $\mu_i = A_i - A_{i+1}$ (or just A_i if $i = n$). To count the number of $-$ spins on the horizontal edges in the i-th row, not counting the right boundary edge, we enumerate them by the paths. We note that one path enters from the top in the column $a_{i,j}$ and exits at the column $a_{i+1,j}$. There are $a_{i,j} - a_{i+1,j}$ $-$ spins on this edge.

The argument requires minor modification for the last path, which exits on the right and contributes a_{n+1-i}. We do not need to consider this an exception if we extend the Gelfand pattern by zero and define $a_{i+1,n+1-i} = 0$. With this convention, $A_{n+1} = 0$.

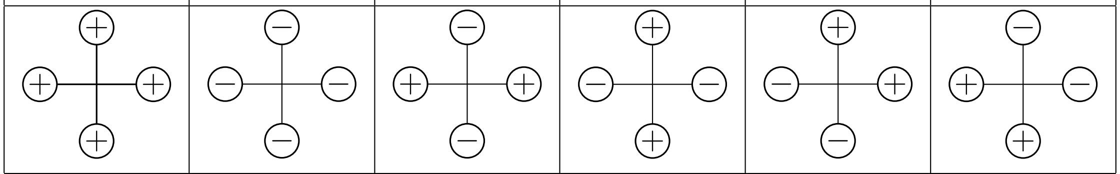
Summing the contributions of all paths,

$$\mu_i = \sum_{j=1}^{n+1-i} a_{i,j} - a_{i+1,j} = A_i - A_{i+1},$$

as required. $\square$

2. Tokuyama Ice: $q = 1$

If either $q = 0$ or $q = 1$, one of the six vertex types in the Tokuyama model disappears. In these two cases, there are only five allowed states of spins adjacent to a vertex, and we will call the resulting models *five-vertex models*. In the case $q = 1$, the Boltzmann weights are:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
					
1	z_i	-1	z_i	0	1

We see that there can no longer be any $\mathtt{c}_1$ patterns. This has a profound effect on the paths and on the Gelfand-Tsetlin patterns.

LEMMA 2.1. *If G is the Gelfand-Tsetlin pattern of a state having no $\mathtt{c}_1$ patterns, then every row of the Gelfand-Tsetlin pattern is a subset of the row above, obtained by deleting one entry.*

PROOF. If the $(i+1)$-st row is *not* obtained from the i-th row by deleting a single entry, then there is an element $a_{i+1,j}$ that is not in the i-th row. Since $a_{i,j} \geqslant a_{i+1,j} \geqslant a_{i,j+1}$ by the definition of a Gelfand-Tsetlin pattern we must have $a_{i,j} > a_{i+1,j} > a_{i,j+1}$. This implies that there is a $\mathtt{c}_1$ pattern at in the i-th row at column $a_{i+1,j}$, which is a contradiction. $\square$

Recall that the "Weyl group" W is the symmetric group S_n.

PROPOSITION 2.2. *When $q = 1$, we have*

$$Z(\mathbf{z}; 1) = \sum_{w \in W} \operatorname{sgn}(w) \mathbf{z}^{w(\lambda+\rho)}. \tag{17}$$

PROOF. There are $n!$ states $\mathfrak{s}$ that omit $\mathtt{c}_1$ patterns, namely those in which each row is obtained from the previous one by dropping a single entry. By Proposition 1.1, the Boltzmann weight $\beta(\mathfrak{s})$ is $\pm\mathbf{z}^\mu$, where $\mu_i = A_i - A_{i+1}$. By the Lemma, this value $A_i - A_{i+1}$ is some element of the i-th row, hence of the top row $\lambda + \rho$. (The sign $-$ is the number of $\mathtt{b}_1$ patterns.) We may therefore write $\mu = w(\lambda + \rho)$ for some permutation $w \in W$, and $\beta(\mathfrak{s}) = \pm\mathbf{z}^{w(\lambda+\rho)}$, where the sign must be determined.

We have proved in Lecture 5 that

$$S_\lambda(\mathbf{z}) = \frac{Z(\mathbf{z}; 1)}{\prod_{i<j}(z_i - z_j)} \tag{18}$$

is symmetric. The denominator is alternating, that is, it changes sign when an odd permutation is applied. Therefore the numerator $Z(\mathbf{z}; 1)$ is also alternating. Now there is one state which has no $\mathtt{b}_1$ patterns: this is the state in which the entry in the i-th row of the Gelfand-Tsetlin pattern G that is dropped is always the first one. For this state, $\beta(\mathfrak{s}) = \mathbf{z}^{\lambda+\rho}$. Therefore $Z(\mathbf{z}; 1)$ is of the form $\sum_{w \in W} \pm\mathbf{z}^{w(\lambda+\rho)}$, is known to be alternating, and one of the terms is $\mathbf{z}^{\lambda+\rho}$. Hence the signs of the other terms are determined. This proves (17). $\square$

Now we recognize the numerator and denominator in the ratio (18)

$$S_\lambda(\mathbf{z}) = \frac{\sum_{w \in W} \pm\mathbf{z}^{w(\lambda+\rho)}}{\prod_{i<j}(z_i - z_j)} = \frac{\det(z_j^{\lambda_i+n-i})}{\det(z_j^{n-i})},$$

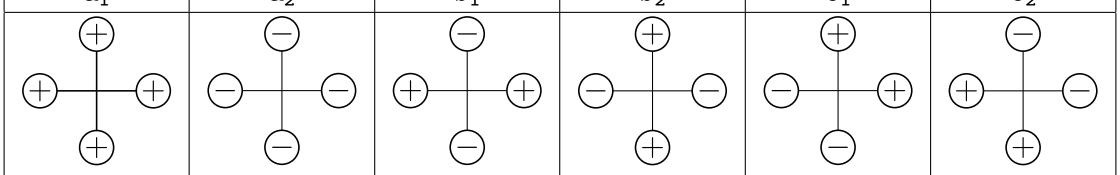
using the Vandermonde identity. This equals the Schur polynomial $s_\lambda(\mathbf{z})$ by the first definition.

3. The Crystal Limit

Before we consider the case $q = 0$, a word about how important this case is. Before the 1980's, an analogy between the representation theory of $\mathrm{GL}(n, \mathbb{C})$ and the theory of semistandard Young tableaux (SSYT) emerged in work of Robinson, Littlewood, Schensted, Knuth, Lascoux and Schützenberger. For example, if λ is a partition, then λ indexes two particular things, an irreducible representation $\pi_\lambda^{\mathrm{GL}(n)}$ of $\mathrm{GL}(n, \mathbb{C})$, and the set $\mathcal{B}_\lambda$ of semistandard Young tableaux. The cardinality of $\mathcal{B}_\lambda$ equals the dimension of $\pi_\lambda^{\mathrm{GL}(n)}$, and this is the beginning of a fruitful parallel. Ultimately Kashiwara, in the theory of crystal bases (crystals) gave an explanation for this: the representation $\pi_\lambda^{\mathrm{GL}(n)}$ can be thought of as being in a family of modules of the quantum groups $U_q(\mathfrak{gl}_n)$. These are somewhat complicated objects, but in the "crystal limit" $q \longrightarrow 0$ much of the complexity disappears, and the combinatorial theory remains. The quantum group $U_q(\mathfrak{gl}_n)$ does not, itself, have a limit when $q = 0$, but some of its operations do survive, giving $\mathcal{B}_\lambda$ some extra structure, that of a crystal. We will therefore refer to the case $q \longrightarrow 0$ as the "crystal limit."

4. The case $q \longrightarrow 0$

When $q = 0$, we have the following Boltzmann weights:



$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	z_i	0	z_i	z_i	1

Now we see that the pattern $\mathtt{b}_1$ no longer appears. This means that every path that comes down to a vertex from the top must bend to the right.

Lemma 4.1. *Let $\mathfrak{s}$ be a state of the system $\mathfrak{S}_\lambda(\mathbf{z}; q)$, and let*

$$G = \left\{ \begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{21} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{n1} & & & \end{array} \right\}$$

be the corresponding strict Gelfand-Tsetlin pattern. Then a necessary and sufficient condition that $\mathfrak{s}$ contains no $\mathtt{b}_1$ patterns is that for every i, j we have $a_{i,j} > a_{i+1,j}$.

Proof. In terms of the paths, one path descends from above to the vertex in the i-th row in column $a_{i,j}$ and leaves downwards in the column $a_{i+1,j}$. Thus if $a_{i,j} = a_{i+1,j}$, that means precisely that the vertex in row i and column $a_{i,j}$ produces a $\mathtt{b}_1$ pattern. $\square$

We will call a Gelfand-Tsetlin pattern *left-strict* if its entries satisfy $a_{i,j} > a_{i+1,j} \geqslant a_{i,j+1}$. (The second inequality is part of the definition of a Gelfand-Tsetlin pattern, so the significant assumption is that $a_{i,j} > a_{i+1,j}$.) We see that the states of the five-vertex model $\mathfrak{S}_\lambda(\mathbf{z}; 0)$ are in bijection with the left-strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$.

Let us denote by ρ_k the vector $(k-1, k-2, \cdots, 0)$ in $\mathbb{Z}^k$, so that $\rho = \rho_n$ in our previous notation. We can make a Gelfand-Tsetlin pattern with rows $\rho_n, \rho_{n-1}, \cdots, \rho_1$ thus:

$$P = \left\{ \begin{array}{ccccccc} n-1 & & n-2 & & \cdots & & 0 \\ & n-2 & & \cdots & & 0 & \\ & & \ddots & & \iddots & & \\ & & & 0 & & & \end{array} \right\}.$$

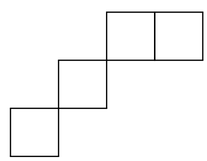
LEMMA 4.2. *The map $G \longrightarrow G - P$ is a bijection between left-strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$ and Gelfand-Tsetlin patterns with top row λ.*

PROOF. This is easy to check. $\square$

LEMMA 4.3. *Let G be a Gelfand-Tsetlin pattern and let T be the corresponding semistandard Young tableau as defined in Section 2. Let $\lambda_1, \cdots, \lambda_n$ be the rows of G and let $A_i = |\lambda_i|$ denote the corresponding row sums. Then*

$$\mathrm{wt}(T) = (A_n, A_{n-1} - A_n, \cdots, A_2 - A_3, A_1 - A_2).$$

PROOF. Let λ, μ be two partitions with Young diagrams λ, μ. If the Young diagram $\mathrm{YD}(\mu)$ is contained in $\mathrm{YD}(\lambda)$, then the pair λ, μ, denoted λ/μ is called a *skew shape*. Its Young diagram is the set-theoretic difference $\mathrm{YD}(\lambda) - \mathrm{YD}(\mu)$. For example $(5,3,2)/(3,2,1)$ is a skew shape and its diagram is



We may use the skew shape terminology to reformulate the relationship between a Gelfand-Tsetlin pattern G and its associated tableau T, first discribed in Lecture 2. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the rows of G. We also let $\lambda_{n+1} = ()$ be the empty partition. Then $\lambda_{n+1-i}/\lambda_{n+2-i}$ is a skew shape, which is the union of all the boxes in the tableau T that contain the entry i.

By definition, $wt(T) = (\mu_1, \mu_2, \cdots, \mu_n)$ where μ_i is the number of boxes that contain the entry i. The these comprise the skew tableau with shape $\lambda_{n+1-i}/\lambda_{n+2-i}$, and since $|\lambda_i| = A_i$, we obtain the advertised formula for $\mathrm{wt}(T)$. $\square$

Example 4.4. To illustrate Lemma 6, suppose $n = 3$ and

$$G = \left\{ \begin{array}{ccccc} 5 & & 3 & & 1 \\ & 4 & & 1 & \\ & & 3 & & \end{array} \right\}.$$

Then the corresponding tableau is

1	1	1	2	3
2	3	3		
3				

Thus $\mathrm{wt}(T) = (3, 5-3, 8-5) = (3, 2, 3)$. The three skew shapes corresponding to $1, 2, 3$ are

$$(3)/\varnothing, \qquad (4,1)/3, \qquad (5,3,1)/(4,1),$$

that is:

$$\begin{array}{cccccc}
 & \boxed{1} & \boxed{1} & \boxed{1}, & & \\
 & & & & \boxed{2} & \\
\boxed{2} & & & & & \\
 & & & & & \boxed{3} \\
 & \boxed{3} & \boxed{3} & & & \\
\boxed{3} & & & & &
\end{array}$$

We've left the letters 1,2,3 in the skew tableau to remind us that these skew shapes came from the original semistandard Young tableau by keeping only the boxes with a given label.

We let w_0 be the "long element" of the Weyl group $W = S_n$, which is the permutation that maps k to $n+1-k$ of $\{1, 2, 3, \cdots, n\}$. If $\mathbf{z} = (z_1, \cdots, z_n) \in (\mathbb{C}^\times)^n$ and $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$, then $w_0\mathbf{z} = (z_n, \cdots, z_1)$ and $w_0\mu = (\mu_n, \cdots, \mu_1)$. Obviously $\mathbf{z}^{w_0\mu} = (w_0\mathbf{z})^\mu$.

Proposition 4.5. *Let $\mathfrak{s}$ be an admissible state of the system $\mathfrak{S}_\lambda(\mathbf{z}; 0)$. Since $\mathfrak{s}$ has no $\mathtt{b}_1$ patterns, the corresponding Gelfand-Tsetlin pattern G is column strict. Let $G^\circ = G - P$, which is a Gelfand-Tsetlin pattern with top row λ. Let T be the semistandard Young tableau associated with G° as in Lecture 2. Then $\beta(\mathfrak{s}) = \mathbf{z}^\rho \cdot (w_0\mathbf{z})^{\mathrm{wt}(T)}$.*

Proof. Since the Boltzmann weights of every vertex can only be 1 or z_i for some i, it is obvious that $\beta(\mathfrak{s})$ is a monomial $\mathbf{z}^\mu$ and we need to compute μ. This is accomplished by Proposition 1.1. Writing $G = P + G^\circ$ the contribution of P is obviously $\mathbf{z}^\rho$, and we must discuss the contribution of G^o but by Lemma 4.3 and Proposition 1.1, this is $\mathbf{z}^{w_0 \ \mathrm{wt}(T)} = (w_0\mathbf{z})^{\mathrm{wt}(T)}$. □

Theorem 4.6. *The polynomial $S_\lambda = s_\lambda$ where s_λ is the Schur function defined by the second combinatorial definition.*

Proof. To summarize what we have done so far, culminating in Proposition 4.5, we have seen that every state $\mathfrak{s}$ of $\mathfrak{S}_\lambda(\mathbf{z}; 0)$ has no $\mathtt{b}_1$ patterns. Such states are parametrized by left-strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$. Each such pattern G can be written as $G^\circ + P$ where G° is a Gelfand-Tsetlin pattern with top row λ. If T is tableau corresponding to G° then $\beta(\mathfrak{s}) = \mathbf{z}^\rho \cdot (w_0\mathbf{z})^{\mathrm{wt}(T)}$. Summing over all states and using the combinatorial definition of the Schur function we obtain

$$Z_\lambda(\mathbf{z}; 0) = \mathbf{z}^\rho \, s_\lambda(w_0\mathbf{z}).$$

On the other hand, we have shown for all q that

$$Z_\lambda(\mathbf{z}; q) = \left(\prod_{i<j} z_i - qz_j\right) S_\lambda(\mathbf{z}).$$

When $q = 0$, the product becomes $z_1^{n-1} z_2^{n-2} \cdots = \mathbf{z}^\rho$. Comparing gives

$$S_\lambda(\mathbf{z}) = s_\lambda(w_0\mathbf{z}).$$

We may replace $\mathbf{z}$ by $w_0\mathbf{z}$ and remember that we proved (using the Yang-Baxter equation) that S_λ is symmetric, so $S_\lambda = s_\lambda$. □

Comparing the evaluations of $S_\lambda(\mathbf{z})$ when $q = 1$ and $q = 0$, we have now proved the equivalence of the two definitions of the Schur function.

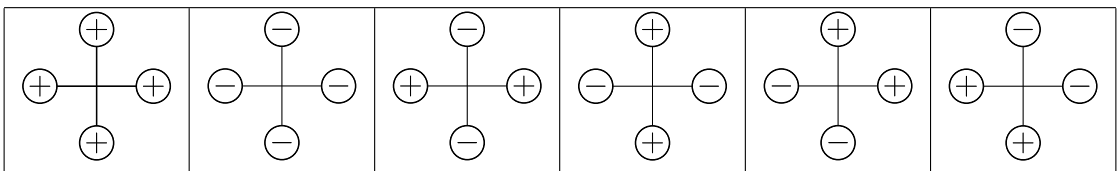
LECTURE 7

The Free-fermionic Yang-Baxter Equation

1. The general free-fermionic six-vertex model

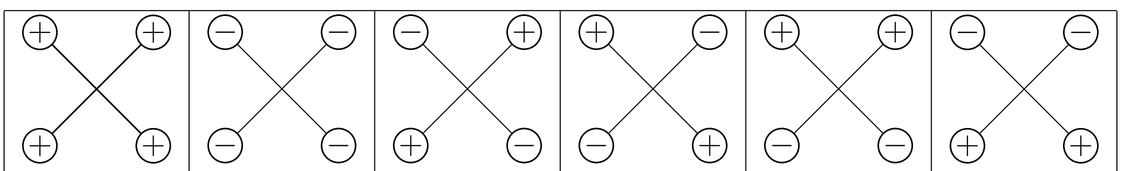
In this section we will consider a very remarkable parametrized Yang-Baxter equation with a nonabelian parameter group $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ which includes the Tokuyama weights as a special case. More typical parameter groups are usually abelian: $\mathbb{C}^\times$, $\mathbb{C}$ or an elliptic curve.

The Tokuyama weights are examples of free-fermionic weights. Label the the Boltzmann weights at a vertex R as follows:



$a_1(R)$	$a_2(R)$	$b_1(R)$	$b_2(R)$	$c_1(R)$	$c_2(R)$

or alternatively:



$a_1(R)$	$a_2(R)$	$b_1(R)$	$b_2(R)$	$c_1(R)$	$c_2(R)$

We call the weights *free-fermionic* if

$$a_1(R)a_2(R) + b_1(R)b_2(R) = c_1(R)c_2(R)$$

and $c_1(R)$, $c_2(R)$ are both nonvanishing.

It turns out that *all* free-fermionic weights fit into a parametrized Yang-Baxter equation with parameter group $\Gamma = \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$. This parametrized Yang-Baxter equation was discovered by Korepin (see [**63**] page 126, and rediscovered by Brubaker, Bump and Friedberg [**19**]). Let

$$\rho : \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \longrightarrow \{\text{free-fermionic vertices}\}$$

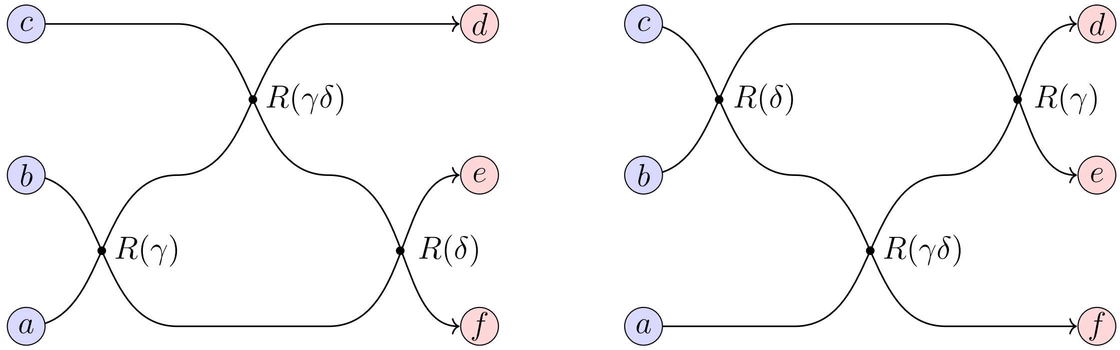
be the map that sends the matrix

$$\gamma = \begin{pmatrix} c_1 & & & \\ & a_1 & b_2 & \\ & -b_1 & a_2 & \\ & & & c_2 \end{pmatrix}$$

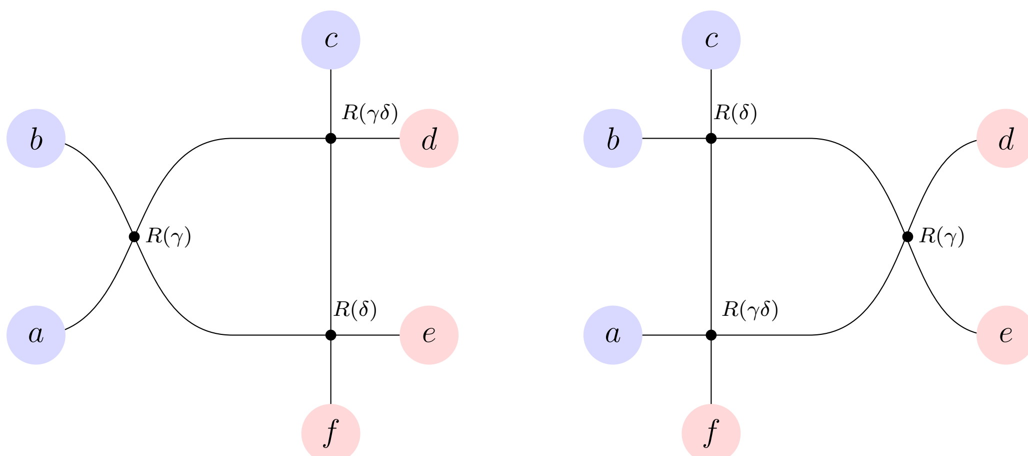
to the vertex with those Boltzmann weights.

THEOREM 1.1. *The map R is a parametrized Yang-Baxter equation with parameter group* $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$.

The parametrized Yang-Baxter equation can be either of the two forms from Lecture 4: we ask that for all a,b,c,d,e,f the two following partition functions are equal:



Alternatively:



PROOF. Let $R=\rho(\gamma)$, $T=R(\delta)$ and $S=R(\gamma\delta)$, where the product $\gamma\delta$ is just matrix multiplication. Thus

$$\gamma=\begin{pmatrix} c_1(R) & & & \\ & a_1(R) & b_2(R) & \\ & -b_1(R) & a_2(R) & \\ & & & c_2(R)\end{pmatrix},\qquad \delta=\begin{pmatrix} c_1(T) & & & \\ & a_1(T) & b_2(T) & \\ & -b_1(T) & a_2(T) & \\ & & & c_2(T)\end{pmatrix},$$

Multiplying the matrices γ and δ and remembering that $S=R(\gamma\delta)$, the S Boltzmann weights are:

$$\begin{aligned} c_1(S)&=c_1(R)c_1(T), & c_2(S)&=c_2(R)c_2(T).\\ a_1(S)&=a_1(R)a_1(T)-b_2(R)b_1(T), & a_2(S)&=-b_1(R)b_2(T)+a_2(R)a_2(T),\\ b_1(S)&=b_1(R)a_1(T)+a_2(R)b_1(T), & b_2(S)&=a_1(R)b_2(T)+b_2(R)a_2(T).\end{aligned}$$

With these values, and taking the values

$$c_2(R)=(a_1(R)a_2(R)+b_1(R)b_2(R))/c_1(R),$$

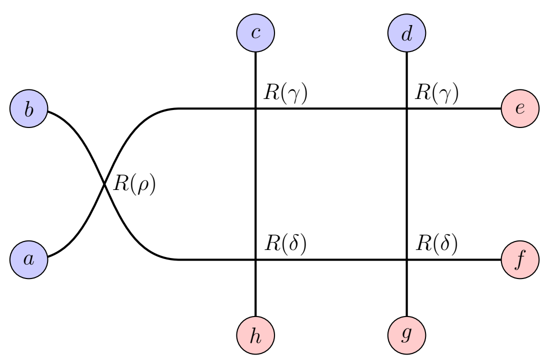
$$c_2(T)=(a_1(T)a_2(T)+b_1(T)b_2(T))/c_1(T),$$

it is straightforward to check all cases of the Yang-Baxter equation. This is done in a computer program called `free-fermionic1.sage`, posted on the class web page. □

2. Column parameters

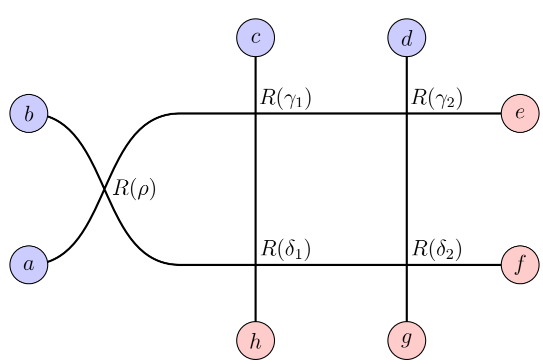
In the Tokuyama models, the Boltzmann weights depended on the rows but not the columns. We may use the parametrized Yang-Baxter equation to predict another model in which the weights do depend on the columns. But let us ask for free-fermionic models that do show column dependence. Such models exist and give for example factorial Schur functions **??**.

However we want to predict their existence by thinking about the $\mathrm{GL}(2) \times \mathrm{GL}(1)$ parametriczed free-fermionic Yang-Baxter equation. Let us postulate a grid, with free-fermionic vertices. We start with two rows of free-fermionic vertices $S = R(\gamma)$ and $T = R(\delta)$. If $R = R(\rho)$ where $\rho\delta = \gamma$, then by Theorem 1.1 we may have the Yang-Baxter equation that we need and can do the train argument. So we want $\rho = \gamma\delta^{-1}$:

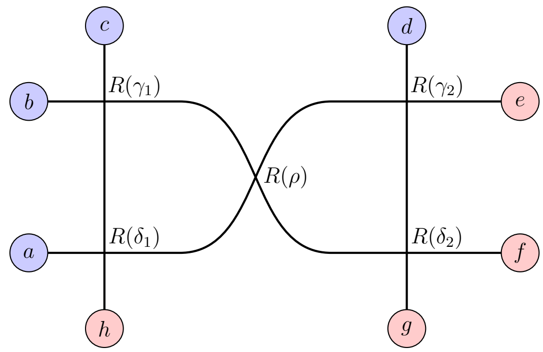


Note that γ and δ could be *arbitrary* free-fermionic vertex types.

Now let us show that Theorem 1.1 allows us to modify the weights S and T so that they are dependent on the columns, not just the rows. This procedure will not affect the R-matrix $R = R(\rho)$. Let $\gamma_1, \gamma_2, \cdots$ be the elements of the parameter group $G = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ such that $R(\gamma_i)$ and $R(\delta_i)$ are to be the row weights in the modified system. We want to be able to attach ths *same* matrix $R = R(\rho)$ for the train argument.



Now we need $\gamma_1 = \rho\delta_1$ so $\rho = \gamma_1\delta_1^{-1}$. Assuming this we can do the Yang-Baxter equation:



But now to do the next step, we need $\delta_2\rho = \gamma_2$. To complete the train argument, we clearly need $\delta_j^{-1}\gamma_j = \rho$ for all j.

To summarize, a necessary and sufficient condition to be able to do the train argument is that $\delta_j^{-1}\gamma_j = \rho$ for all j, and we already have γ and δ such that $\delta^{-1}\gamma = \rho$.

Now to arrange this, let us fix an element $\alpha_j \in G$ for every column j, and we define $\gamma_j = \gamma\alpha_j$ and $\delta_j = \delta\alpha_j$. Then the conditions are satisfied.

Let us do an example. We want γ to correspond to Tokuyama weights at the vertex S with parameter z. This means that the weights of γ are given by the following table:

$a_1(S)$	$a_2(S)$	$b_1(S)$	$b_2(S)$	$c_1(S)$	$c_2(S)$
1	z	$-q$	z	$(1-q)z$	1

Therefore

$$\gamma = \begin{pmatrix} c_1(S) & & & \\ & a_1(S) & b_2(S) & \\ & -b_1(S) & a_2(S) & \\ & & & c_2(S) \end{pmatrix} = \begin{pmatrix} (1-q)z & & & \\ & 1 & z & \\ & q & z & \\ & & & 1 \end{pmatrix}$$

Similarly let δ correspond to Tokuyama weights with parameter w, so

$$\delta = \begin{pmatrix} (1-q)w & & & \\ & 1 & w & \\ & q & w & \\ & & & 1 \end{pmatrix}.$$

Now to choose the perturbing matrices α_j, let $a_1, a_2, \cdots$ be an arbitrary sequence of integers and take

$$\alpha_j = \begin{pmatrix} 1 & & & \\ & 1 & a_j & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Note that this matrix satisfies the free-fermionic condition. Now

$$\gamma\alpha_j = \begin{pmatrix} (1-q)z & & & \\ & 1 & z & \\ & q & z & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & a_j & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} (1-q)z & & & \\ & 1 & z + a_j & \\ & q & z + qa_j & \\ & & & 1 \end{pmatrix}$$

This leads to the following modification of the Tokuyama weights:

$a_1(S)$	$a_2(S)$	$b_1(S)$	$b_2(S)$	$c_1(S)$	$c_2(S)$
1	$z + qa_j$	$-q$	$z + a_j$	$(1 - q)z$	1

The partition functions of the modified models replace the Schur functions of the Tokuyama model with *factorial Schur functions*. See [**25**] for more information, and [**82**] for more general free-fermionic models that generalize Schur functions.

LECTURE 8

Bosonic Models I

Lecture 8

We will continue looking at some examples that introduce new concepts. In this Chapter we will look (briefly) at some bosonic models before turning to a new major topic, colored models. In this lecture we will look at perhaps the simplest colored models, and take the theory up to the point were we see Demazure operators emerging from the Yang-Baxter equation. In Lecture 9 we will take this theory further. The appearance of Demazure operators gives us a point of contact with representation theory, since they generate a Hecke algebra.

1. Bosonic Models

The paths in lattice models can be thought of as the trajectories of particles. In the six-vertex model as we have been treating it, these move downwards and to the right.

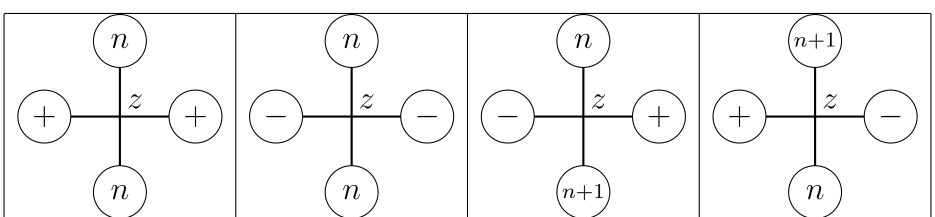
In physics, there is a distinction between particles which are called *bosons* and particles called *fermions*. The distinction is that no two fermions are allowed to occupy the same state: this is called the *Pauli exclusion principle*. Bosons, on the other hand, are allowed to occupy the same state.

The spinset for all edges in the six-vertex model just consists of $\{+, -\}$ where we interpret $+$ to be the absence of a particle, and $-$ to be the presence. An alternative spinset consist of the nonnegative integers $\{0, 1, 2, \cdots\}$ where the integer value indicates the number of identical particles.

We consider a simple type of model, invented by Kulish [**65**], with the partition functions computed by Korff [**64**]. We will call these models the bosonic *Hall-Littlewood models*.

The horizontal edges will have the fermionic spinset $\{+, -\}$, but the vertical edges will have the bosonic spinset $\{0, 1, 2, \cdots\}$.

Paths are still relevant but now a single vertical edge can carry more than one path. The fermionic horizontal edges can only carry a single path. We thus arrive at the following vertex types, for which we have assigned Boltzmann weights:



1	z	$z(1-t^{n+1})$	1

The R-matrix is:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$\oplus \quad \oplus$ $\oplus \quad \oplus$	$\ominus \quad \ominus$ $\ominus \quad \ominus$	$\ominus \quad \oplus$ $\oplus \quad \ominus$	$\oplus \quad \ominus$ $\ominus \quad \oplus$	$\oplus \quad \oplus$ $\ominus \quad \ominus$	$\ominus \quad \ominus$ $\oplus \quad \oplus$
$z - tw$	$z - tw$	$t(z - w)$	$z - w$	$(1 - t)z$	$(1 - t)w$

For comparison, the R-matrix for the Tokuyama model is identical *except* for the $\mathtt{a}_1$ weight. The parameter t is analogous to the parameter we have been calling q in other models we have been looking at.

Now for the boundary conditions, these are similar to the Tokuyama models, with one exception. We choose a partition λ. As in the Tokuyama models, the columns are labeled 0 to N from right to left, for sufficiently large N, and the rows are labeled 1 to n from top to bottom.

Now we put the following spins on boundary edges, same as the Tokuyama models, with one exception. We put $+$ on the left (horizontal) boundary edges, $-$ on the right boundary edges, 0 on the bottom (vertical boundary edges). For the top vertical edge in column j, we put spin k, where k is the number of parts λ_i equal to j. This last choice differs from the Tokuyama models where we put the spins in the columns $\lambda_i + n - i$. That had the effect of preventing two spins from landing on the same edge. Since this model is bosonic, it is unnecessary to do that. The i-th row of the pattern is labeled by $z = z_i$, and we use the above Boltzmann weights.

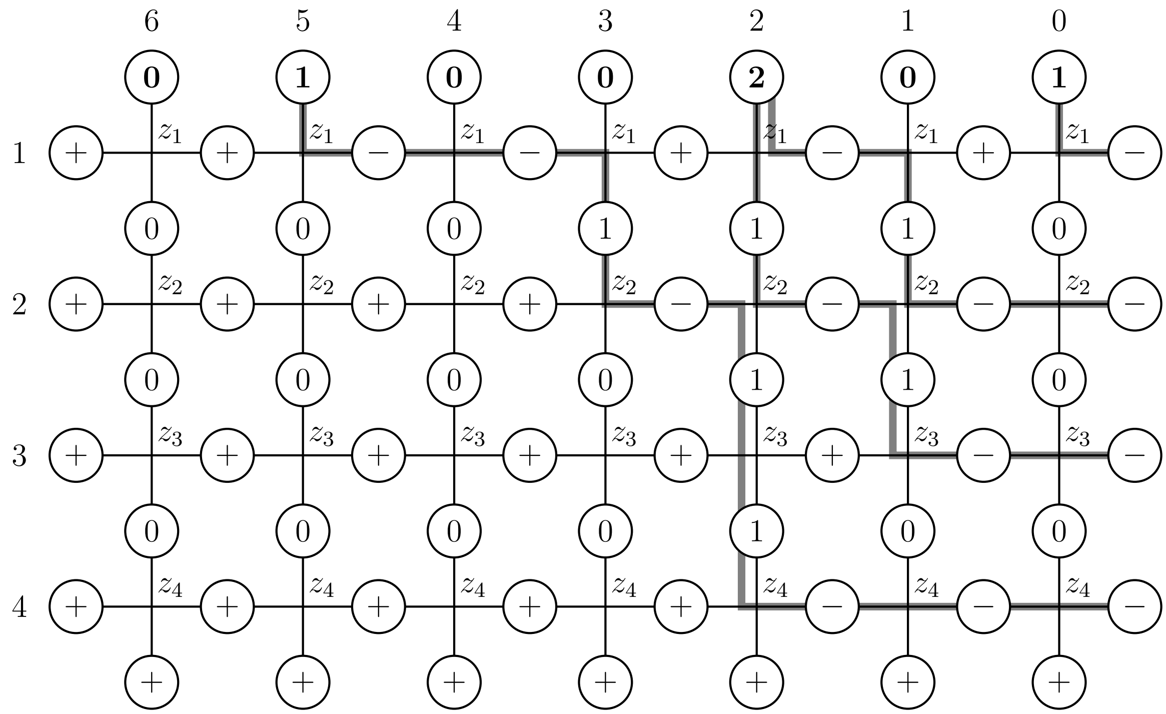
THEOREM 1.1 (Korff [**64**]). *The partition function of this model equals the Hall-Littlewood symmetric polynomial $P_\lambda(z_1, \cdots, z_n; t)$.*

We will not digress now to define the Hall-Littlewood polynomials, but see Macdonald [**76**] Chapter 3 for their definitions and properties. We will point out that the information that we get from the Yang-Baxter equation is precisely a symmetric function, and in contrast with the Tokuyama models, that information does not seem to be enough to evaluate the partition function.

Similarly to the Tokuyama case, we parametrize the states by Gelfand-Tsetlin patterns of size n with top row λ. For example, suppose that:

$$\lambda = \left\{ \begin{array}{ccccccc} 5 & & 2 & & 2 & & 0 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{array} \right\}$$

The entries of this Gelfand-Tsetlin pattern are precisely the vertical edges that carry paths, and we easily arrive at the following state:



The paths are seen to double up in Column 2.

The quantum group underlying thes bosonic models is $U_q(\widehat{\mathfrak{sl}}_2)$. The vertical edges correspond to "Verma modules", which are infinite-dimensional representations of $\mathfrak{sl}_2$ (or its affine quantization) that do not lift to representations of $\mathrm{SL}_2(\mathbb{C})$.

2. The Symmetric group

A *Coxeter group* is a group W with generators $s_1, \cdots, s_r$ subject to relations $s_i^2 = 1$, and *braid relations* which have the form

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

where for some $n_{i,j}$ depending on i and j there are exactly $n_{i,j}$ terms on both sides. For example if $n_{i,j} = 3$ then

$$s_i s_j s_i = s_j s_i s_j$$

and if $n_{i,j} = 2$, then s_i and s_j commute. It is assumed in the definition of the Coxeter group that these relations are a presentation of W.

In discussing the colored models we will start to need some properties of the $\mathrm{GL}(r, \mathbb{C})$ Weyl group, which is the *symmetric group* S_r. Let $s_1, \cdots, s_{r-1}$ be the *simple reflections*, so s_i is the transposition $(i, i+1)$.

THEOREM 2.1. *The group S_r is a Coxeter group with generators s_i. This means that the s_i generate S_r and satisfy the quadratic relations*

$$s_i^2 = 1$$

and the braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \text{ if } |i - j| > 1,$$

and that moreover these relations give a presentation of S_r.

You can find proofs of this in many places such as my Lie groups book (second edition) Theorem 25.1.

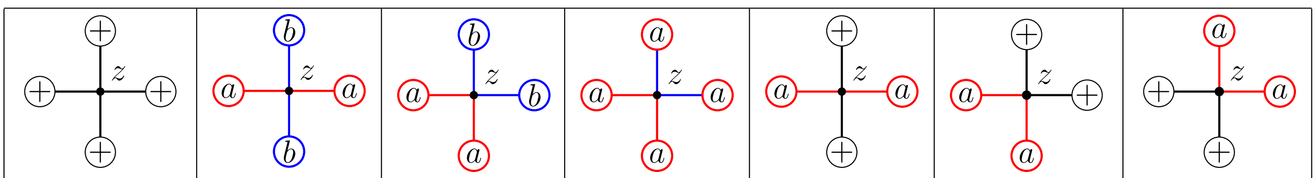
3. Open Colored Models

Borodin and Wheeler [**10**] instigated the current mania for colored models. Their models were bosonic, but fermionic models are also possible. We will look at the two very simplest examples, which we call the *open* and *closed* models.

Formulated in terms of paths, the idea behind the colored models is very simple: instead of one type of path there will be m types, where m is some sufficiently large number. These different types are called *colors*. It is important that they have an order, so let $c_1 > c_2 > \cdots > c_m$ be the m colors. Actually the largest number of colors that we can make use of is the number of rows of the grid, so we can take m to be the number r of rows in the grid. If there are more than m colors, there is no harm in taking $m = r$.

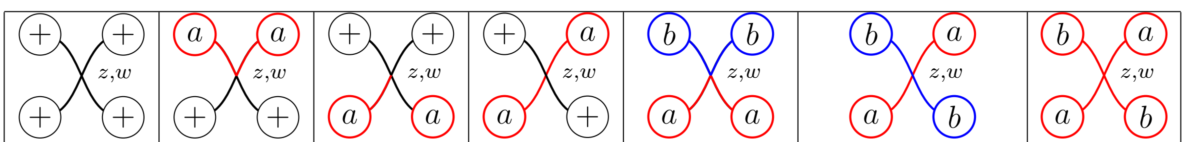
There is a feeling that if we have an uncolored model (e.g. six-vertex model) that we should be able to find a colorized version. The relationship between colored model and the uncolored model often shed light on the uncolored model

We could start with the Tokuyama model for this, but to get the simplest possible theory, we will start with the crystal limit 5-vertex model, which we encountered in Lecture 6. There are *two* theories which we will call *open* and *closed*. The open model was studied in [**17**], and we will look at it in this section.



1	z $a>b$ 0 $a<b$	0 $a>b$ z $a<b$	z	z	z	1

Here is the R-matrix.



w	w	z	$z-w$	w $a>b$ z $a<b$	$z-w$ $a>b$ 0 $a<b$	z

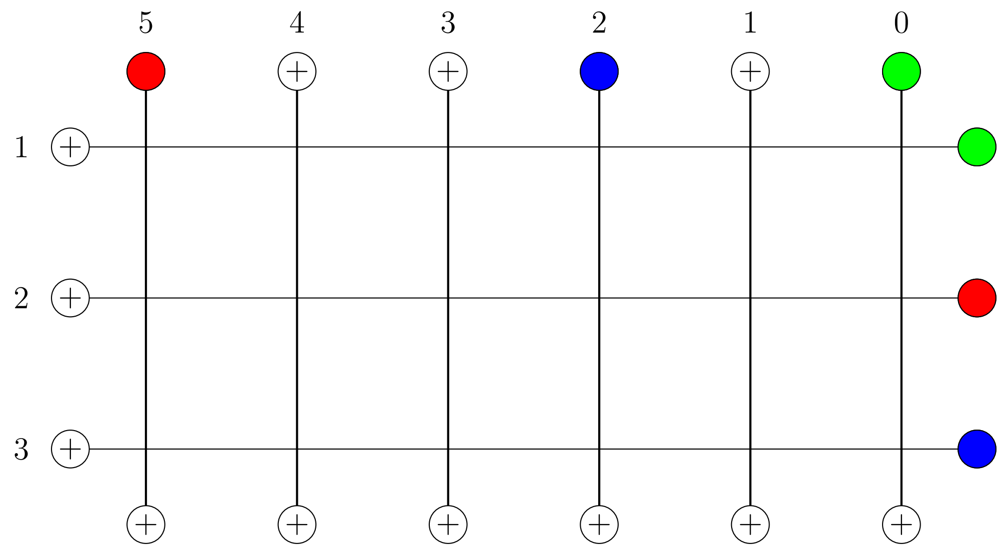
Let us specify boundary conditions. Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition. The model will be similar to the Tokuyama model with columns labeled $0, \cdots, N$ from right to left and rows labeled 1 to N from top to bottom. We put + spins on the top and bottom edges. For the top edge, we will put color c_i in column $\lambda_i + n - i$.

Note that this implies that the colors on the top edge are in decreasing order from right to left. For the right edge, we choose a *flag* $\mathbf{d} = (d_1, \cdots, d_r)$ where $d_1, \cdots, d_r$ are the colors $c_1, \cdots, c_r$ that we use on the top edge, in some order. We will denote by $\mathbf{c}_0 = (c_1, \cdots, c_r)$ where, we remind the reader, we have ordered $c_1 > \cdots > c_r$. Then there is a permutation $w \in W$ such that $\mathbf{d} = w\mathbf{c}_0$.

Here is an example. The three colors are ordered

$$\text{red } (\bullet) > \text{blue } (\bullet) > \text{green } (\bullet)$$

So if $w = (123) = s_1 s_2$ then $\mathbf{c} = s_1 s_2 \mathbf{c}_0 = (\bullet, \bullet, \bullet)$. Then if λ is the partition $(3, 1, 0)$, so $\lambda + \rho = (5, 2, 0)$. Here are the boundary conditions as we have described them:



Let $Z_\lambda(\mathbf{z}; \mathbf{d})$ or $Z_\lambda(\mathbf{z}; w)$ denote the partition function.

Now let us define an operator δ_i° on functions $f(\mathbf{z})$ by:

$$\delta_i^\circ f(\mathbf{z}) = \frac{z_{i+1} f(\mathbf{z}) - z_i f(s_i \mathbf{z})}{z_i - z_{i+1}}.$$

This is a *divided difference operator*, of a type used by Demazure [**33**] and Bernstein-Gelfand-Gelfand [**6**] in algebraic geometry. They are also important in algebraic combinatorics.

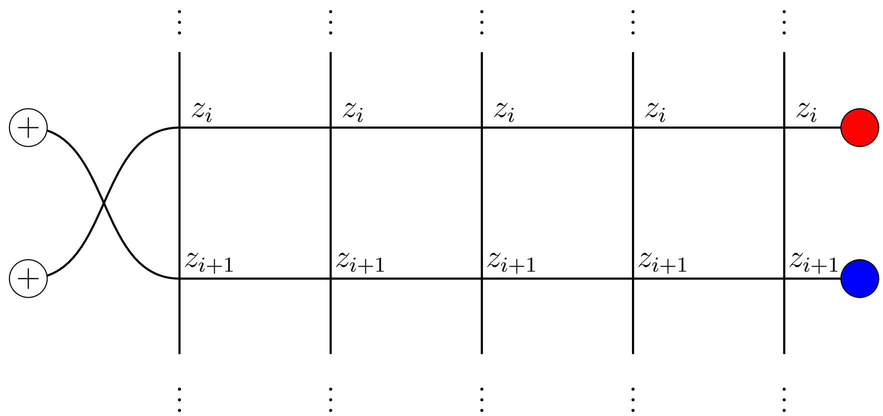
Lemma 3.1. *If f is holomorphic as a function of $\mathbf{z}$, so is $\delta_i^\circ f$.*

Proof. We need to show that the numerator is divisible by the denominator. The numerator vanishes where the denominator does, because if $z_i = z_{i+1}$ then $\mathbf{z} = s_i \mathbf{z}$. The vanishing of the numerator $z_{i+1} f(\mathbf{z}) - z_i f(s_i \mathbf{z})$ along the hyperplane $z_i = z_{i+1}$ implies that the denominator divides the numerator. $\square$

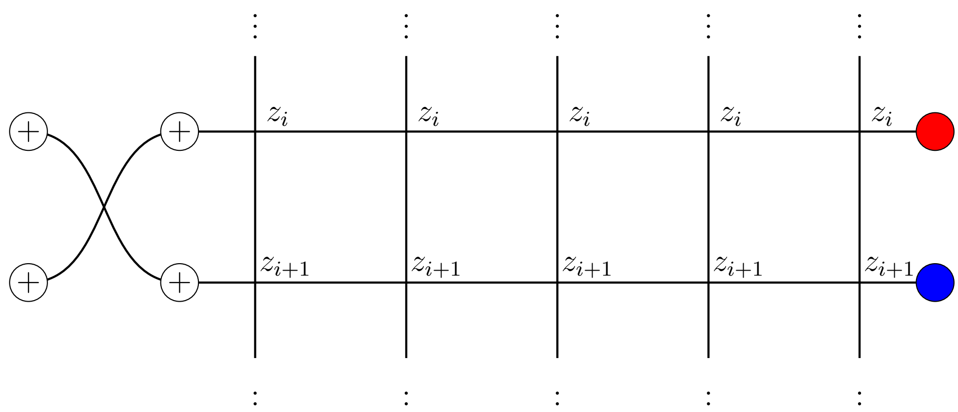
Proposition 3.2. *Suppose that $d_i > d_{i+1}$. Then the partition function $Z_\lambda(\mathbf{z}; s_i \mathbf{d})$ satisfies*

$$Z_\lambda(\mathbf{z}; s_i \mathbf{d}) = \delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}).$$

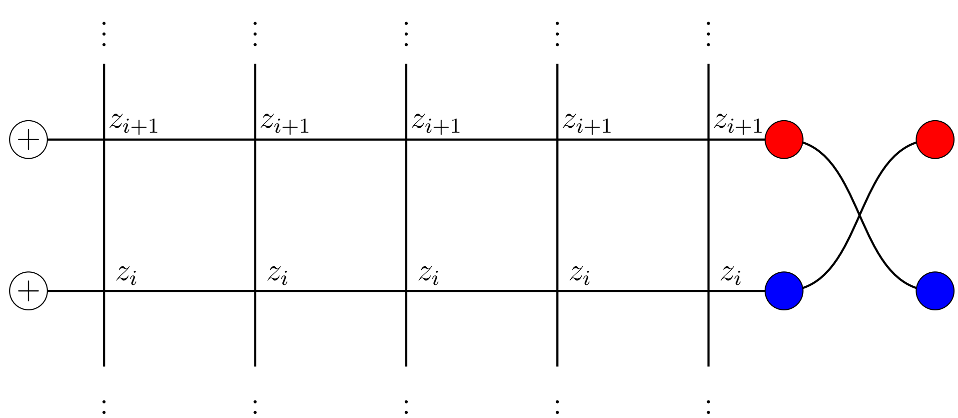
PROOF. Let us attach the R-matrix to the left:



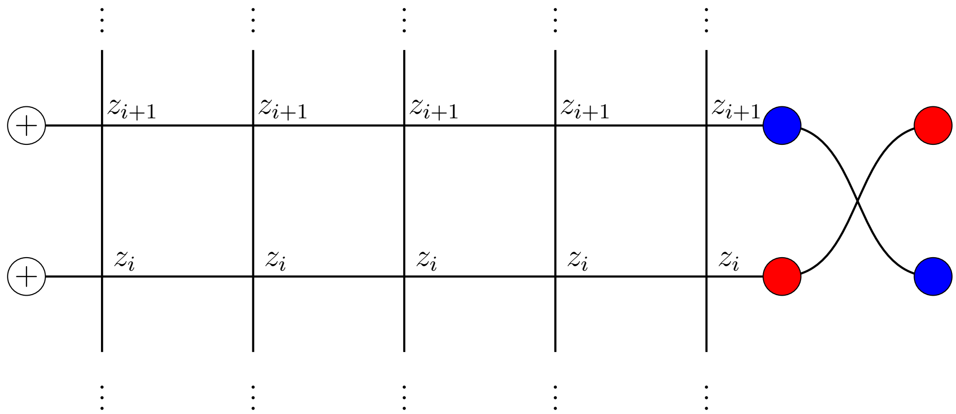
Given the spins $+$, $+$ on the left edge the spins on the R-matrix can only be all $+$, so we may assume that the configuration is as follow:



the partition function of this system is $Z_\lambda(\mathbf{z};\mathbf{d})$ times the value z_{i+1} of the R-matrix. Running the train argument, it turns out there are two possible configurations on the right-hand side, namely



and



Consulting the Boltzmann weights for the R-matrix, the partition functions for these configurations are

$$z_i Z_\lambda(s_i\mathbf{z}; \mathbf{d})$$

and (since the colors get switched for the second one):

$$(z_i - z_{i+1}) Z_\lambda(s_i\mathbf{z}; s_i\mathbf{d}).$$

Hence we obtain the identity

$$z_{i+1} Z_\lambda(\mathbf{z}; \mathbf{d}) = z_i Z_\lambda(s_i\mathbf{z}; \mathbf{d}) + (z_i - z_{i+1}) Z_\lambda(s_i\mathbf{z}; s_i\mathbf{z}).$$

We want to interchange z_i and z_{i+1}, so replace $\mathbf{z}$ by $s_i\mathbf{z}$. Then

$$z_i Z_\lambda(s_i\mathbf{z}; \mathbf{d}) = z_{i+1} Z_\lambda(\mathbf{z}; \mathbf{d}) + (z_{i+1} - z_i) Z_\lambda(\mathbf{z}; s_i\mathbf{d}).$$

Reorganizing this gives

$$Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \frac{z_{i+1} Z_\lambda(\mathbf{z}; \mathbf{d}) - z_i Z_\lambda(s_i\mathbf{z}; \mathbf{d})}{z_i - z_{i+1}}$$

as required. $\square$

LECTURE 9

Bosonic Models II

1. Some Lie Theory

If G is a (reductive) Lie group, we may associate with G a *Weyl group* W, a *maximal torus* T, a *root system* Φ and a *weight lattice* Λ. Everything in this section generalizes to that setup. But we will specialize to the case $G = \mathrm{GL}(n, \mathbb{C})$.

Let $G = \mathrm{GL}(n, \mathbb{C})$, and let $T = (\mathbb{C}^\times)^n$, which we will embed in G via

$$\mathbf{z} = (z_1, \cdots, z_n) \longmapsto \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}.$$

The *weight lattice* $\Lambda = \mathbb{Z}^n$. If $\mathbf{z} \in T$ and $\mu = (\mu_1, \cdots, \mu_n) \in \Lambda$, we will denote

$$\mathbf{z}^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}.$$

Let $\mathcal{O}(T)$ be the ring of regular functions on T. This as the space spanned by the functions $\mathbf{z}^\mu$ with $\mu \in \Lambda$. Occasionally we may want to embed Λ in the vector space $\mathbb{R} \otimes \Lambda \cong \mathbb{R}^n$.

Let $\mathbf{e}_i$ be the standard basis of $\Lambda = \mathbb{Z}^n$. The *root system* $\Phi \subseteq \Lambda$ consists of the $n(n-1)$ vectors $\mathbf{e}_i - \mathbf{e}_j$ with $i \neq j$. Then $\Phi = \Phi^+ \cup \Phi^-$ (disjoint) where Φ^+ consists of the $\frac{1}{2}n(n-1)$ vectors $\mathbf{e}_i - \mathbf{e}_j$ with $i < j$, and Φ^- is the complement. The elements of Φ^+ and Φ^- are called *positive* and *negative roots*. The particular positive roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ with $1 \leqslant i \leqslant n-1$ are called *simple roots*. Every positive root may be written as a sum of simple roots.

The *Weyl group* W of $\mathrm{GL}(n, \mathbb{C})$ is the symmetric group S_n. It acts on T, Λ and Φ by permuting the coordinates.

Let W be a group with a fixed set I of generators, $I = \{s_1, \cdots, s_r\}$. Then W is called a *Coxeter group* if the following relations are satisfied. First, the *quadratic relations*

$$s_i^2 = 1 \tag{19}$$

and the *braid relations*

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \tag{20}$$

where there are $n_{i,j}$ entries on both sides, where $n_{i,j}$ is the order of $s_i s_j$; and furthermore that these relations are a presentation of W. This means that if Γ is any group with generators t_i satisfying the quadratic and braid relations, then there is a homomorphism $W \longrightarrow \Gamma$ such that $s_i \longmapsto t_i$.

For the symmetric group $W = S_n$ we take $I = \{s_1, \cdots, s_n\}$ where s_i is the transposition $(i, i+1)$. The element s_i is called a *simple reflection*.

Theorem 1.1. *The Weyl group is a Coxeter group.*

Proof. This is true for the Weyl group of any Lie group, though we are specializing to the case of the symmetric group. See [**24**] Theorem 25.1 or [**46**] Theorem 19.1. □

There is a close relationship between the Weyl group and the root system. In particular, the simple reflections are related to the simple roots by the following property.

LEMMA 1.2. *The reflection s_i sends α_i to its negative, and permutes other positive roots. In other words s_i maps $\Phi^+ - \{\alpha_i\}$ to itself.*

PROOF. This simple but important property is easily checked for the symmetric group. $\square$

Definition 1. A *Weyl vector* is a vector $\rho \in \Lambda$ or $\mathbb{R} \otimes \Lambda$ such that $\rho - s_i(\rho) = \alpha_i$ for simple roots α_i and corresponding simple reflections α_i.

Example 1.3. We could take ρ to be half the sum of the positive roots. Then the fact that $\rho - s_i(\rho) = \alpha_i$ follows easily from Lemma 1.2. However for $W = S_n$ and $\Lambda = \mathbb{Z}^n$ we prefer to take

$$\rho = (n-1, n-2, \cdots, 0). \tag{21}$$

Defining ρ to be half the sum of the positive roots would give $\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2}\right)$, and if n is even, this vector has denominators that we can avoid by the choice (21).

2. Matsumoto's Theorem

The Weyl group has a *length function* $\ell : W \longrightarrow \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Two possible definitions can be given which are equivalent.

Definition 2. The length $\ell(w)$ is the smallest length k of a word $w = s_{i_1} \cdots s_{i_k}$ expressing w as a product of simple reflections. Alternatively, $\ell(w)$ is the cardinality of the set

$$\{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}.$$

For the equivalence of the two definitions see [**24**] Proposition 20.5.

An expression $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$ is called *reduced*. There may be many reduced expressions for w. For example if $W = S_4$ and $w_0 = (1, 4)(2, 3)$ is the longest word, there are 14 reduced expressions for w_0.

Matsumoto's theorem, found independently by H. Matsumoto and Tits in the 1960's is an extremely useful fact. Roughly it says if $\ell(w) = k$ and if

$$w = s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$$

are two reduced expressions, then the equivalence of the two relations can be proved using only the braid relations (20) and not the quadratic relations (19). To give an example, if $W = S_4$ then $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$ and $w_0 = s_3 s_2 s_3 s_1 s_2 s_3$ are two reduced expression. The braid relations are:

$$s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3, \qquad s_1 s_3 = s_3 s_1.$$

Matsumoto's theorem asserts that we can prove $s_1 s_2 s_1 s_3 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3$ using the braid relations and not the quadratic relations. Let us write 121321 instead of $s_1 s_2 s_1 s_3 s_2 s_1$. Using the braid relations:

$$121321 = 212321 = 213231 = 231213 = 232123 = 323123.$$

To formulate Matsumoto's theorem rigorously, we introduce the *braid group* $B(W)$ of a Coxeter group W. This is the group with generators u_i (in bijection with the s_i) that satsify the braid relations but not the quadratic relations.

THEOREM 2.1 (Matsumoto [**78**]). *If $s_{i_1} \cdots s_{i_k}$ and $s_{j_1} \cdots s_{j_k}$ are reduced expressions for the same element of W, then the corresponding elements $u_{i_1} \cdots u_{i_k}$ and $u_{j_1} \cdots u_{j_k}$ are equal in the braid group $B(W)$.*

PROOF. For a proof using some geometric ideas, see [**24**], Theorem 25.2. □

3. The ground state

We return to the open models. Recall that the order of colors at the top, from right to left (in descending column number) is:

$$\begin{array}{cc} \text{column} & \text{color} \\ \lambda_1 + n - 1 & c_1 \\ \lambda_2 + n - 1 & c_2 \\ \vdots & \vdots \\ \lambda_n & c_n \end{array}$$

where $c_1 > c_2 > \cdots > c_n$. The order of colors at the right is determined by a permutation $\mathbf{d}$ of

$$\mathbf{c}_0 = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

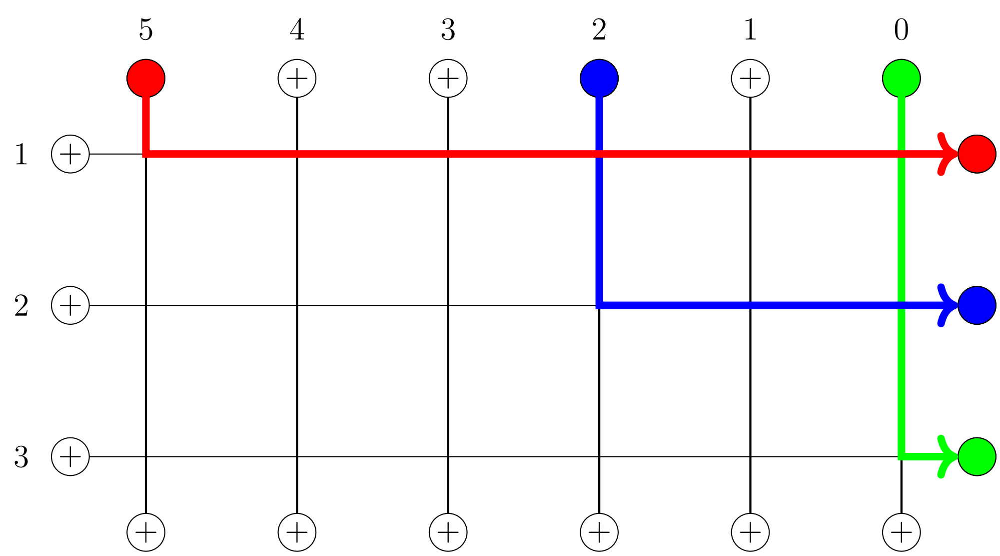
the vector of colors.

We call a system *monostatic* if it has but one state.

PROPOSITION 3.1. *If $\mathbf{d} = \mathbf{c}_0$ the open model is monostatic. The partition function is*

$$Z_\lambda(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^{\lambda+\rho}.$$

PROOF. A glance at an example will convincingly persuade you that this is true.



The red path must follow the given course, given that the path can only move down and to the right. Then the blue path, since the top row is blocked to it, can only follow the indicated path, and so on. So this system has only one state. We leave the reader to check that the Boltzmann weight of that state is $\mathbf{z}^{\lambda+\rho}$. □

We proved in the last Lecture that if $d_i > d_{i+1}$ then

$$Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}) \tag{22}$$

where δ_i° is the divided difference operator

$$\delta_i^\circ f(\mathbf{z}) = \frac{z_{i+1} f(\mathbf{z}) - z_i f(s_i\mathbf{z})}{z_i - z_{i+1}}.$$

We may take f to be any function in $\mathcal{O}(T)$.

Let us divide the numerator and denominator by z_{i+1}. Remembering that $z_i/z_{i+1} = \mathbf{z}^{\alpha_i}$, we may write

$$\delta_i^\circ f = \frac{f - \mathbf{z}^{\alpha_i} s_i(f)}{\mathbf{z}^{\alpha_i} - 1}.$$

Thus we may write

$$\delta_i^\circ = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - \mathbf{z}^{\alpha_i} s_i).$$

Let us consider where this operator actually lives. Let $\mathcal{M}(T)$ be the field of fractions of $\mathcal{O}(T)$, which is an integral domain. Then $\mathcal{M}(T)$ acts as operators on itself (by multiplication). Also W acts on $\mathcal{M}(T)$ by the formula $(wf)(\mathbf{z}) = f(w^{-1}\mathbf{z})$. So δ_i° lives in the ring

$$\mathcal{R} = \bigoplus_{w \in W} \mathcal{M}(T) w.$$

Taking this point of view if $f \in \mathcal{M}(T)$ then $wfw^{-1} = w(f)$. The operator δ_i° is actually special since although its coefficients have denominators, δ_i° maps $\mathcal{O}(T)$ into itself. (See Lemma 3.1 of Lecture 8.) So δ_i° actually lives in the subring $\mathcal{R}_{\mathcal{O}}$ of $\mathcal{R}$ that preserves $\mathcal{O}(T)$.

Let us normalize the partition function $Z_\lambda(\mathbf{z}; \mathbf{d})$ as follows:

$$Y_\lambda(\mathbf{z}; \mathbf{d}) = \mathbf{z}^{-\rho} Z_y(\mathbf{z}; \mathbf{d}).$$

Then (22) can be rewritten as follows.

PROPOSITION 3.2. *If $d_i > d_{i+1}$ then*

$$Y_\lambda(\mathbf{z}; s_i\mathbf{d}) = \partial_i^\circ Y_\lambda(\mathbf{z}; \mathbf{d})$$

where

$$\partial_i^\circ f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i\mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}.$$

PROOF. Let us check that $\partial_i^\circ = \mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho$. Indeed, since $s_i(\rho) = \rho - \alpha_i$ we $\mathbf{z}^{-\rho} s_i \mathbf{z}^\rho = \mathbf{z}^{-\alpha_i} s_i$ in the ring $\mathcal{R}$. Thus

$$\mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho = \mathbf{z}^{-\rho}(\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - \mathbf{z}^{\alpha_i} s_i)\mathbf{z}^\rho = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - s_i) = \partial_i^\circ.$$

Now

$$Y_\lambda(\mathbf{z}; s_i\mathbf{d}) = \mathbf{z}^{-\rho} Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \mathbf{z}^{-\rho}\delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}) = \mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho Y_\lambda(\mathbf{z}; \mathbf{d}),$$

as required. $\square$

4. Looking ahead

The open models illustrate a scenario that is very common with the colored models: There are monostatic systems, whose partition functions are very simple, and there are Demazure recursion relations between the models. The beauty of the open models is that the recursions are in terms of operators ∂_i° that may be shown to satisfy the braid relations for the symmetric group, but not the quadratic relation since you may check that $(\partial_i^\circ)^2 = -\partial_i^\circ$. They generate a *degenerate Hecke algebra*. We will look at the implications of this next time.

LECTURE 10

The Bruhat order and Demazure Operators

1. Bruhat order

Most of the facts that I need about the Bruhat order are covered in Chapter 25 of [**24**]. There is a typo in the second edition: in (25.7) the wrong font is used and D should be ∂. For this section W can be the symmetric group, or a more general Coxeter group, though the geometric

I will give geometric proofs of the fact that S_n is a Coxeter group, that is, that it has a presentation:

$$S_n \cong \langle s_i | s_i^2 = 1, \text{braid relations} \rangle$$

and Matsumoto's theorem (Lecture 9). Referring to the book, these proofs are Theorem 25.1 (page 214) and Theorem 25.2 (page 217). These types of geometric arguments might be unsatisfactory since the results can be proved by purely algebraic methods in greater generality. However the technique is very powerful and useful. See [**32**] for applications of such geometric ideas.

I will also give a similar geometric proof of the *exchange principle* which is Proposition 20.3 or Proposition 20.4.

Proposition 1.1. *Let $w = s_{i_1} \cdots s_{i_k}$ be a product of k simple reflections such that $\ell(w) < k$. Then it is possible to omit two of the factors and get another reduced expression:*

$$w = s_{i_1} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k},$$

where the "hat" means a factor is omitted, with $1 \leqslant a < b \leqslant k$.

Proof. This is Proposition 20.4 in [**24**], and in class I will give a geometric proof similar to the geometric proofs of the Coxeter property and Matsumoto's theorem mentioned above. The exchange property is valid for any Coxeter group, and a purely algebraic proof may be found in [**11**], Section IV.1.5. Another proof can be found in [**46**] Section 1.7 (pages 13–15). □

Proposition 1.2 (Exchange principle). *Suppose that $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression and s_j a simple reflection such that $\ell(s_j w) < \ell(w)$. (Reduced means that $k = \ell(w)$.) Then we may find another reduced expression*

$$w = s_j s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k} \tag{23}$$

for some $1 \leqslant a \leqslant k$, where the "hat" means a factor is omitted.

Proof. Let us observe how this follows from Proposition 1.1. We have

$$s_j w = s_j s_{i_1} \cdots s_{i_k} = s_{i_0} s_{i_1} \cdots s_{i_k}, \qquad i_0 := j.$$

Since $\ell(s_j w) < k$ this expression is not reduced. Therefore we may omit two factors on the right and obtain a reduced expression for $s_j w$:

$$s_j w = s_{i_0} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k}.$$

Now we claim that $a = 0$, since if not, we have

$$w = s_{i_1} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k}$$

contradicting our assumption that $\ell(w) = k$. Thus

$$s_j w = s_{i_1} \cdots \widehat{s_{i_b}} \cdots s_{i_k},$$

proving (23). □

PROPOSITION 1.3. *Suppose that s is a simple reflection and $\ell(sw) < \ell(w)$. Then w has a reduced expression $w = s_{i_1} \cdots s_{i_k}$ such that $s_{i_1} = s$.*

PROOF. Let $w = s_{j_1} \cdots s_{j_k}$ be a reduced expression. Then by the exchange principle, $w = s s_{j_1} \cdots \hat{s}_{j_a} \cdots s_{j_k}$ for some a, and this is the required reduced expression. □

Next we come to the *Bruhat order* on $W = S_n$ (or a more general Coxeter group). This is defined on page 222 of [**24**]. See [**8**] for more information about this very important concept.

Let $u, v \in W$, and let $v = s_{i_1} \cdots s_{i_k}$ be a reduced expression. We write $u \leqslant v$ if there is a subsequence $(j_1, \cdots, j_l)$ of $(i_1, \cdots, i_k)$ such that $u = s_{j_1} \cdots s_{j_l}$.

PROPOSITION 1.4. *(i) This definition does not depend on the choice of reduced expression $v = s_{i_1} \cdots s_{i_k}$.*

(ii) If there exists any sequence $(j_1, \cdots, j_l)$ such that $u = s_{j_1} \cdots s_{j_l}$ then there exists such a sequence such that this is a reduced expression.

PROOF. For (i) see [**24**], Proposition 25.4 for a deduction of this from Matsumoto's theorem. For (ii), if the expression $u = s_{j_1} \cdots s_{j_l}$ is found and is not reduced (so $\ell(u) < l$) then by Proposition 1.1 we may discard entries in pairs to shorten the expression until it is reduced. □

LEMMA 1.5. *If s is a simple reflection and $w \in W$ then either $sw > w$ or $sw < w$. Indeed $sw < w$ if and only if $\ell(sw) < \ell(w)$, in which case $\ell(sw) = \ell(w) - 1$; and $sw > w$ if and only if $\ell(sw) = \ell(w) + 1$.*

PROOF. It follows easily from the definition that $\ell(w)$ is the length of the shortest expression of w as a product of simple reflections that $\ell(sw) = \ell(w) \pm 1$.

Write $w = s_{i_1} \cdots s_{i_k}$. If $\ell(sw) > \ell(w)$ then $sw = s s_{i_1} \cdots s_{i_k}$ is a reduced expression so that $w < sw$ in the Bruhat order. On the other hand if $\ell(sw) < \ell(w)$ by the exchange principle, we may write $sw = s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k}$ so $sw < w$. □

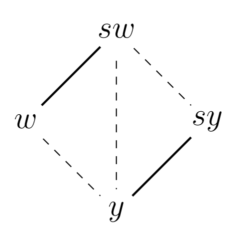
PROPOSITION 1.6 (Deodhar's Property Z [**34**]). *Let $y, w \in W$ and let s be a simple reflection. Assume that $w < sw$ and $y < sy$. Then the following are equivalent:*

(i) $y \leqslant w$;

(ii) $y \leqslant sw$;

(iii) $sy \leqslant sw$.

Here is a lattice diagram illustrating this fact:



The solid lines represent the assumed inequalities $w < sw$ and $y < sy$. Then the dotted lines are the three equivalent statements.

Proof. (i) $\Rightarrow$ (iii): Assume $y \leqslant w$. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression for w and let $y = s_{j_1} \cdots s_{j_l}$ be a reduced expression for y such that $(j_1, \cdots, j_l)$ is a subword of $(i_1, \cdots, i_k)$. Since $\ell(sw) = \ell(w) + 1$ the expression $ss_1 \cdots s_{i_k}$ is a reduced expression for sw, and $ss_{j_1} \cdots s_{j_l}$ is a subexpression representing sy, so $sy \leqslant sw$.

(iii) $\Rightarrow$ (ii): Assume $sy \leqslant sw$. Then $y < sy \leqslant sw$, as required.

(ii) $\Rightarrow$ (i). Assume $y < sw$. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression for w. Since $\ell(sw) = \ell(w) + 1 = k + 1$, the expression $sw = ss_{i_1} \cdots s_{i_k}$ is reduced, and y can be obtained from this by discarding factors. So if we take $s_{i_0} = s$, then we have a reduced expression $y = s_{j_1} \cdots s_{j_l}$ where $(j_1, \cdots, j_l)$ is a subsequence of $(i_0, i_1, \cdots, i_k)$. Now j_1 cannot be i_0 because this would imply that $sy = s_{j_1} y < y$. Therefore $(j_1, \cdots, j_l)$ is a subsequence of $(i_1, \cdots, i_k)$ which implies that $y \leqslant w$. □

2. The relationship between ∂_w° and ∂_w

Let $\partial_i^\circ = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - s_i)$ and $\partial_i = (1 - \mathbf{z}^{-\alpha_i})^{-1}(1 - \mathbf{z}^{\alpha_i} s_i)$ as before. We proved in Lecture 9 that both species of Demazure operators satisfy the braid relation, and so we may define

$$\partial_w^\circ = \partial_{i_1}^\circ \cdots \partial_{i_k}^\circ, \qquad \partial_w^\circ = \partial_{i_1} \cdots \partial_{i_k}$$

where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression, and by Matsumoto's theorem these are well-defined.

Theorem 2.1. *We have*

$$\partial_w = \sum_{y \leqslant w} \partial_y^\circ \tag{24}$$

Proof. (From [**17**].) We prove this by induction on $\ell(w)$. If $w = 1$, then $\partial_1 = \partial_1^\circ$ is the identity operator and (24) is certainly true. So assume (24). Let s be a simple reflection such that $\ell(sw) > \ell(w)$. This is equivalent to $sw > w$ in the Bruhat order. We must prove

$$\partial_{sw} = \sum_{y \leqslant sw} \partial_y^\circ. \tag{25}$$

Using our induction hypothesis

$$\partial_{sw} = \partial_s \partial_w = \partial_s \sum_{y \leqslant w} \partial_y^\circ.$$

Now suppose that $sy < y$. Then $\partial_y^\circ = \partial_{s \cdot sy}^\circ = \partial_s^\circ \partial_{sy}^\circ$ and since $\partial_s \partial_s^\circ = 0$ (as is easily checked) we have $\partial_s \partial_y^\circ = 0$. We may thus discard such terms from the sum and obtain

$$\partial_{sw} = \partial_s \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ. \tag{26}$$

We can divide W up into pairs $\{y, sy\}$ such that $y < sy$. These pairs are just the left cosets of W by the 2 element group $\langle s \rangle$. So we may write

$$\sum_{y \leqslant sw} \partial_y^\circ = \sum_{\substack{y \leqslant sw \\ y < sy}} \partial_y^\circ + \sum_{\substack{y \leqslant sw \\ sy < y}} \partial_y^\circ = \sum_{\substack{y \leqslant sw \\ y < sy}} \partial_y^\circ + \sum_{\substack{sy \leqslant sw \\ y < sy}} \partial_{sy}^\circ,$$

where we have made a variable change $y \to sy$ in the second term. By Deodhar's Property Z, if $y < sy$ then

$$y \leqslant w \quad \Leftrightarrow \quad y \leqslant sw \quad \Leftrightarrow \quad sy \leqslant sw$$

and if this is true then $\partial_{sy}^\circ = \partial_s^\circ \partial_y^\circ$. Also $\partial_s = 1 + \partial_s^\circ$.

(27)
$$\sum_{y \leqslant sw} \partial_y^\circ = \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ + \sum_{\substack{y \leqslant w \\ y < sy}} \partial_s^\circ \partial_y^\circ = (1 + \partial_s^\circ) \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ = \partial_s \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ.$$

Combining this with (26) gives (25), and we are done. ☐

LECTURE 11

The Open and Closed Models

1. Open models and Demazure atoms

Let $\mathfrak{S}_\lambda(\mathbf{z};q)$ be the Tokuyama models as in Lectures 5 and 6. We proved that the partition function $Z_\lambda(\mathbf{z};q)$ of these models equals

$$Z_\lambda(\mathbf{z};q) = \prod_{i<j}(z_i - qz_j)s_\lambda(\mathbf{z})$$

Let $Y_\lambda(\mathbf{z};q) = \mathbf{z}^{-\rho}Z_\lambda(\mathbf{z};q)$. Dividing the last equation by $\mathbf{z}^\rho = z_1^{n-1}z_2^{n-2}\cdots$ gives

$$Y_\lambda(\mathbf{z};q) = \prod_{\alpha\in\Phi^+}(1 - q\mathbf{z}^{-\alpha})s_\lambda(\mathbf{z}).$$

We are specializing to the case $q = 0$ in order to view some phenomena concerned with colored models in their simplest cases. These are:

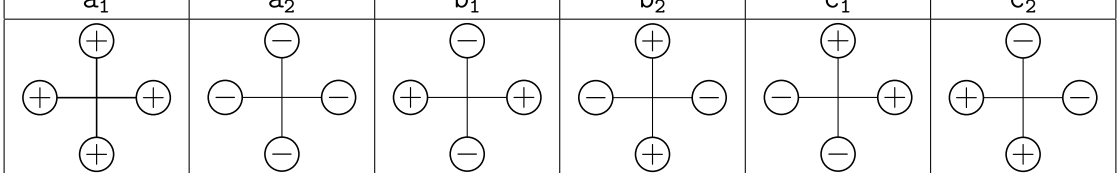
- There is a relationship between the colored and uncolored models;
- In a family of colored models there are some that are *monostatic* meaning that the system has only one state, and the partition function is therefore easy to evaluate;
- The Yang-Baxter equation gives us a recursion relations between partition functions of models in the category involving Demazure operators.

Such phenomena can be seen in other models such as [**10, 14, 2, 26**] and many other examples. This is therefore an important paradigm.

The Tokuyama models have at least two variants, which we are calling the open and closed models. The open models are investigated in [**17**], and the closed models are (as far as I know) not in any published literature.

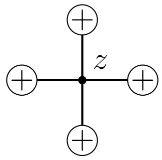
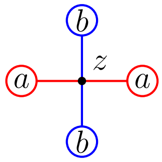
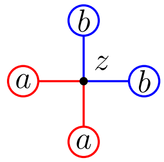
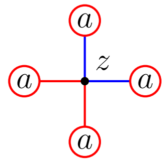
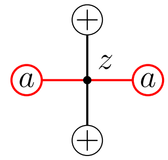
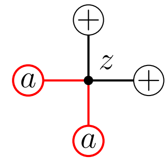
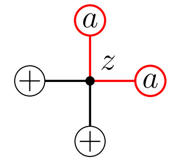
Let $\mathfrak{S}^\circ_\lambda(\mathbf{z};\mathbf{d})$ be the colored model, where $\mathbf{d}$ is a flag of colors, which we covered in Lectures 8 and 9. We may also write $\mathbf{d} = w\mathbf{c}_0$ where $\mathbf{c}_0 = (c_1, \cdots, c_n)$ is the "standard flag" of colors in decreasing order: $c_1 > \ldots > c_n$, and $w \in W = S_n$ is a permutation. We recall the results that we have already proved. We will denote $Y^\circ_\lambda(\mathbf{z};\mathbf{d}) = \mathbf{z}^{-\rho}Z^\circ_\lambda(\mathbf{z};\mathbf{d})$.

For convenience, here are the Boltzmann weights of the two types of systems. For the Tokuyama with $q = 0$:



$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	z_i	0	z_i	z_i	1

For the closely related open models:

						
1	$z \quad a > b$ $0 \quad a < b$	$0 \quad a > b$ $z \quad a < b$	z	z	z	1

One relationship between the two models is easy to see: if we take the open model but only use a single color, we recover the Tokuyama models with $q = 0$.

But we will show a *deeper* relationship, by showing that the partition function of the uncolored Tokuyama $q = 0$ model can be decomposed into a sum (over flags $\mathbf{d}$) of partition functions of the uncolored model.

PROPOSITION 1.1. *The model $\mathfrak{S}_\lambda(\mathbf{z}; \mathbf{c}_0)$ has only one state. The normalized partition function*

$$Y_\lambda^\circ(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^\lambda.$$

PROOF. This is Proposition 3.1 in Lecture 9. □

LEMMA 1.2. *If $s_i w > w$ then*

$$Y_\lambda(\mathbf{z}; s_i w \mathbf{c}_0) = \partial_i^\circ Y_\lambda(\mathbf{z}; w\mathbf{c}_0)$$

PROOF. This is proved (using the Yang-Baxter equation) as Proposition 3.2 in Lecture 9, with the assumption $d_i > d_{i+1}$. This assumption is equivalent to $s_i w > w$ by Exercise 8(i). □

PROPOSITION 1.3. *If $s_i w > w$, then*

$$Y_\lambda^\circ(\mathbf{z}; \mathbf{d}) = \partial_w^\circ \mathbf{z}^\lambda.$$

PROOF. This follows by induction from the last two propositions. □

The polynomial $\partial_w^\circ \mathbf{z}^\lambda$ is called a *Demazure atom*.

PROPOSITION 1.4. *We have*

$$s_\lambda(\mathbf{z}) = \sum_{w \in W} \partial_w^\circ \mathbf{z}^\lambda. \tag{28}$$

PROOF. Take $w = w_0$ (the longest element in W) in Theorem 2.1 of Lecture 10. Since $w \leqslant w_0$ for all $w \in W$, this gives

$$\sum_{w \in W} \partial_w^\circ \mathbf{z}^\lambda = \partial_{w_0} \mathbf{z}^\lambda.$$

This equals the Schur polynomial $s_\lambda(\mathbf{z})$ by the Demazure character formula ([**24**] Theorem 25.3). □

Now in equation (28) the left-hand side is the partition function of the uncolored $q = 0$ model by Proposition 1.1, and the right hand side is the *sum* of the partition functions of the open models. This shows that there is a relationship between the open and uncolored models. Let us prove this directly.

THEOREM 1.5. *We have*

$$Z_\lambda(\mathbf{z};0) = \sum_{w\in W} Z^\circ_\lambda(\mathbf{z}; w\mathbf{c}_0).$$

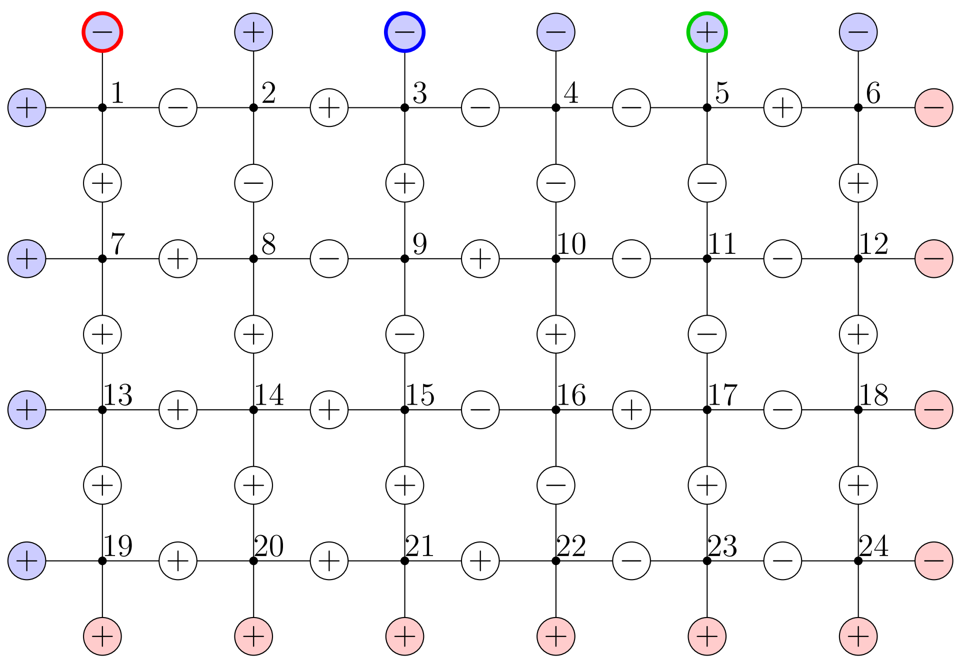
PROOF. We will show that there is a bijection

$$\mathfrak{S}_\lambda(\mathbf{z};0) \quad\longleftrightarrow\quad \bigsqcup_{\mathbf{d}} \mathfrak{S}^\circ_\lambda(\mathbf{z};\mathbf{d})$$

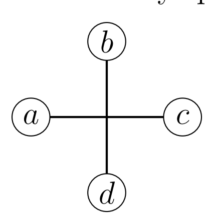
in which corresponding states have the same Boltzmann weights. The existence of maps $\phi_{\mathbf{d}} : \mathfrak{S}^\circ_\lambda(\mathbf{z};\mathbf{d}) \longrightarrow \mathfrak{S}_\lambda(\mathbf{z};0)$ is easy: we just take a state $\mathfrak{s}^\circ$ of any one of the models $\mathfrak{S}^\circ_\lambda(\mathbf{z};\mathbf{d})$ and replace every colored spin by $-$ to obtain a state of $\mathfrak{S}_\lambda(\mathbf{z};0)$.

So what we need to show is that every state $\mathfrak{s}$ of $\mathfrak{S}_\lambda(\mathbf{z};0)$ is $\phi_{\mathbf{d}}(\mathfrak{s}^\circ)$ for a unique state $\mathfrak{s}^\circ$ of one of the systems $\mathfrak{S}^\circ_\lambda(\mathbf{z};\mathbf{d})$. About the desired $\mathfrak{s}^\circ$ we know which edges will be colored. Of the boundary edges, we know the colors of the edges on the top, since the boundary conditions put color c_i in the λ_i+n-i column, and $+$ elsewhere. We do not know the colors of the edges at the right, since we do not know for which $\mathbf{d}$ we will have $\mathfrak{s} \in \mathfrak{S}_\lambda(\mathbf{z};\mathbf{d})$.

Our procedure will be to "color" the uncolored state $\mathfrak{s}$ by replacing the $-$ spins in order by colors. We order the edges of the grid from left to right and from top to bottom as follows.



visiting each vertex in this order, let the boundary spins be labeled in this order:



Because we have already considered all prior vertices, the spins a and b in the colored state $\mathfrak{s}^\circ$ are already determined. As for the colored spins c and d, they are not determined, but we will argue that they have a unique possible assignment. Because we know the uncolored state $\mathfrak{s}$, we know whether c and d are colors or $+$, and we know what colors (or $+$) a and b

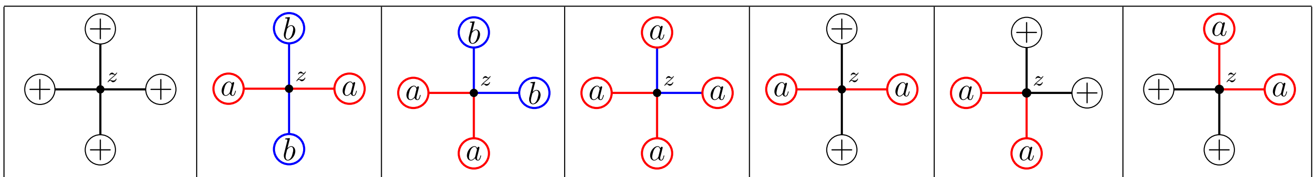
are. From the Boltzmann weights, there is a unique assignment of colors (or $+$) to c and d. For example, if a and b are both colors, c and d will be the same two colors, and c will be the larger of the two.

Visiting all the colors in order, we find there is a unique colored spin $\mathfrak{s}^\circ$ such that $\phi(\mathfrak{s}^\circ) = \mathfrak{s}$. We can read off the flag $\mathbf{d}$ from the colors on the right edge. $\square$

Remark 6. Theorem 1.5 implies Proposition 1.4, which was not used in its proof. Thus we have a proof of the Demazure character formula for $\mathrm{GL}(n)$.

2. Closed models

Here for convenience are the Boltzmann weights for the closed models.



1	z $a>b$ 0 $a<b$	z $a>b$ 0 $a<b$	z	z	z	1

Boundary conditions are the same as for the open model. Let $\mathfrak{S}^\bullet_\lambda(\mathbf{z};\mathbf{d})$ be the closed model, and let $Z^\bullet_\lambda(\mathbf{z};\mathbf{d})$ be its partition function. We will again denote by

$$Y^\bullet_\lambda(\mathbf{z};\mathbf{d}) = \mathbf{z}^{-\rho} Z^\bullet_\lambda(\mathbf{z};\mathbf{d})$$

the normalized partition function.

Proposition 2.1. *We have*

$$Y_\lambda(\mathbf{z};\mathbf{d}) = \partial_w \mathbf{z}^\lambda.$$

Proof. This is Exercise 9. $\square$

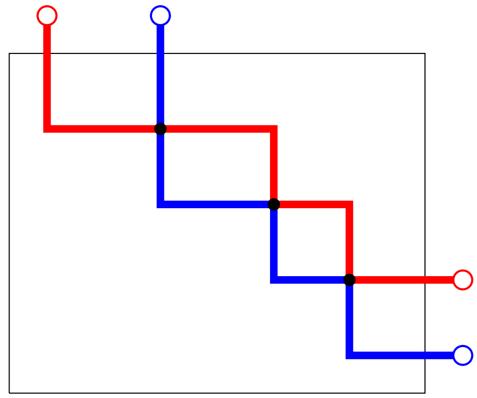
The polynomial $\partial_w \mathbf{z}^\lambda$ is called a *Demazure character* or *key polynomial*. In Lecture 10 we proved

$$\partial_w \mathbf{z}^\lambda = \sum_{y \leqslant w} \partial^\circ_y \mathbf{z}^\lambda,$$

expressing the Demazure character as a sum of Demazure atoms. In the extreme case that $w = w_0$ is the long element, the Demazure character formula ([**24**] Theorem 25.3) asserts that $\partial_{w_0} \mathbf{z}^\lambda = s_\lambda(\mathbf{z})$, so the closed model and the Tokuyama $q=0$ model have the same partition function.

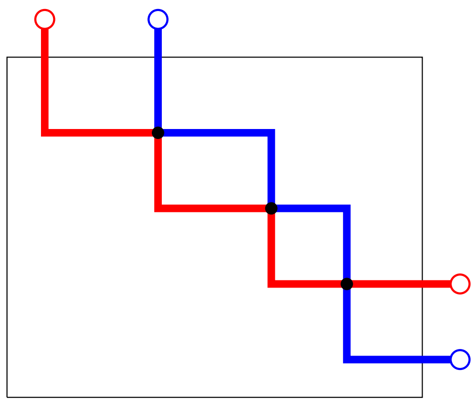
We can get an intimation of the reason for the difference between the open and closed models from Exercises 5 and 6. Let us consider two colored paths in the open models. Let us assume that the paths meet several times, as in Exercise 5.

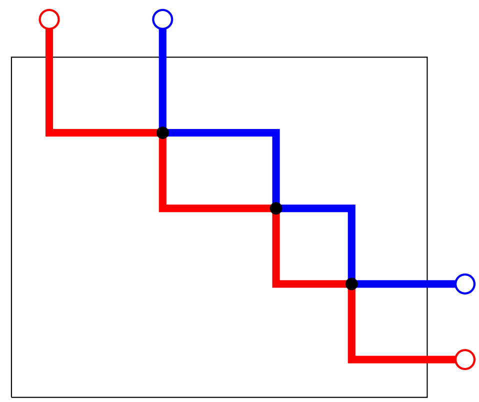
For the open model, the paths must be colored as follows (with red $>$ blue).



Thus in the open model paths must cross the first time they meet, but may not cross again. This may be seen easily from consideration of the Boltzmann weights.

Now in the closed models, there are two possibilities (Exercise 5).





The paths may cross, or not cross, but if they do cross, it must be the *last* time they meet. This extra flexibility causes the difference between the open and closed models.

3. Generalizations

The open and closed models may be described as refinements of the Tokuyama $q = 0$. Refinements of the general Tokuyama model for general q are described in [**14**].

We recall that the normalized partition function $Y(\mathbf{z}; q) = \mathbf{z}^{-\rho} Z(\mathbf{z}; q)$ equals

$$\prod_{\alpha\in\Phi^+} (1 - q^{-1}\mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z}). \tag{29}$$

This product appears in a very different context: a famous formula of Casselman and Shalika [**28**] for Whittaker functions on p-adic groups (particularly $\mathrm{GL}(n)$, where the formula was found earlier by Shintani) gives (29) as the values of the spherical Whittaker function.

The models in [**14**] are open models in that they break this function into pieces $Y(\mathbf{z}; q; w\mathbf{c}_0)$ whose sum equals (29). There are some unexpected phenomena.

- Some vertical edges may carry more than one color; that is, two colored paths may travel along the same vertical edge. However the models are still fermionic since each vertical edge may not carry more than one instance of the same color.

- There are some unexpected cancellations: terms coming from different w may appear as "twins" with opposite sign that cancel out in the sum (29).
- Generalizing the Casselman-Shalika formula, the partition functions can be identified with the values of Iwahori Whittaker functions on $\mathrm{GL}(n, F)$ where F is a p-adic field.

Although there are new phenomena, the general framework for understanding these models is the same as for the open models that we have devoted a few lectures to. That is, if $\mathbf{d} = \mathbf{c}_0$ the systems are monostatic, and the partition functions are easy to evaluate; and in the general case, they satisfy Demazure recursion that can be proved using the Yang-Baxter equation. These are sufficient to evaluate the partition functions in a useful way.

Other related fermionic models appear in [**15, 2**]. There are other related models including bosonic variants ([**10, 26**]) and models with column parameters.

4. Open questions about closed models

Question 1. Are there also closed models that are related to the Tokuyama models for general q? This is not known if $q \neq 0$.

Question 2. The proof of Theorem 1.5 gives maps $\mathfrak{S}^{\circ}_{\lambda}(\mathbf{z}; w\mathbf{c}_0) \longrightarrow \mathfrak{S}_{\lambda}(\mathbf{z}; 0)$ and similarly we can find maps $\mathfrak{S}^{\bullet}_{\lambda}(\mathbf{z}; w\mathbf{c}_0) \longrightarrow \mathfrak{S}_{\lambda}(\mathbf{z}; 0)$. Let us therefore identify $\mathfrak{S}^{\circ}_{\lambda}(\mathbf{z}; w\mathbf{c}_0)$ and $\mathfrak{S}^{\bullet}_{\lambda}(\mathbf{z}; w\mathbf{c}_0)$ with their images in $\mathfrak{S}_{\lambda}(\mathbf{z}; 0)$. Since

$$Y^{\bullet}_{\lambda}(\mathbf{z}; w\mathbf{c}_0) = \sum_{y \leqslant w} Y^{\circ}_{\lambda}(\mathbf{z}; w\mathbf{c}_0)$$

(which is equivalent to the last result in Lecture 10) we expect that

$$\mathfrak{S}^{\bullet}_{\lambda}(\mathbf{z}; w\mathbf{c}_0) = \bigcup_{y \leqslant w} \mathfrak{S}^{\circ}_{\lambda}(\mathbf{z}; y\mathbf{c}_0) \qquad \text{(disjoint)}.$$

The problem is to prove this. The path discussion in Section 2 is a promising starting point, but I do not think this is obvious.

LECTURE 12

Hecke Algebras and Demazure-Lusztig Operators

1. Hecke algebras and Demazure-Lusztig operators

Let $W = S_n$ or more generally, let W be any Coxeter group. The theory in this section works in generality.

The Demazure operators ∂_w span a $|W|$-dimensional algebra $\mathcal{H}_0$, generated by the ∂_i. The ∂_w° give another basis of this ring, with generators $\partial_i^\circ = \partial_i - 1$. The generators satisfy the braid relations (so $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ and $\partial_w^\circ = \partial_{i_1}^\circ \cdots \partial_{i_k}^\circ$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression) and quadratic relations $\partial_i^2 = \partial_i$.

A more general ring $\mathcal{H}(W)$ depends on a parameter q. This ring has generators T_i that satisfy the braid relations and quadratic relations

$$T_i^2 = (q-1)T_i + q.$$

These first appeared in the work of Iwahori and Matsumoto [**49**] determining the Iwahori Hecke algebra of a p-adic group, for example $G = \mathrm{GL}(n, \mathbb{Q}_p)$. For this, we there is a Coxeter group $\tilde{W}$ called the *affine Weyl group*. The main result of Iwahori and Matsumoto is that $\mathcal{H}(\tilde{W})$ can be realized as a convolution ring of functions on G. Actually there is a slightly larger ring than $\mathcal{H}(\tilde{W})$ that can be realized this way, the *extended* affine Weyl group, but we will not discuss that.

In addition to the appearance of the Iwahori Hecke algebra as a convolution ring of functions on the p-adic group G, the same Hecke algebra appears in the theory in a seemingly different way in the theory of intertwining operators between different induce representations on the group [**86**].

Lusztig [**73**] realized that the same Hecke algebra appears in a different context, namely in the equivariant K-theory of the complex flag varieties. (See [**30**] for context.) Kazhdan and Lusztig [**60**] exploited the fact that the affine Iwahori Hecke algebra $\mathcal{H}(\tilde{W})$ appears in two different contexts to translate statements about representations of p-adic groups into algebraic geometry, where they could be proved.

The Hecke algebra $\mathcal{H}(W)$ has the following representation on functions. For definiteness we will work with just $W = S_n$ but everything would work if W is the Weyl group of any reductive algebraic group over $\mathbb{C}$. As in earlier lectures, let $T = (\mathbb{C}^\times)^n$ and let $\mathcal{O}(T)$ be the ring of functions on T spanned by the functions $\mathbf{z}^\mu$. Let $\mathcal{L}_i$ be the operator

$$\mathcal{L}_i = (\mathbf{z}^{\alpha_i} - 1)^{-1}[(1 - s_i) - q(1 - \mathbf{z}^{\alpha_i})s_i].$$

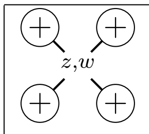
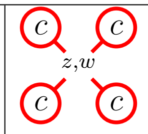
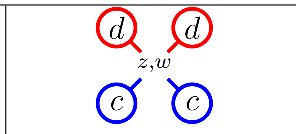
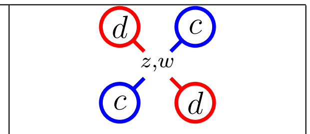
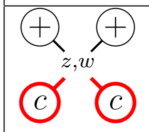
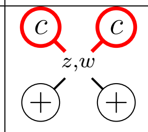
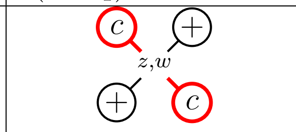
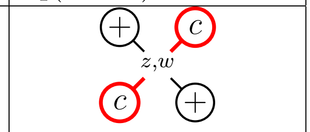
THEOREM 1.1 (Lusztig). *The operators $\mathcal{L}_i$ satisfy the braid and quadratic relations*

$$\mathcal{L}_i^2 = (q-1)\mathcal{L}_i + q,$$

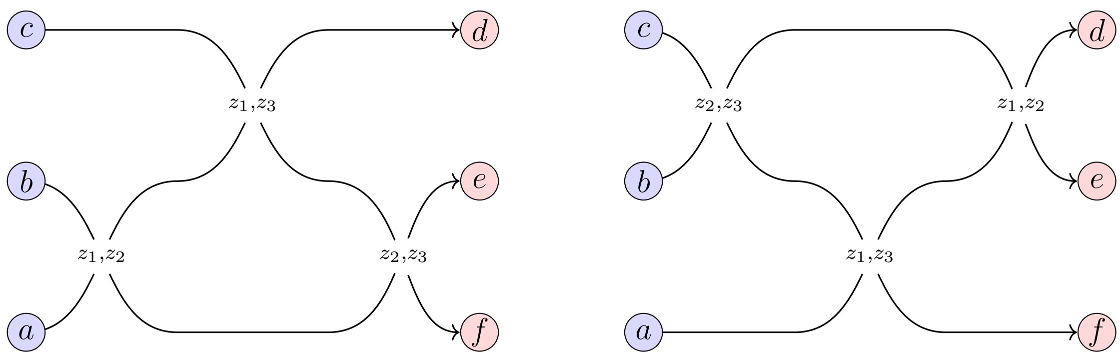
hence generate an algebra isomorphic to $\mathcal{H}(W)$.

PROOF. This is just a calculation. The braid relation is somewhat tedious to check [**73**]. The quadratic relation is Exercise 10. ∎

But let us show how these operators may appear in a lattice model. We will use the following R-matrix (due to Jimbo [**51**] in the notation of [**26**])

			
$z-qw$	$z-qw$	$(1-q)z$ if $c<d$ $(1-q)w$ if $c>d$	$z-w$ if $c>d$ $q(z-w)$ if $c<d$
			
$(1-q)z$	$(1-q)w$	$q(z-w)$	$z-w$

This satisfies a parametrized Yang-Baxter equation thus:



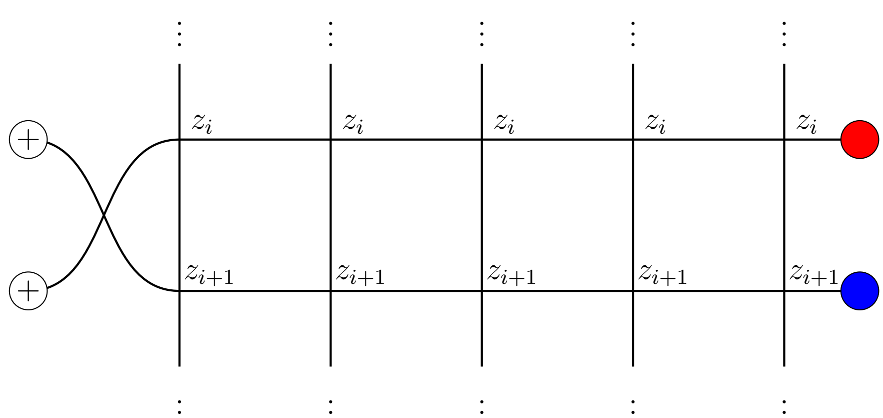
We will postulate a lattice model that uses this R-matrix. We do not need to specify the Boltzmann weights for the model itself, and indeed there are multiple choices. But let $Z(\mathbf{z};\mathbf{d})$ be the partition function, where as in our discussion of the open model, $\mathbf{d}=(d_1,\cdots,d_n)$ is a flag describing the boundary conditions on the right edge.

Proposition 1.2. *If $d_i > d_{i+1}$ then*

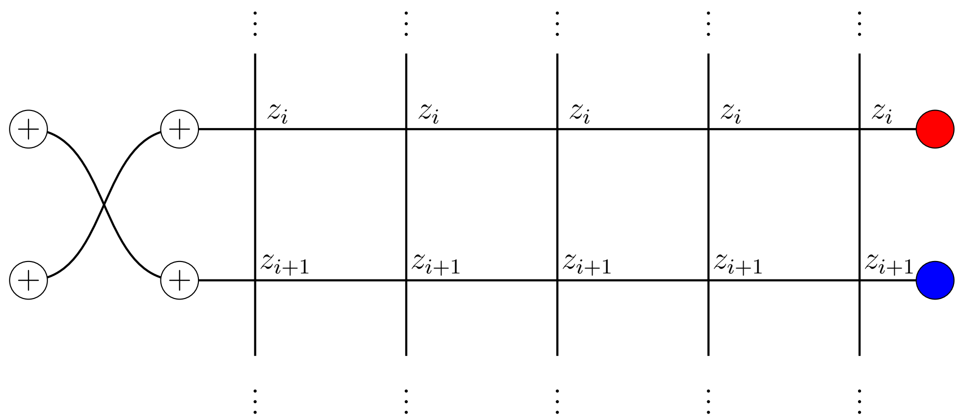
$$Z(\mathbf{z};s_i\mathbf{d}) = \mathcal{L}_i Z(\mathbf{z};\mathbf{d}),$$

where $\mathcal{L}_i$ is the Demazure-Lusztig operator.

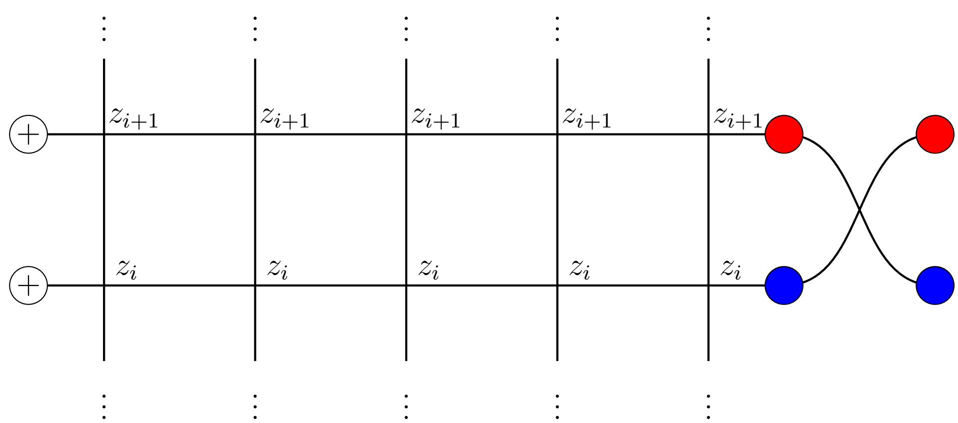
Proof. Imitating the argument in Lecture 8, Let us attach the R-matrix to the left:



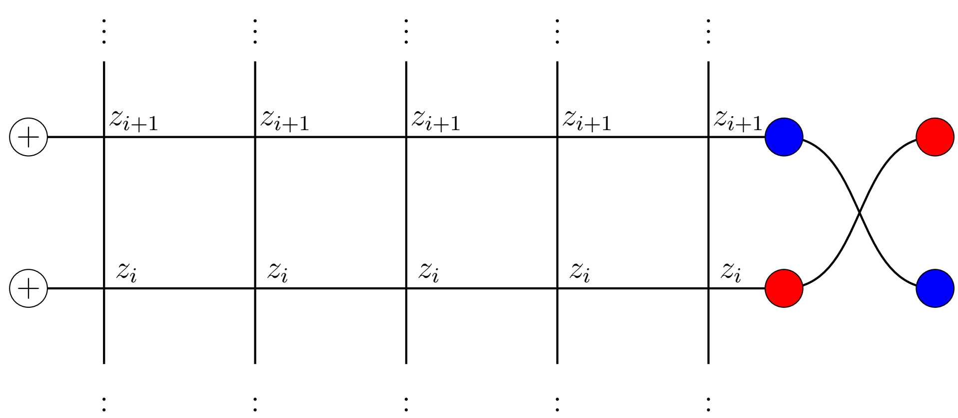
Given the spins +, + on the left edge the spins on the R-matrix can only be all +, so we may assume that the configuration is as follow:



the partition function of this system is $Z(\mathbf{z};\mathbf{d})$ times the value $z_i - qz_{i+1}$ of the R-matrix. (We are using $z = z_i$ and $w = z_{i+1}$ in the table.) Running the train argument, it turns out there are two possible configurations on the right-hand side, namely



and



Inserting the values of the R-matrices for these two configurations gives the identity

$$(z_i - qz_{i+1})Z(\mathbf{z};\mathbf{d}) = (1-q)z_iZ(s_i\mathbf{z};\mathbf{d}) + (z_i - z_{i+1})Z(s_i\mathbf{z};s_i\mathbf{d}).$$

It will be convenient to replace $\mathbf{z} \to s_i\mathbf{z}$ so $z_i \leftrightarrow z_{i+1}$ and rewrite this identity

$$(z_{i+1} - qz_{i+1})Z(s_i\mathbf{z};\mathbf{d}) = (1-q)z_{i+1}Z(\mathbf{z};\mathbf{d}) + (z_{i+1} - z_i)Z(\mathbf{z}; s_i\mathbf{d}).$$

Reorganizing,

$$Z(\mathbf{z}; s_i\mathbf{d}) = \frac{(1-q)z_jZ(\mathbf{z};\mathbf{d}) - (z_{i+1} - qz_i)Z(s_i\mathbf{z};\mathbf{d})}{z_i - z_{i+1}} = \mathcal{L}_iZ(\mathbf{z};\mathbf{d}).$$

□

We see that the same Iwahori Hecke algebra, by its representation by Demazure-Lusztig operators appears in at least three completely different places: the representation theory of p-adic groups (Iwahori-Matsumoto) the equivariant K-theory of the complex flag variety (Kazhdan-Lusztig) and now the theory of solvable lattice models.

2. Groups

Solutions to the Yang-Baxter equation have at least two types of applications:

- Solvable lattice models;
- Knot invariants, such as the Jones and Alexander polynomials.

Therefore it is of interest that there is a mechanism that produces a variety of solutions to the Yang-Baxter equation, underlying most of the examples that we need. This is the theory of *quantum groups*, invented by Drinfeld [**35**] and Jimbo [**50**]. For modern treatments [**59, 77**].

Quantum groups are actually *Hopf algebras*. In this lecture we will introduce Hopf algebras and show how groups can produce Hopf algebras.

Please review the notions of *monoidal category* and *braided category* in Lecture 4.

The notion of a *Hopf algebra* is extremely similar to the notion of a group, but more flexible. Quantum groups, as defined by Drinfeld, are actually Hopf algebras, so we start with those.

To motivate the definition, let us formulate the axioms of a group "categorically." We work in the category of sets, which has products and a terminal object, namely the set $I = \{1_I\}$ with one element. The category of sets is a monoidal category with $\times$ its operation and I its unit element. It is actually a *symmetric* monoidal category, which is a special case of a braided monoidal category. In a braided category, there are morphisms $c_{A,B} : A \times B \longrightarrow B \times A$ for objects A and B. In a symmetric monoidal category, we assume further that the composition

$$A \times B \xrightarrow{c_{A,B}} B \times A \xrightarrow{c_{B,A}} A \times B$$

is the identity map.

The reason for this digression into the definition of a group, the idea is that if we formulate the notion correctly we can apply it in other symmetric monoidal categories, particularly the category of vector spaces. We will find that if we take the definition of a group, formulated categorically, and apply it in the category of vector spaces, we get a useful notion, that of a Hopf algebra.

Let G be a group. Let $\varepsilon : I \longrightarrow G$ be the map that sends 1_I to 1_G. Let $\mu : G \times G \longrightarrow G$ be multiplication, and $S : G \longrightarrow G$ the inverse map. We will also need to make use of the diagonal map $\Delta : G \longrightarrow G \times G$ that sends g to (g, g), and the map $\eta : G \longrightarrow I$ that sends $g \mapsto 1_I$.

Then the following properties are satisfied.

The Associative Law: The following diagram is commutative.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

The Unit axiom: The following diagrams is commutative. The maps $I \times G \cong G$ and $G \times I \cong G$ are the obvious ones. These maps are part of the data making the category of sets into a monoidal category.

$$\begin{tikzcd}
I \times G \arrow[r, "\varepsilon\times 1"] \arrow[d, "\cong"] & G \times G \arrow[dl, "\mu"] \\
G &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G \times I \arrow[r, "1\times\varepsilon"] \arrow[d, "\cong"] & G \times G \arrow[dl, "\mu"] \\
G &
\end{tikzcd}$$

The diagonal map Δ and counit η have dual properties to the associative law and unit axiom.

Coassociativity: The following diagram is commutative.

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d, "\Delta"] & G \times G \arrow[d, "1\times\Delta"] \\
G \times G \arrow[r, "\Delta\times 1"] & G \times G \times G
\end{tikzcd}$$

Counit: The following diagrams are commutative.

$$\begin{tikzcd}
I \otimes G \arrow[d, "\cong"] & G \times G \arrow[l, "\eta\times 1"'] \\
G \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G \otimes I \arrow[d, "\cong"] & G \times G \arrow[l, "1\times\eta"'] \\
G \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

The group law requires $g \cdot S(g) = S(g) \cdot g = 1_G$, and we may formulate these diagramatically thus:

Antipode: The following diagram is commutative.

$$\begin{tikzcd}
G \times G \arrow[rr, "1\times S"] & & G \times G \arrow[d, "\mu"] \\
G \arrow[u, "\Delta"] \arrow[r, "\eta"] & I \arrow[r, "\varepsilon"] & G
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G \times G \arrow[rr, "S\times 1"] & & G \times G \arrow[d, "\mu"] \\
G \arrow[u, "\Delta"] \arrow[r, "\eta"] & I \arrow[r, "\varepsilon"] & G
\end{tikzcd}$$

There is one more property that is needed, and this requires the "flip" map $\tau : G\times G \longrightarrow G \times G$ that sends $(x, y) \mapsto (y, x)$. This is just a new notation for $c_{G,G}$ in the definition of a symmetric monoidal category.

Hopf: The following diagram commutes.

$$\begin{tikzcd} G \times G \arrow[r, "\Delta\times\Delta"] \arrow[d, "\mu"] & G\times G\times G\times G \arrow[r, "1\times\tau\times1"] & G\times G\times G\times G \arrow[d, "\mu\times\mu"] \\ G \arrow[rr, "\Delta"] & & G\times G \end{tikzcd}$$

Indeed both compositions are the map $(g,h) \longmapsto (gh, gh)$.

3. Hopf Algebras

The same axioms but applied in the category of vector spaces produces the notion of a *Hopf algebra*. We work over a field F. Then category of F-vector spaces is a monoidal category with unit element F and monoidal operation $\otimes$ (tensor product).

But let us start with just the first two axioms. We need a vector space H with a vector space homomorphism $\mu : H \otimes H \longrightarrow H$ and a homomorphism $\varepsilon : F \longrightarrow H$. Note that these data are equivalent to a bilinear operation $H \times H \longrightarrow H$ and a distinguished element $1_H := \varepsilon(1_F)$.

Associative:

$$\begin{tikzcd} H\otimes H\otimes H \arrow[r, "\mu\otimes1"] \arrow[d, "1\otimes\mu"] & H\otimes H \arrow[d, "\mu"] \\ H\otimes H \arrow[r, "\mu"] & H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} \mathbb{C}\otimes H \arrow[r, "\varepsilon\otimes1"] \arrow[d, "\cong"] & H\otimes H \arrow[dl, "\mu"] \\ H & \end{tikzcd} \qquad \begin{tikzcd} H\otimes I \arrow[r, "1\otimes\varepsilon"] \arrow[d, "\cong"] & H\otimes H \arrow[dl, "\mu"] \\ H & \end{tikzcd}$$

The isomorphisms $H \cong F \otimes H \cong H \otimes F$ are part of the data making the category of vector spaces into a monoidal category.

Given such μ we can define a bilinear composition law $H \times H \longrightarrow H$ by $x \cdot y = \mu(x \otimes y)$. The two axioms mean that H is an associative F-algebra.

We return to generalizing the group axioms. The role of the diagonal map is played by a lnear map $\Delta : H \longrightarrow H \otimes H$, and we also need a *counit* which is a linear map $\eta : H \longrightarrow F$. We need:

Coassociative:

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"] & H\otimes H \arrow[d, "1\otimes\Delta"] \\ H\otimes H \arrow[r, "\Delta\otimes1"] & H\otimes H\otimes H \end{tikzcd}$$

Counit:

$$
\begin{tikzcd}
I \otimes H \arrow[d, "\cong"] & H \otimes H \arrow[l, "\eta\otimes 1"'] \\
H \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H \otimes I \arrow[d, "\cong"] & H \otimes H \arrow[l, "1\otimes\eta"'] \\
H \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

A vector space with maps Δ and η satisfying these two axioms is called a *coalgebra*. Now we need a linear map $S : H \longrightarrow H$ and two more axioms:

Antipode:

$$
\begin{tikzcd}
H \otimes H \arrow[rr, "1\otimes S"] & & H \otimes H \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\eta"] & I \arrow[r, "\varepsilon"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[rr, "S\otimes 1"] & & H \otimes H \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\eta"] & I \arrow[r, "\varepsilon"] & H
\end{tikzcd}
$$

Hopf:

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"] & H \otimes H \otimes H \otimes H \arrow[r, "1\otimes\tau\otimes 1"] & H \otimes H \otimes H \otimes H \arrow[d, "\mu\otimes\mu"] \\
H \arrow[rr, "\Delta"] & & H \otimes H
\end{tikzcd}
$$

If A, B are algebras, so is $A \otimes B$. If A, B are co-algebras, so is $A \otimes B$. The Hopf axiom can be expressed as saying either that the comultiplication $\Delta : H \longrightarrow H \otimes H$ is an algebra homomorphism, or (equivalently) that the multiplication $\mu : H \otimes H \longrightarrow H$ is a coalgebra homomorphism.

Let us give some examples of Hopf algebras that arise from groups. First, let G be a finite group, and let $\mathbb{C}[G]$ be its group algebra. This is a Hopf algebra. To define the comultiplication, we extend the diagonal map $\Delta : G \longrightarrow G \times G$ to a map $\mathbb{C}[G] \longrightarrow \mathbb{C}[G \times G] \cong \mathbb{C}[G] \otimes \mathbb{C}[G]$ by linearity. The counit is the augmentation map $\mathbb{C}[G] \longrightarrow \mathbb{C}$. The multiplication of G is encoded in the multiplication of $\mathbb{C}[G]$.

On the other hand, let $\mathcal{O}(G)$ be the commutative algebra of functions on G. The multiplication is just pointwise multiplication. For the comultiplication, let $\delta_g \in \mathcal{O}(G)$ be the basis of $\mathcal{O}(G)$ defined for $g \in G$ by

$$
\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\Delta(\delta_g) = \bigoplus_{\substack{(x,y)\in G \\ xy=g}} \delta_x \otimes \delta_y .
$$

These two Hopf algebras are in duality. This means we have a dual pairing $\mathbb{C}[G] \otimes \mathcal{O}(G) \longrightarrow \mathbb{C}$ defined by $g \otimes f \longmapsto f(g)$, which make the multiplication in $\mathbb{C}[G]$ dual to the comultiplication in $\mathcal{O}(G)$.

Similarly if G is a complex Lie group, such as $\mathrm{GL}(n, \mathbb{C})$, we have two types of Hopf algebras. On the one hand, there is the universal enveloping algebra $U(\mathfrak{g})$ where $\mathfrak{g} = \mathrm{Lie}(G)$. On the other hand, regarding G as an affine algebraic group, there is the commutative algebra

$\mathcal{O}(G)$ of polynomial functions on G. These are Hopf algebras, and they are related to each other by a dual pairing.

Both $U(\mathfrak{g})$ and $\mathcal{O}(G)$ have deformations (depending on a parameter q). Denoting these as $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, the category of modules over $U_q(\mathfrak{g})$ is braided, or the category of comodules for $\mathcal{O}_q(G)$. As a variant, instead of the finite-dimensional algebra $U_q(\mathfrak{g})$ we may use $U_q(\hat{\mathfrak{g}})$ where $\hat{\mathfrak{g}}$ is an affine Lie algebra.

This gives rise to many examples of the Yang-Baxter equation. For example, the R-matrix that we ended the last section is associated with pairs of representations of $U_q(\widehat{\mathfrak{gl}}(n+1))$.

An important use of the comultiplication in a Hopf algebra is that the category of finite-dimensional representations is a monoidal category. That is, given H-modules V and W we can give $V \otimes W$ the structure of an H-module.

Naturally $V \otimes W$ is a module over the algebra $H \otimes H$, and this may be true for modules over any associative algebra. But we want it to be a module over H, and we may do this using an algebra homomorphism $H \to H \otimes H$, and for this we use the comultiplication.

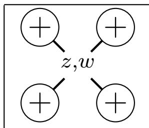
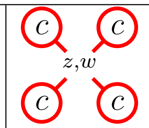
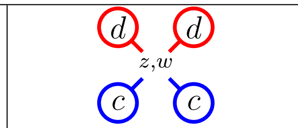
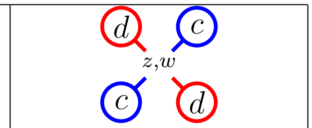
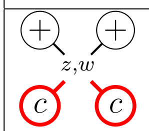
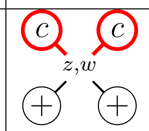
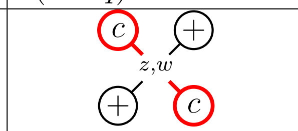
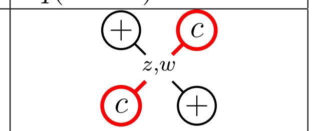
Drinfeld's accomplishment was to show how to define a class of Hopf algebras called *quasitriangular*. The beauty of quasitriangular Hopf algebras is that the module category is not just monoidal, it is braided. The braiding is described thus: there is, in $H \otimes H$ an invertible element R called the *universal R-matrix* which has the property that if U, V are H-modules then the map $c_{U,V} : U \otimes V \to V \otimes U$ defined by τR is a braiding.

The Hopf algebra $U_q(\mathfrak{g})$ (which we have not yet defined) is not quasitriangular, since the universal R-matrix is given by an infinite sum and hence lives not in $H \otimes H$ but in a completion. This is not really a problem since there are various workarounds, so $U_q(\mathfrak{g})$ is "morally quasitriangular," and the module category is braided.

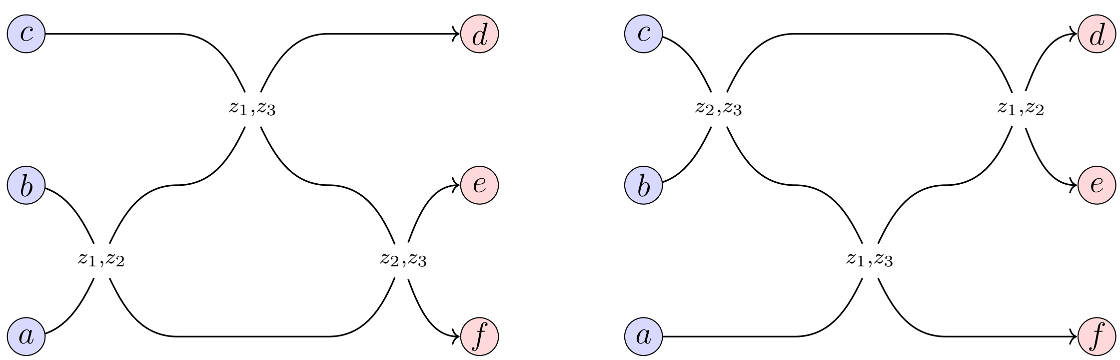
Quantum Groups and the Yang-Baxter Equation

1. Examples of colored models

In Lecture 12, we considered the following R-matrix:

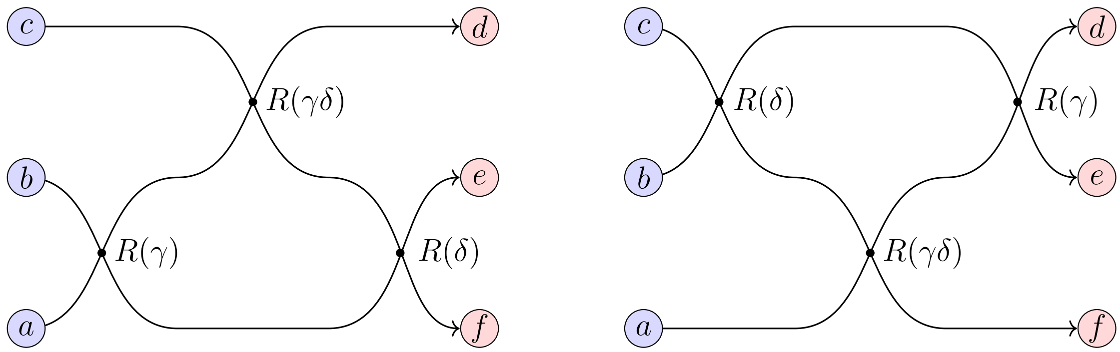
			
$z - qw$	$z - qw$	$(1-q)z$ if $c < d$ $(1-q)w$ if $c > d$	$z - w$ if $c > d$ $q(z-w)$ if $c < d$
			
$(1-q)z$	$(1-q)w$	$q(z-w)$	$z - w$

We mentioned that this satisfies a Yang-Baxter equation as follows:

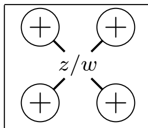
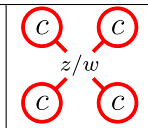
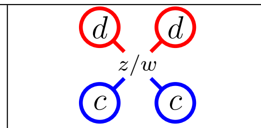
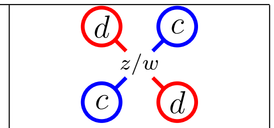
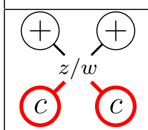
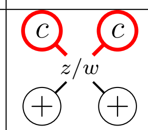
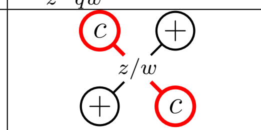
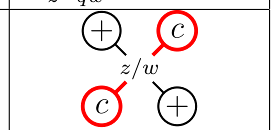


We will refer to this as the RRR equation since it involves three copies of the R-matrix.We described this as a parametrized Yang-Baxter equation, but it requires a bit of explanation why this is an instance of a parametrized Yang-Baxter equation. We recall that this requires a group Γ and a map R from Γ to the set of Boltzmann weights such that the following

Yang-Baxter equation holds:

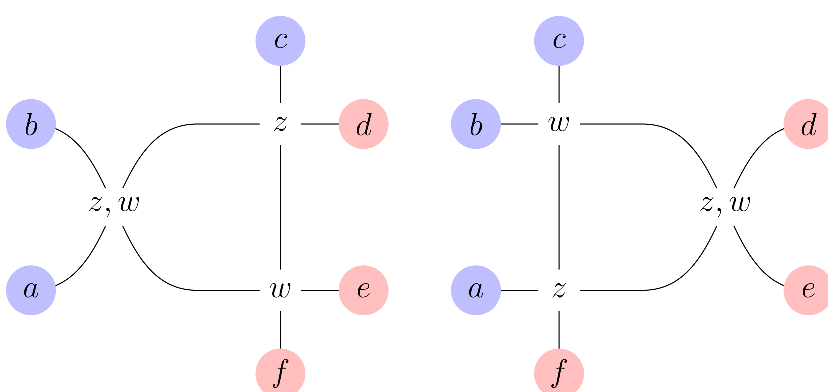


One way to interpret our Yang-Baxter equation as a parametrized one is to divide the Boltzmann weights by $z - qw$, and use these weights instead:

			
1	1	$\frac{(1-q)z}{z-qw}$ if $c<d$ $\frac{(1-q)w}{z-qw}$ if $c>d$	$\frac{z-w}{z-qw}$ if $c>d$ $\frac{(1-q)w}{z-qw}$ if $c<d$
			
$\frac{(1-t)z}{z-tw}$	$\frac{(1-t)w}{z-tw}$	$\frac{t(z-w)}{z-tw}$	$\frac{z-w}{z-tw}$

This change does not affect the validity of the Yang-Baxter equation since it divides both sides by the same constant $(z_1-qz_2)(z_1-qz_3)(z_2-qz_3)$. But with this change the Boltzmann weights only depend on z/w and we have indicated this in the notation by labeling the R-matrix with $z/w \in \mathbb{C}^\times$. We then recognize the Yang-Baxter equation as a parametrized Yang-Baxter equation with parameter group $\mathbb{C}^\times$.

In Lecture 12 we considered partition functions assuming we have a Yang-Baxter equation as follows:



We will refer to this as the RTT equation, which can be written symbolically as RTT=TTR. The letter T refers to the vertex types labeled z and w. We did not specify the Boltzmann weights at the “T” vertices except to remark that there are multiple possibilities.

And if we form the partition function of a system $Z(\mathbf{z};\mathbf{d})$ with boundary conditions as in the open models, then these satisfy a recursion

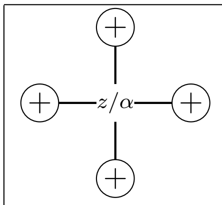
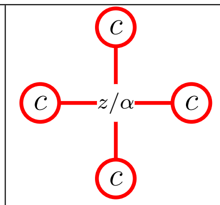
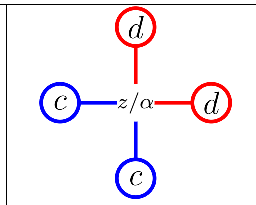
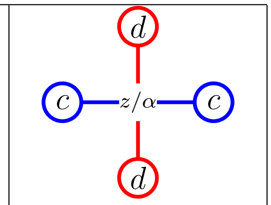
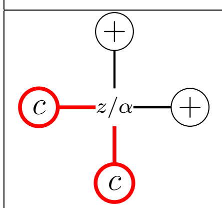
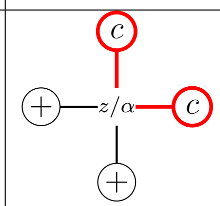
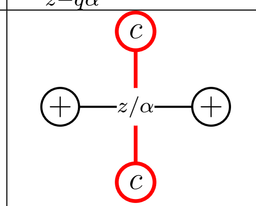
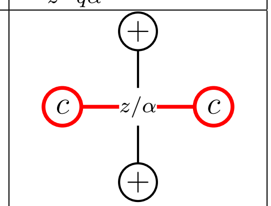
$$Z(\mathbf{z}; s_i\mathbf{d}) = \mathcal{L}_i Z(\mathbf{z};\mathbf{d})$$

where $\mathcal{L}_i$ is the Demazure-Lusztig operator, assuming $d_i > d_{i+1}$.

Let us investigate some choices for the T weights.

Example 1.1. First, we can just use the same vertex types as with the R-matrix, but rotated by 45° (clockwise).

To explain this, we rotate the R-matrix and replace the parameter w by a new parameter α which can depend on the column, and obtain these weights:

			
1	1	$\frac{(1-q)z}{z-q\alpha}$ if $c<d$ $\frac{(1-q)\alpha}{z-q\alpha}$ if $c>d$	$\frac{z-\alpha}{z-q\alpha}$ if $c>d$ $\frac{(1-q)\alpha}{z-q\alpha}$ if $c<d$
			
$\frac{(1-q)z}{z-q\alpha}$	$\frac{(1-q)\alpha}{z-q\alpha}$	$\frac{q(z-\alpha)}{z-q\alpha}$	$\frac{z-\alpha}{z-q\alpha}$

Here α can be arbitrary but in the partition function α must be constant in the column. Note that the RRR parametrized Yang-Baxter equation is equivalent to the RTT equation.

Example 1.2. Another possibility, and an interesting one, is the bosonic models used in [**26**], which are special cases of more general ones in [**10**]. In these models, every vertical edge can carry an arbitrary number of bosons for every color. Thus if $c_1, \cdots, c_n$ are the colors, the spinset of the vertical edges is $\mathbb{N}^n$ where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and if $\mu \in \mathbb{N}^n$ we may write $\mathbf{c}_0^\mu$ for the spin with μ_i bosons of color c_i, where $\mathbf{c}_0 = (c_1, \cdots, c_n)$ is the standard flag. We will not describe the Boltzmann weights here, but see [**26**] for details. The partition functions are nonsymmetric Hall-Littlewood polynomials, and in [**10**] there are similar bosonic models whose partition functions are more general nonsymmetric Macdonald polynomials.

Our point is that there are multiple choices for the edges in the models for a very good reason. In the paradigm we are considering, every edge of the model corresponds to an object in a braided category. In this case, we will see (later) that this category is the category of $U_{\sqrt{q}}(\widehat{\mathfrak{sl}}_{n+1})$-modules. And if U, V are any two objects of this category, then there is a braiding $c_{U,V} : U \otimes V \longrightarrow V \otimes U$, and these all satisfy the Yang-Baxter equation (Lecture 4).

2. Back to quantum groups

The theory of quantum groups gives an explanation of where the Yang-Baxter equation comes from, and what instances we may expect. Our goal is to give a taste of this.

Please review Lecture 12. We saw that a vector space H over a field F (for us usually $\mathbb{C}$) equipped with map $\mu : H \otimes H \longrightarrow H$ and $\varepsilon : F \longrightarrow H$ satisfying the associativity and unit axioms is the same as an associative algebra, with multiplication $x \cdot y = \mu(x \otimes y)$ and identity element $\varepsilon(1_F)$. Similarly, a vector space H equipped with a linear map $\Delta : H \longrightarrow H \otimes H$ (called comultiplication) and $\eta : H \longrightarrow F$ satisfying the coassociativity and counit axioms is called a coalgebra. A Hopf algebra is thus both an algebra and a coalgebra.

If A and B are algebras, so is $A \otimes B$ and the Hopf axiom can be interpreted as saying that $\Delta : H \longrightarrow H \otimes H$ is an algebra homomorphism. So is the counit $\eta : H \longrightarrow F$. It is equivalent to say that $\mu : H \otimes H \longrightarrow H$ is a homomorphism of coalgebras.

Proposition 2.1. *Let H be a Hopf algebra. Then the category of H-modules is monoidal.*

Proof. For an associative algebra A, if V and W are A-modules, then $V \otimes W$ is not naturally an A-module. It is, however, very naturally an $A \otimes A$-module.

Now let V and W be H-modules. We need to put an H-module structure on $V \otimes W$. For this, we use the comultiplication, which is an algebra homomorphism $H \longrightarrow H \otimes H$. ☐

There are two important and related types of Hopf algberas that have deformations into "quantum groups." Let G be a reductive algebraic group over $\mathbb{C}$ such as $\mathrm{GL}(n)$. Let $\mathcal{O}(G)$ be the ring of polynomial functions on G. This algebra is of course commutative. The multiplication map $G \times G \longrightarrow G$ is a morphism hence induces an algebra homorphism $\mathcal{O}(G) \longrightarrow \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. This is the comultiplication, making $\mathcal{O}(G)$ into a Hopf algebra. A deformation of this will be called a *deformed function algebra*.

On the other hand, let us recall the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. This is an associative algebra $U(\mathfrak{g})$ that contains a copy of $\mathfrak{g}$ as a vector subspace, such that if $X, Y \in \mathfrak{g}$ then

$$[X, Y] = X \cdot Y - Y \cdot X. \qquad (\,\cdot = \text{multiplication in } U(\mathfrak{g})\,)$$

It has the universal property that if $f : \mathfrak{g} \longrightarrow A$ of $\mathfrak{g}$ into an associative algebra A such that

$$f([X, Y]) = f(X)f(Y) - f(Y)f(X),$$

then f extends uniquely to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow A$. Then $U(\mathfrak{g})$ is a cocommutative Hopf algebra whose comultiplication satisfies

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad (X \in \mathfrak{g}).$$

What Drinfeld and Jimbo showed ([**35, 50**]) was that it is possible to deform the enveloping algebra $U(\mathfrak{g})$, after expanding it slightly to include some group-like elements. The deformation $U_q(\mathfrak{g})$, with q a complex parameter, is called a *quantized enveloping algebra*.

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field F with a bilinear "bracket" operation $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, for which we use the notation $[X, Y]$, that satisfies

$$[Y, X] = -[X, Y], \qquad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

The second relation is called the *Jacobi relation*. The Lie algebra $\mathfrak{gl}_n$ is $\mathrm{Mat}_n(\mathbb{C})$ with the bracket operation

$$[X, Y] = XY - YX. \tag{30}$$

It can be easily checked that this is a Lie algebra. Alternatively, if V is a vector space, $\mathfrak{gl}(V)$ is the endomorphism ring of V with bracket operation (30). The Lie algebra $\mathfrak{sl}_n$ is the vector subspace $\mathfrak{gl}_n$ consisting of matrices of trace zero.

Definition 3. A *representation* of the Lie algebra $\mathfrak{g}$ is a homomorphism $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Thus it is a linear map to $\mathrm{End}(V)$ that satsifies

$$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

Example 2.2. If $\pi : \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(V)$ is a representation, then we obtain a representation $d\pi : \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ by differentiating. Thus

$$d\pi(X)v = \frac{d}{dt} e^{tX} v|_{t=0}.$$

It can be checked that this is a representation ([**24**], Proposition 7.2).

The *universal enveloping algebra* $U(\mathfrak{g})$ is the algebra generated by $\mathfrak{g}$ subject to relations

$$X \cdot Y - Y \cdot X = [X,Y]. \tag{31}$$

This resembles (30) but note that in (30) the multiplication is matrix multiplication and in (31) the multiplication is the multiplication in $U(\mathfrak{g})$. Now if $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a representation, then since by the definition of a representation the relations (31) are satisfied by $\pi(X)$, $\pi(Y)$ and $\pi([X,Y])$, the linear map π extends to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$.

To summarize:

- Representations of a Lie group G become representations of its Lie algebra $\mathfrak{g}$, by differentiation. A representation of $\mathfrak{g}$ that comes from a representation of G is called *integrable*.
- Representations of a Lie algebra $\mathfrak{g}$ extend to representations of the associative algebra $U(\mathfrak{g})$.

So the enveloping algebra captures the representations of a Lie group or Lie algebra. We caution that the Lie algebra of a Lie group has representations that are not integrable, such as Verma modules, so its representation theory is slightly richer than G. Quantum versions of these "non-integrable" representations can still figure in the Yang-Baxter equation. For example, Verma modules of $U_{\sqrt{q}}(\mathfrak{sl}_{n+1})$ underlie Example 1.2.

PROPOSITION 2.3. *The enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with comultiplication satisfying*

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad (X \in \mathfrak{g}).$$

PROOF. We take $\Delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$ to be defined by (2.3) when $X \in \mathfrak{g}$. We must show that this definition extends to $U(\mathfrak{g})$. First let us note that if $X, Y \in \mathfrak{g}$ then

$$\Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) = XY \otimes 1 - YX \otimes 1 + 1 \otimes XY - 1 \otimes YX.$$

Indeed, expanding the left-hand side gives eight terms but four cancel in pairs. In $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ we therefore have

$$\Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) = \Delta([X,Y]).$$

The elements $\Delta(X)$ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ thus satisfy the generating relations of $U(\mathfrak{g})$, which was defined by generators $X \in \mathfrak{g}$ and relations (31). It follows that they extend to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$. As for the counit $\eta : U(\mathfrak{g}) \longrightarrow F$, this is obtained by extending the zero map $\mathfrak{g} \longrightarrow F$ to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow F$.

The antipode is an antimultiplicative map $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ that satisfies $S(X) = -X$ for $X \in \mathfrak{g}$. To see that this map exists, if $U(\mathfrak{g})^{\mathrm{opp}}$ is the opposite ring then the generators $-X$ satisfy the defining relations for $U(\mathfrak{g})$, so there is a homomorphism $S : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\mathrm{opp}}$ that sends X to $-X$, and this is the antipode.

We leave checking the axioms to the reader. □

3. $U_q(\mathfrak{sl}_2)$

The very simplest and most important case is $\mathfrak{g} = \mathfrak{sl}_2$. It has a basis consisting of:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$[H, E] = 2E, \quad [H, F] = -2F, \qquad [E, F] = H.$$

Thus the enveloping algebra $U(\mathfrak{sl}_2)$ is a noncommutative polynomial ring with generators E, F, H modulo the ideal generated by the relations

$$HE - EH = 2E, \qquad HF - FH = -2F, \qquad EF - FE = H.$$

The comultiplication, we have already seen, is

$$\Delta X = X \otimes 1 + 1 \otimes X, \qquad X \in \mathfrak{g},$$

and the antipode satsifies $S(X) = -X$ for $X \in \mathfrak{g}$.

Now let us explain how to deform $U(\mathfrak{g})$. Let $q \in \mathbb{C}$. We will first define $U_q(\mathfrak{g})$ as an associative algebra, then prove it has a comultiplication. In place of H we make use of a “grouplike” element K which we can think of as the matrix

$$\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}.$$

We can express $H = (q - q^{-1})^{-1}(K - K^{-1})$ and so we do not need H among the generators. The algebra is then generated by E, F and K with relations

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F, \qquad EF - FE = (q - q^{-1})^{-1}(K - K^{-1}).$$

We should also take K^{-1} among the generators of $U_q(\mathfrak{sl}_2)$ with obvious relations.

Proposition 3.1. *The ring $U_q(\mathfrak{g})$ admits a comultiplication $\Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ such that*

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

There is also an antipode S that satisfies

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1},$$

and a counit satisfying $\eta(F) = \eta(E) = 0$, so $U_q(\mathfrak{g})$ is a Hopf algebra.

Proof. The proof consists of showing that the elements $K \otimes K$, $E \otimes K + 1 \otimes E$ and $F \otimes 1 + K^{-1} \otimes F$ satisfy the same relations as K, E and F. We will omit this verification, or the verification of the antipode. □

4. R-matrices

Drinfeld [**35**] defined the notion of a *quasitriangular Hopf algebra*. This is a Hopf algebra H with an invertible element $R \in H \otimes H$ satisfying certain axioms. The first axiom is that for $h \in H$ we have

$$\tau(\Delta h) = R(\Delta h)R^{-1},$$

where $\tau : H \otimes H \longrightarrow H \otimes H$ is the flip map $\tau(x \otimes y) = y \otimes x$. It is not hard to check that this implies that if U, V are H-modules, then the map

$$u \otimes v \longmapsto \tau(R(u \otimes v))$$

is an H-module homorphism $U \otimes V \longrightarrow V \otimes U$. Then there are two more axioms that guarantee that this map is a braiding. See [**77**] Chapter 5 for further details. The element R of $H \otimes H$ is called the *universal R-matrix*.

THEOREM 4.1. *Assume that q is not a root of unity. The category of finite-dimensional modules a quantized enveloping algebra such as $U_q(\mathfrak{sl}_2)$ is braided.*

PROOF. Unfortunately $H = U_q(\mathfrak{g})$ is *not* a quasitriangular Hopf algebra. There *is* a universal R-matrix, but it is given by an infinite series and so it does not live in $H \otimes H$ but rather in a completion. There are various ways of avoiding this difficulty. One way is to work with a quantized function algebra that is in duality with H, and show that this Hopf algebra is dual quasitriangular. □

So even though $U_q(\mathfrak{sl}_2)$ is not quasitriangular, it is almost as good. But rather than try to work with the universal R-matrix, it is usually possible to work directly with equations to find the braiding. So let us see how that works in this particular case.

Let $V = \mathbb{C}^2$ be the two-dimensional standard module, with basis $\{x, y\}$ such that E, F and K are represented by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}.$$

We will begin by determining the endomorphisms of $V \otimes V$. The tensor product module is not irreducible, but splits into two irreducible submodules, of dimensions 1 and 3. So the endomorphism ring will turn out to be two dimensional.

We recall that the action of H on $V \otimes V$ is via the comultiplication. In particular $\Delta K = K \otimes K$, so

$$K \cdot (x \otimes y) = Kx \otimes Ky.$$

Hence the eigenspaces of K corresponding to the eigenvalues q^2, 1 and q^{-2} have bases $\{x \otimes x\}$, $\{x \otimes y, y \otimes x\}$ and $\{y \otimes y\}$. These must be invariant by any endomorphism ϕ of $V \otimes V$, so with respect to the basis $x \otimes x, x \otimes y, y \otimes x, y \otimes y$, the matrix of ϕ has the form

$$\begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix}.$$

Assuming that ϕ is invertible, we may scale it so that

$$\phi(x \otimes x) = x \otimes x,$$

$$\phi(x \otimes y) = ax \otimes y + cy \otimes x, \tag{32}$$

$$\phi(y \otimes x) = bx \otimes y + dy \otimes x, \tag{33}$$

$$\phi(y \otimes y) = \lambda y \otimes y$$

for some nonzero constant y.

LEMMA 4.2. *We have*

$$a + qc = 1, \qquad b + dq = q, \tag{34}$$

$$q^{-1}a + b = q^{-1}, \qquad q^{-1}c + d = 1. \tag{35}$$

Moreover $b = c$, $\lambda = 1$.

PROOF. From $\Delta E = E \otimes K + 1 \otimes E$ we have $E(x \otimes y) = Ex \otimes Ky + x \otimes Ey = x \otimes x$ and similarly $E(y \otimes x) = qx \otimes x$. Then

$$x \otimes x = \phi(x \otimes x) = \phi(E(x \otimes y)) = E\phi(x \otimes y) = aE(x \otimes y) + cE(y \otimes x) = (a + cq)x \otimes x,$$

proving that $a + cq = 1$. Similarly

$$qx \otimes x = \phi(E(y \otimes x)) = E\phi(y \otimes x) = bE(x \otimes y) + dE(y \otimes x) = (b + dq)x \otimes x,$$

proving that $b + dq = q$. We have proved

Starting with $\phi(x \otimes x) = x \otimes x$ and noting that $F(x \otimes x) = q^{-1}x \otimes y + y \otimes x$ we get

$$q^{-1}x \otimes y + y \otimes x = F(x \otimes x) = F\phi(x \otimes x) = \phi F(x \otimes x) = \phi(q^{-1}x \otimes y + y \otimes x).$$

Expanding this using (32) and (33), then comparing coefficients gives the identities (35). Comparing (34) and (35) gives $b = c$.

Proceeding similarly but starting with $y \otimes y$ instead of $x \otimes x$ gives the same identities (34) and (35) but contingent on $\lambda = 1$. $\square$

THEOREM 4.3. *There are two $U_q(\mathfrak{sl}_2)$-module endomorphisms R and R' of $V \otimes V$ that satisfy the Yang-Baxter equation in the form*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{36}$$

They are the endomorphisms with matrices

$$R = \begin{pmatrix} 1 & & & \\ & 1-q^2 & q & \\ & q & 0 & \\ & & & 1 \end{pmatrix}, \qquad R' = \begin{pmatrix} 1 & & & \\ & 0 & q^{-1} & \\ & q^{-1} & 1-q^{-2} & \\ & & & 1 \end{pmatrix}.$$

Remark 7. The notation is as follows: if $R \in \mathrm{End}(V \otimes V)$ then $R_{i,j} \in \mathrm{End}(V \otimes V \otimes V)$ is R applied to the i- and j-th components of $V \otimes V \otimes V$. The Yang-Baxter equation is often written

$$R_{12}R_{13}R_{22} = R_{23}R_{13}R_{12}. \tag{37}$$

The relationship between the two versions is that if R satisfied (37), then τR satisfies (36), where as usual $\tau(x \otimes y) = y \otimes x$.

PROOF OF THEOREM 4.3. We have seen in the Lemma that every invertible H-module homomorphism $V \otimes V \longrightarrow V \otimes V$ is a scalar multiple of one of the form

$$\begin{pmatrix} 1 & & & \\ & a & b & \\ & b & d & \\ & & & 1 \end{pmatrix}$$

with $a + qb = 1$ and $b + dq = q$. Such a matrix is a linear combination of two standard ones. With $d = 0$, we have $b = q$ and hence $a = 1 - q^2$. On the other hand, with $a = 0$, we have $b = q^{-1}$ and so $d = 1 - q^{-2}$. These give R and R' as a basis of the two-dimensional vector space $\mathrm{End}_H(V \otimes V)$.

Now, for the Yang-Baxter equation, we can take a linear combination $tR + uR'$ and check whether it satisfies the Yang-Baxter equation. This can be checked using a computer. We find three solutions, but one is the scalar matrix $qR' - q^{-1}R = (q - q^{-1})I_{V\otimes V}$. The other solutions of the Yang-Baxter equation are just R and R' (or constant multiples). □

5. Parametrized Yang-Baxter equations

THEOREM 5.1. *Let $q \in \mathbb{C}^\times$ be fixed. Let R and R' be as in Theorem 4.3. For $z \in \mathbb{C}^\times$ let*

$$R(z) = R - zq^2 R'.$$

Then we have a parametrized Yang-Baxter equation

$$R(z)_{12} R(zw)_{23} R(w)_{12} = R(w)_{23} R(zw)_{12} R(z)_{23}.$$

PROOF. This can be checked by hand, or by computer (see `sl2param.sage`, posted on the class web page). □

This parametrized Yang-Baxter equation is equivalent to the one in Lecture 4. The colored equation from Lecture 12 (and the beginning of this lecture) is a generalization due to Jimbo [**51**]. (To compare them replace $q \to \sqrt{q}$ in Theorem 5.1.)

Jimbo[**51**] also gave generalizations to the other classical Cartan types. These Yang-Baxter equations come from the quantized enveloping algebras of affine Lie algebras, which we will consider briefly in future lectures. The Lie algebra $\widehat{\mathfrak{sl}}_2$ or $U_q(\widehat{\mathfrak{sl}}_2)$, at least when q is not a root of unity, has one two-dimensional irreducible representation V_z for each $z \in \mathbb{C}^\times$. In one way imitating the proof of Theorem 4.3 is actually simpler in the affine case, for if z and w are in general position, the representation $V_z \otimes V_w$ is irreducible, so the R-matrix $V_z \otimes V_w \longrightarrow V_w \otimes V_z$ is determined up to scalar multiple. See [**43**] Proposition 9.2.4.

LECTURE 14

Affine Weyl Groups, Lie Algebras and Hecke Algebras

1. A survey of some possibilities

The spinsets of lattice models that we care about usually correspond to modules of various quantum groups. It is not necessary to use the quantum group theory such as the universal R-matrix to compute the R-matrices, since this can be done by other methods. (Using a computer is sometimes useful.) However knowing that the edges of the grid correspond to modules of a quantum group seems an important point, and understanding this fact has predictive power.

Our goal is to survey some of the various spinsets that we encounter and explain how these are related to various modules of particular quantum groups. Here is an overview of what we want to cover. We will cover none of these topics in any depth.

- We have encountered several examples of parametrized Yang-Baxter equations with parameter $\mathbb{C}^\times$. Although we were able to obtain such a parametrized Yang-Baxter equation from $U_q(\mathfrak{sl}_2)$ in Lecture 13, it is better to regard this as the R-matrix for the *affine* quantum group $U_q(\widehat{\mathfrak{sl}}_2)$.
- Affine Lie algebras come with Weyl groups, which are Coxeter groups that are also associated with Hecke algebras. We have seen that the Hecke algebra of $\mathfrak{gl}_n$ acts on partition functions of colored models, and these actions can be extended to the affine Hecke algebra.
- We have briefly discussed bosonic models, in which the spinsets are infinite. These often correspond to *Verma modules*, which are usually infinite-dimensional.
- In addition to quantized enveloping algebras of Lie algebras, we encounter enveloping algebras of Lie superalgebras such as $\mathfrak{gl}(m|n)$. So we want to touch on this topic.
- Lie superalgebras have a special kind of Verma module called *Kac modules* that are finite-dimensional. We believe these to be important for this topic. For example, $U_q(\mathfrak{gl}(m|n))$ has Kac modules that have dimension 2^{mn}. In particular the Kac modules for $U_q(\mathfrak{gl}(1|1))$ are 2-dimensional modules that differ from the 2-dimensional standard modules. These account for the vertical edges in the Tokuyama models.

In this and the next lectures we will briefly introduce each of these topics.

2. Affine Lie algebras

We have seen that quantum groups are sources of solutions to the Yang-Baxter equation. From the quantum group $U_q(\mathfrak{sl}_2)$ we obtained two R-matrices R' and R'' and obtained a parametrized Yang-Baxter equation by taking a linear combination of these. This is an *ad hoc* procedure that works for $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, but which would require modification for other Cartan types.

An alternative, better approach is to work with the (untwisted) *affine* Lie algebra $\hat{\mathfrak{g}}$, for any complex reductive Lie algebra $\mathfrak{g}$. Since appear in a great deal of mathematics, it is worth digressing to introduce them. They are special cases of *Kac-Moody* Lie algebras, for which the standard work is [**56**].

Affine Lie algebras and more general Kac-Moody Lie algebras were only discovered as recently as the 1970's. Now however they are everywhere. For us, they underlie the most important parametrized Yang-Baxter equations we have seen, and so we will spend a lecture on them.

Kac-Moody Lie algebras have much in common with simple complex Lie algebras. They have a Weyl group, a weight lattice, and for the integrable highest weight representations, an analog of the Weyl character formula. The characters of affine Lie algebras turn out to be modular forms.

Like the finite-dimensional simple Lie algebras, affine Lie algebras are a very special case of the more general Kac-Moody Lie algebras, and it is worth while treating them separately. Every finite dimensional simple Lie algebra $\mathfrak{g}$ gives rise to an (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$.

This has two different descriptions. First, it can be described by generators and relations. But an alternative description in Chapter 6 of [**56**] shows the relationship between $\mathfrak{g}$ and $\hat{\mathfrak{g}}$.

If $\mathfrak{g}$ is a Lie algebra and A is an associative algebra then $A \otimes \mathfrak{g}$ is naturally a Lie algebra with bracket

$$[a \otimes X, b \otimes Y] = ab \otimes [X, Y].$$

So we may construct the Lie algebra

$$\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g},$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring. If $\mathfrak{g}$ is simple this has a central extension by a one-dimensional abelian Lie algebra spanned by K. Thus we have a Lie algebra $\hat{\mathfrak{g}}'$ with a short-exact sequence

$$0 \longrightarrow \mathbb{C} \cdot K \longrightarrow \hat{\mathfrak{g}}' \longrightarrow \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \longrightarrow 0.$$

It is possible to enlarge this one more time by adjoining a derivation d of $\hat{\mathfrak{g}}'$, so that

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C}d.$$

The main subtlety is in constructing a cocycle that produces the central extension $\hat{\mathfrak{g}}'$. The difference between $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{g}}$ is important, but we can ignore it for our purposes.

If V is any irreducible $\hat{\mathfrak{g}}$-module, then by Schur's Lemma the central element K acts by a scalar. The algebra $\hat{\mathfrak{g}}$ has an important family of infinite-dimensional representations, the integrable highest weight representations, in which K acts by a nonzero scalar. As far as I know these have not been used in lattice models but maybe they should be. One particular integrable highest weight representation, called the *basic representation* is of particular importance, showing up in diverse places such as string theory and the modular representations of the symmetric group.

If V is any $\mathfrak{g}$-module, and if $z \in \mathbb{C}^\times$, then V becomes a $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ module in which t acts by the scalar z. We can then pull this back to $\hat{\mathfrak{g}}'$ and obtain a family of modules V_z in which K acts by zero. At least when $\mathfrak{g}$ is a classical group, the R-matrices for these were computed by Jimbo [**51**]. For $\widehat{\mathfrak{sl}}_2$, this gives the parametrized R-matrices that were computed in Lecture 13.

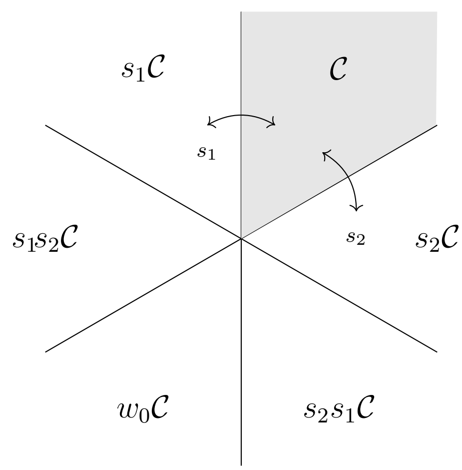
3. Affine Weyl group

The affine Lie algebra $\hat{\mathfrak{g}}$ has a Weyl group W_{aff} that is a Coxeter group. As before, let $\mathfrak{g}$ be a complex simple Lie algebra, with Weyl group W, root lattice Λ and root system Φ. The weight lattice can be embedded in a Euclidean space, that is, a real vector space V with a positive definite inner product that is W-invariant. The lattice Λ_{root} is of finite index in Λ. The weight lattice Λ can be characterized as

$$\left\{\lambda \in V | \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \text{ for } \alpha \in \Phi\right\}.$$

Let α_i be the simple positive roots, and $s_i \in W$ the corresponding simple reflections. If $\alpha \in \Phi$, let H_α be the hyperplane through the origin orthogonal to α. The set $V - \bigcup H_\alpha$ is disconnected, and the connected components are called *Weyl chambers*. One particular one is $\mathcal{C}_+^\circ = \{x \in V | \langle x,\alpha\rangle > 0 \text{ for } \alpha \in \Phi^+\} = \{x \in V | \langle x,\alpha\rangle > 0 \text{ for } \alpha \in \Phi^+\}$. Let $\mathcal{C}_+$ be the closure of $\mathcal{C}_+^\circ$. It is a fundamental domain for the action of W on V, in that every orbit of W intersects $\mathcal{C}_+$ in a unique point.

Then W can be defined as the group generated by the reflections in the hyperplanes H_α ($\alpha \in \Phi$). The simple reflection s_i is the reflection in H_{α_i}. The group W is actually generated by the subset $\{s_1, \cdots, s_r\}$ generated by the simple reflections. The hyperplanes H_{α_i} are just the walls of $\mathcal{C}_+$. For $\mathfrak{g} = \mathfrak{sl}_3$, here is a picture of the six Weyl chambers, with $\mathcal{C}_+$ shaded.



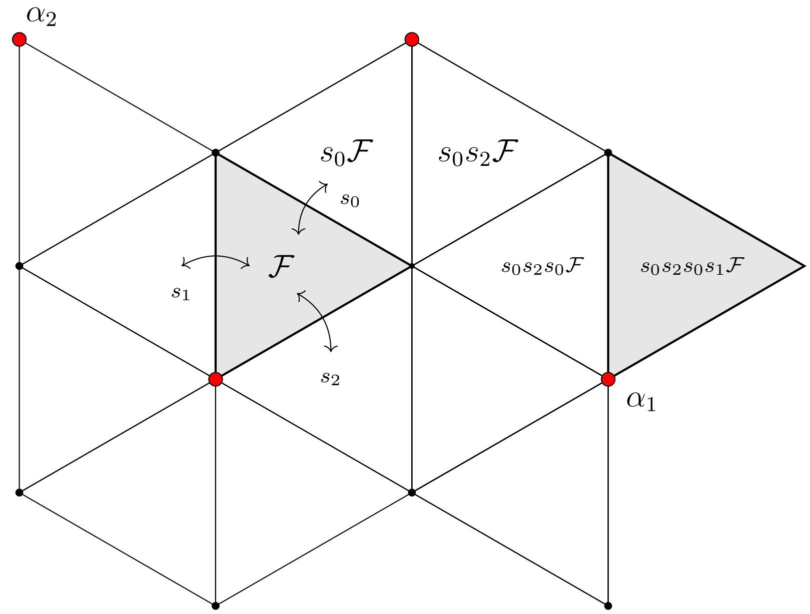
We now turn to W_{aff}.

If $k \in \mathbb{Z}$ let $H_{\alpha,k} = \{x \in V | \langle x,\alpha\rangle = k\}$. Again we may consider the complement of $\bigcup H_{\alpha,k}$. The closure of one connected component of this complement is called an *alcove*. In particular

$$\mathcal{F} = \{x \in V | \langle x,\alpha_i\rangle \geqslant 0, \langle x,\alpha_\ell\rangle \leqslant 1\},$$

where α_ℓ is the highest root is called the *fundamental alcove*. The *affine reflection* s_0 is the reflection in the hyperplane $H_{\alpha_\ell,1}$.

For $\mathfrak{g} = \mathfrak{sl}_3$, $\alpha_\ell = \alpha_1 + \alpha_2$. Here is a picture showing some of the alcoves.



This figure also shows weight lattice (small black dots at the corners of the alcove) and some elements of the root lattice (larger red dots).

The group W_{aff} can be defined as the group generated by the reflections in the hyperplanes $H_{\alpha,k}$. But actually it is generated by $\langle s_0, s_1, \cdots, s_r\rangle$, and it is a Coxeter group with these generators.

The group W_{aff} contains the subgroup Λ_{root} of translations by elements of the root lattice. In the above picture, it is shown that $s_0s_2s_0s_1$ takes the fundamental alcove $\mathcal{F}$ into $\mathcal{F} + \alpha_1$. Indeed there is an isomorphism Θ of Λ_{root} into W, and W_{aff} is the semidirect product of W with the normal subgroup $\Theta(\Lambda_{\text{root}})$.

The group $W_{\text{aff}} \cong W \ltimes \Theta(\Lambda_{\text{root}})$ can be expanded by adding the group of translations by Λ. This expanded group is called the *extended affine Weyl group*.

The Hecke algebra $\mathcal{H}(W_{\text{aff}})$ of W_{aff} has generators $T_0, T_1, \cdots, T_r$ subject to the quadratic relations

$$T_i^2 = (q-1)T_i + q \tag{38}$$

and the braid relations. It has an alternative presentation, due to Bernstein, that is generated by $T_1, \cdots, T_r$ and an abelian subalgebra isomorphic to Λ_{root}.

To describe the Bernstein presentation, we make use of a complex torus T such that the group of rational characters of T is identified with the weight lattice Λ. If $\mathbf{z} \in T$, let $\mathbf{z}^\lambda$ be the character λ evaluated at $\mathbf{z}$. Let $\mathcal{H}(W) = \langle T_1, \cdots, T_r\rangle$ be the finite Hecke algebra, with generators omitting T_0, subject to the quadratic relations and braid relations (which may be read off from the Dynkin diagram). We omit T_0 in this presentation. Now we consider the algebra $\mathcal{H}(W) \otimes \mathbf{z}^\Lambda$, where the generators of $\mathcal{H}(W)$ commute with the weight lattice by the *Bernstein relation*

$$\mathbf{z}^\lambda T_i - T_i \mathbf{z}^{s_i\lambda} = \frac{q-1}{1-\mathbf{z}^{-\alpha_i}}(\mathbf{z}^\lambda - \mathbf{z}^{s_i\lambda}).$$

Just as the affine Weyl group is smaller than the extended affine Weyl group, the algebra $\mathcal{H}(W) \otimes \mathbf{z}^\Lambda$ is alsoslightly bigger than the Coxeter group $\langle T_0, \cdots, T_r\rangle$. It is called the *extended affine Hecke algebra*. To recover the Coxeter group, we restrict the elements $\mathbf{z}^\lambda$ to the root lattice.

THEOREM 3.1 (Bernstein, Zelevinsky, Lusztig). *The subalgebra $\mathcal{H}(W)\otimes\mathbf{z}^{\Lambda_{\text{root}}}$ is isomorphic to the Coxeter group $\mathcal{H}(W_{\text{aff}})$.*

See [**74, 40, 44**] for more information.

In Lecture 12, Theorem 1.1 we saw that there is an action of $\mathcal{H}(W)$ on $\mathcal{O}(T)$ in which $T_1, \cdots, T_r$ act by Demazure-Lusztig operators. In the special case where $W = S_n$ is the Weyl group of $\mathrm{GL}(n)$ we applied this to study the partition functions of colored lattice models.

THEOREM 3.2 (Lusztig [**73**]). *This action extends to the affine Hecke algebra $\mathcal{H}(W)\otimes\mathbf{z}^\Lambda$.*

In this action we let $\mathbf{z}^\lambda$ act by its inverse $\mathbf{z}^{-\lambda}$. To prove this, one must check the Bernstein relation.

4. The Poincaré-Birkhoff-Witt theorem

As we saw in the last lecture, many examples of the Yang-Baxter equation come from quantum groups. It is also possible to work backwards from the Yang-Baxter equation and produce a quantum group ([**85**] or [**59**] Section 8.6). If we understand the term "quantum group" to mean a quasitriangular Hopf algebra, many instances turn out to be quantized enveloping algebras. Recall the $H = U_q(\mathfrak{g})$ is actually not quasitriangular (though if q is a root of unity it has a quasitriangular quotient), but it is "morally" quasitriangular, meaning that there is a universal R-matrix, but it is not in $H \otimes H$ but in a completion which might be denoted $H\hat{\otimes}H$. There are various ways of handling this difficulty.

The notion of Hopf algebra is self-dual, but quasitriangularity is not, so there are also dual quasitriangular Hopf algebras ([**77, 59**]). Quantized function algebras are dual quasitriangular. For the purpose of investigating the Yang-Baxter equation, whether to work with quasitriangular or dual quasitriangular Hopf algebra is a matter of taste.

In preparation for discussing Verma modules, we will introduce here a tool, the *Poincaré-Birkhoff-Witt* (PBW) theorem.

We work with $U(\mathfrak{g})$, not $U_q(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. We assume that $\mathfrak{g}$ is finite-dimensional, though this hypothesis is easily lifted. Let $X_1, \cdots, X_d$ be a basis of $\mathfrak{g}$. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$.

THEOREM 4.1 (PBW). *A basis of $U(\mathfrak{g})$ as a vector space consists of the elements $X_1^{k_1} \cdots X_d^{k_d}$ as $\mathbf{k} = (k_1, \cdots, k_d)$ runs through $\mathbb{N}^d$.*

PROOF. See [**45**] Section 17.3. □

As an application, let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, which we recall is $\mathrm{Mat}_n(\mathbb{C})$ with the bracket operation $[X, Y] = XY - YX$ (matrix multiplication). This Lie algebra has 3 subalgebras, the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, and the subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ nilpotent upper triangular and lower triangular matrices, respectively.

PROPOSITION 4.2. *The multiplication map $U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \longrightarrow U(\mathfrak{g})$ is a vector space isomorphism.*

PROOF. This follows from the PBW theorem by choosing the basis $X_1, \cdots, X_n$ so that the first $\frac{1}{2}n(n-1)$ elements are in $\mathfrak{n}_-$, the next n are in $\mathfrak{h}$ and the last $\frac{1}{2}n(n-1)$ are in $\mathfrak{n}_+$. Then every basis element of $\mathfrak{g}$ is uniquely the product of basis elements of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$, from which the statement is clear. □

In Lecture 15 we will use this to describe certain infinite-dimensional representations of $\mathfrak{g}$ called *Verma modules*. The natural habitat for these is the Bernstein-Gelfand-Gelfand *Category* $\mathcal{O}$ ([**7, 47**], [**56**] Chapter 9). They are not integrable, meaning that they do not lift to representations of $\mathrm{GL}(n, \mathbb{C})$. However Verma modules are still important for us because they do have analogs for quantum groups, and these have applications to lattice models. See [**83**].

LECTURE 15

Verma Modules and Bosonic Models

1. Bosonic Models

Let us return to the bosonic models in Lecture 8. The R-matrix tells us that the quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$ or $U_q(\widehat{\mathfrak{gl}}_2)$. The horizontal edges in the model correspond to 2-dimensional modules $V_{\mathbf{z}}$ where $\mathbf{z} \in \mathbb{C}^\times$.

The vertical edges, however, have spinset $\mathbb{N}$. From the point of view of Kulish [**65**], where the bosonic models first appeared, these spins correspond to the energy levels of the quantum mechanical harmonic oscillator, or rather, a q-deformation of that. But from the point of view we are taking, these edges should correspond to a module of $U_q(\widehat{\mathfrak{sl}}_2)$. This quantum group module is a Verma module.

We will not discuss Verma modules for quantized enveloping algebras, but at least we will look at Verma modules for $U(\mathfrak{g})$. The theory is standard. The books [**47**] and [**56**] Chapter 9 are good references.

To get R-matrices out of Verma modules, one must extend this theory to $U_q(\mathfrak{g})$. For this, see [**83**]. A paper where quantum Verma modules are used to compute R-matrices is [**9**].

2. Verma modules continued

We continue from Lecture 14, where we introduced the PBW theorem. We will review a few ideas about highest weight modules and the BGG Category $\mathcal{O}$. See [**56**] Chapter 9 for more information about these topics.

We make use of the tensor product for noncommutative rings. This is a topic omitted in Lang's *Algebra* but as a reference see Mac Lane's *Homology*, Section 5.1. If R is a noncommutative ring, and M is a *right* R-module and N is a *left* R-module, and if T is an abelian group, a map $\beta : M \times N \longrightarrow T$ is called *balanced* if it is $\mathbb{Z}$-bilinear and $\beta(ma, n) = \beta(m, an)$ for $a \in A$, $m \in M$ and $n \in N$. Then $M \otimes_A N$ is defined to be an abelian group with a balanced map $\otimes : M \times N \longrightarrow M \otimes_A N$ such that any balanced map $\beta : M \times N \longrightarrow T$ factors uniquely through $M \otimes_A N$. We naturally write $m \otimes n$ instead of $\otimes(m, n)$.

There is no natural way to make $M \otimes N$ to an A-module. However a common special case is where M is a bi-module. If B and A are rings, a (B, A)-*bimodule* is an M that is simultaneously a left B-module and a right A-module, such that these actions commute: $b(ma) = (bm)a$ for $m \in M$, $b \in B$ and $a \in A$. In this case, if N is a left A-module then $M \otimes N$ becomes a left B-module.

As an example, suppose that A is a ring and B a ring containing A. Then B is a left B-module and a right A-module, so it is a bimodule and

$$N \mapsto B \otimes_A N$$

is a functor from the category of left A-modules to left B-modules. This functor is called *extension of scalars*.

We now return to the setting at the end of Lecture 14. Let $\mathfrak{g}$ be a simple complex Lie algebra such as $\mathfrak{sl}_n$. We saw that it has a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The Cartan subalgebra $\mathfrak{h}$ is abelian, so any simple $\mathfrak{h}$-module is one-dimensional.

In the Lie algebra setting, weights are elements of $\mathfrak{h}^*$, which we equip with a W-invariant inner product. The root system Φ can then be characterized as the set of nonzero $\alpha \in \mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{h}\} \tag{39}$$

is nonzero. In this case $\mathfrak{g}_\alpha$ is one-dimensional. Let X_α be a generator. For the simple roots $\alpha_1, \cdots, \alpha_r$ we denote $X_{\alpha_i} = E_i$ and $X_{-\alpha_i} = F_i$.

If V is any module, and $\mu \in \mathfrak{h}^*$ let

$$V_\mu = \{v \in V | Hv = \mu(H)v \text{ for all } H \in \mathfrak{h}\} \tag{40}$$

be the corresponding weight space. We will always assume that V is the direct sum of its weight spaces.

The Lie algebra $\mathfrak{g}$ is itself a $\mathfrak{g}$-module with respect to the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$, where $\mathrm{ad}(X)$ is the endomorphism $\mathrm{ad}(X)Y = [X, Y]$. Then the roots are just the nonzero weights in the adjoint representation, and the definition (39) is seen to be a special case of the definition (40).

LEMMA 2.1. *We have $X_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$.*

PROOF. If $H \in \mathfrak{h}$ and $v \in V_\lambda$ then

$$\begin{aligned} HX_\alpha v = [H, X_\alpha]v + X_\alpha Hv &= \alpha(H)X_\alpha v + X_\alpha \lambda(H)v \\ &= (\alpha + \lambda)(H)X_\alpha v. \end{aligned}$$

□

We will call elements of $\mathfrak{h}$ such that $V_\mu \neq 0$ the *weights* of the representation. A weight λ is *integral* if

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

for all $\alpha \in \Phi$. The set of integral weights is the *weight lattice* Λ. If $\mathfrak{g}$ is the Lie algebra of a simply-connected complex Lie group G, this weight lattice can be identified with the weight lattice of G.

Definition 4. Let V be a module. A vector $v \in V$ is a *highest weight vector* with weight $\lambda \in \mathfrak{h}^*$ if $v \in V_\lambda$ and $\mathfrak{n}_+ v = 0$. If V is generated by v, then V is called a *highest weight module* for the weight λ.

For example, if V is a finite-dimensional irreducible representation, then by the Weyl theory V has a highest weight vector that is up to scalar multiple for a unique λ, which is a dominant integral weight.

LEMMA 2.2. *If V is a highest weight module for λ, then $V = U(\mathfrak{n}_-)v$.*

PROOF. By the PBW theorem we have

$$U(\mathfrak{g}) = U(\mathfrak{n}_-)U(\mathfrak{b}).$$

Although we do not need this fact, the PBW theorem actually implies that the multiplication map $U(\mathfrak{n}_-) \times U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$ induces a vector space isomorphism $U(\mathfrak{n}_-) \otimes U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$. Then $V = U(\mathfrak{n}_-)U(\mathfrak{b})v$ and we can discard the $U(\mathfrak{b})$ since clearly $U(\mathfrak{b})v = v$. □

THEOREM 2.3. *Let $\lambda \in \mathfrak{h}^*$. Then $\mathfrak{g}$ has a universal highest weight module $M(\lambda)$, with a highest weight vector m_λ, such that if V is any module and $v \in V$ is a highest weight vector with weight λ, then there is a unique homomorphism $M(\lambda) \longrightarrow V$ taking m_λ to v.*

PROOF. Let $\mathbb{C}_\lambda$ be the $\mathbb{C}$ equipped with the $\mathfrak{h}$-module structure affording the character λ. We can extend this character of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero. This gives us a $U(\mathfrak{b})$-module. Now let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

It is easy to see that $M(\lambda)$ is a highest weight module with $m_\lambda = 1_{U(\mathfrak{g})} \otimes 1_{\mathbb{C}_\lambda}$. To check the universal property, note that the map $\beta : U(\mathfrak{g}) \times \mathbb{C}_\lambda \longrightarrow V$ defined by $\beta(\xi \otimes a) = \xi\, a\, v$ is balanced, hence induces a unique map $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \longrightarrow V$. This is the unique homomorphism. $\square$

If $\lambda \in \mathfrak{h}^*$ let e^λ be a formal symbol such that $e^\lambda e^\mu = e^{\lambda+\mu}$. In this setting the "exponential" e^λ is just a formal device for writing the weight lattice multiplicatively. The *character* of a module V is

$$\chi_V = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, e^\mu.$$

PROPOSITION 2.4. *Let $\lambda \in \mathfrak{h}^*$. Then $\xi \mapsto \xi m_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$. The character of $M(\lambda)$ is*

$$e^\lambda \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.$$

It is understood that we expand the geometric series and collect the terms:

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1} = \prod_{\alpha \in \Phi^+} \sum_{k_\alpha = 0}^{\infty} e^{-\sum k_\alpha \alpha} = \sum_{\mu} \wp(\mu) e^{-\mu} \tag{41}$$

where $\wp(\mu)$ is the number of ways of writing $\mu = \sum_{\alpha \in \Phi^+} k_\alpha \alpha$ for some vector $(k_\alpha | \alpha \in \Phi^+)$ of nonnegative integers. The function $\wp$ is called the *Kostant partition function*.

- `https://en.wikipedia.org/wiki/Kostant_partition_function`

PROOF. This is a stronger statement than Lemma 2.2, which asserts that the map $\xi \mapsto \xi m_\lambda$ is surjective $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$. For this, standard isomorphisms give

$$M(\lambda) = U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda \cong U(\mathfrak{n}_-)$$

as a vector space.

We want to show that the character of $U(\mathfrak{n}_-)$ as an $\mathfrak{h}$-module is (41). By the PBW Theorem a basis of $U(\mathfrak{n}_-)$ consists of elements of the form

$$\prod_{\alpha \in \Phi^+} X_{-\alpha}^{k_\alpha} \in U(\mathfrak{n}_-)$$

and the weight of this is $-\sum k_\alpha \alpha$. $\square$

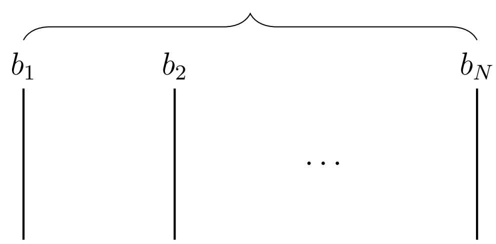
LECTURE 16

Fusion. Colored Bosonic Models

1. Fusion

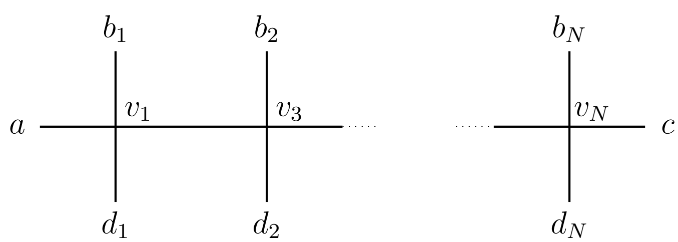
Roughly the fusion operation in lattice models corresponds to the tensor product of modules. But just as the tensor product has different applications in representation theory, so there are different kinds of thing that are called fusion.

Suppose that we have a sequence of vertices as follows. We assume that a sequence of vertical edges labeled $b_1, \cdots, b_N$ have spinsets $\Sigma_1, \cdots, \Sigma_N$.

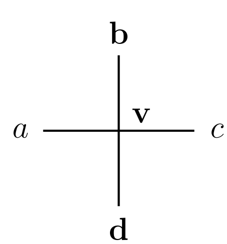


We may associate with these a single vertical edge with spinset $\Sigma_1 \times \cdots \times \Sigma_N$. Assigning spins $b_i \in \Sigma_i$ for $i = 1, \cdots, N$ is equivalent to assigning a spin $\mathbf{b} = (b_1, \cdots, b_N)$ to the fused edge.

If there are vertices on these edges, we may also fuse these into a single vertex. Thus:



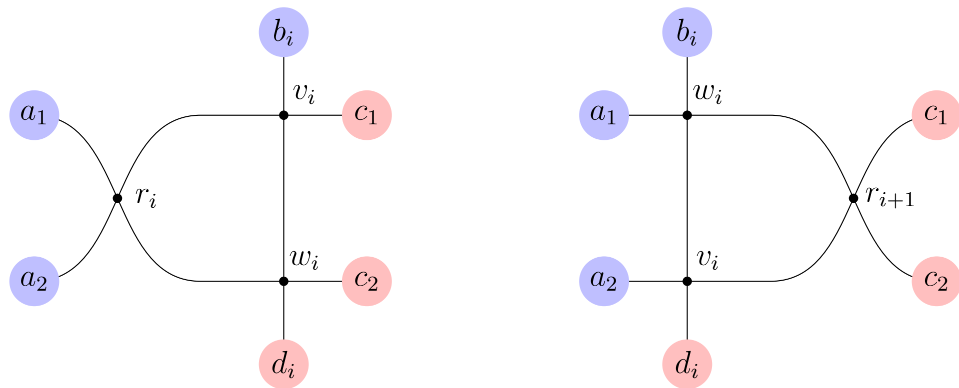
becomes:



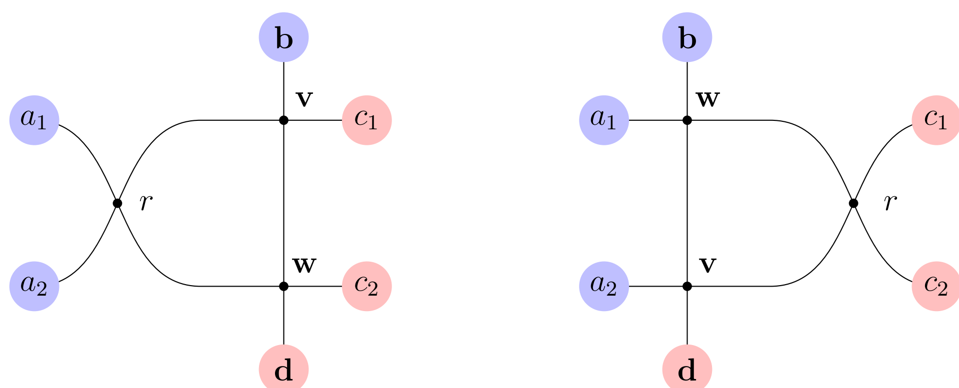
We will call the vertices $v_1, \cdots, v_N$ *unfused* and the vertex $\mathbf{v}$ *fused*.

Now we will encounter exotic versions of the Yang-Baxter equations in which the R-matrix changes when it moves past the vertex. If we are dealing with N unfused vertices

$v_1, \cdots, v_N$ and $w_1, \cdots, w_N$, we may encounter a Yang-Baxter equation that looks like this:



Assuming the periodicity $r_{N+1} = r_1$, and denoting this vertex as just r, we obtain the usual kind of Yang-Baxter equation for the fused vertices:



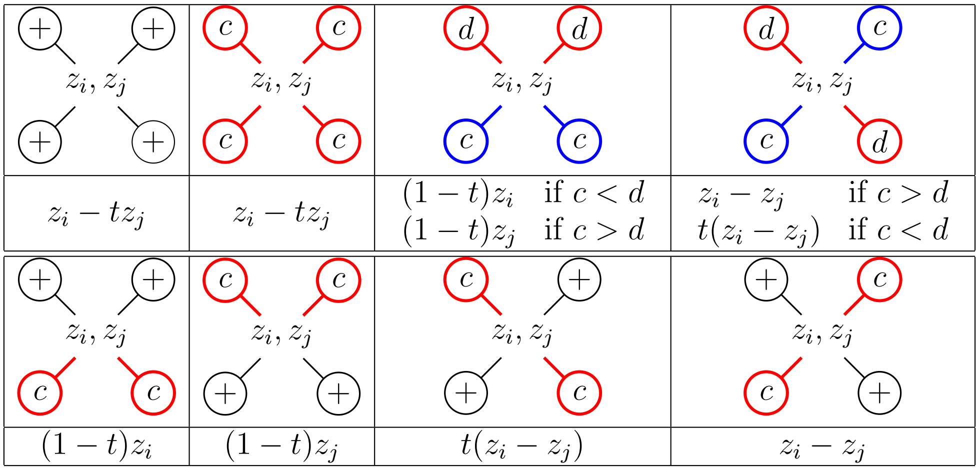
It is expected when this happens that the *fused* edges should have a quantum group interpretation.

2. Example

We know several examples of this factorization phenomenon. The ones we describe are in [**26**]. More general colored models in [**10**] do not factorize this way. The fermionic models in [**14**] also have such a factorization.

$c_1 > c_2 > \cdots > c_N$. The unfused vertical edges are *monochrome* in that each is only allowed to carry a single color. They are *bosonic* in that each vertical edge is allowed to carry multiple instances of its designated color. The partition functions are nonsymmetric

Hall-Littlewood polynomials. Before we describe the fused vertices, here is the R-matrix:



This tells us that the quantum group is $U_q(\widehat{\mathfrak{gl}}_{N+1})$ or equivalently (for this purpose) $U_q(\widehat{\mathfrak{sl}}_{N+1})$.

The horizontal edges in this model are only allowed to carry one color c_i, or no color, designated $+$. Thus the spinset of the horizontal edges is $\{c_1, \cdots, c_N, +\}$.

The states of the fused vertical edges can be described in terms of N bosons, one of each color. The spinset of the fused vertical edges is thus $\mathbb{N}^N$, where where $\mathbb{N} = \{0, 1, 2, \cdots\}$. If $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{N}^N$, we think of this as a state in which the edge carries k_i bosons of color c_i.

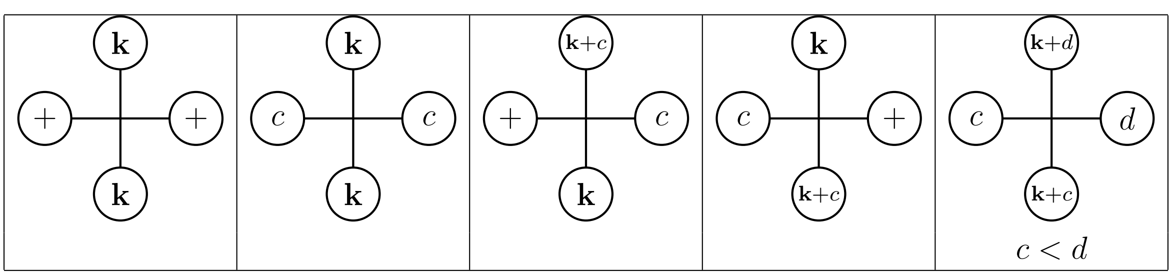
To describe the admissible states, let us introduce this notation. Let $\mathbf{k} \in \mathbb{N}^N$, which is the vertical edge spinset. By $\mathbf{k} + c_i$ we mean

$$(k_1, k_2, \cdots, k_i + 1, \cdots k_N).$$

Here the k_i component, which is interpreted as the number of bosons of color c_i, is increased by 1. Similarly

$$\mathbf{k} - c_i = (k_1, k_2, \cdots, k_i - 1, \cdots k_N).$$

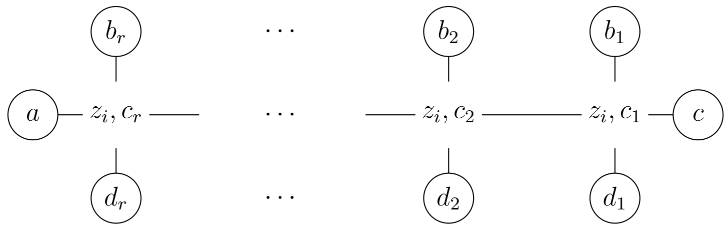
Here are the admissible states.



The striking thing to note here is that the last state is only allowed if $c < d$. More general models in [**10**] do not have this property.

Now the vertical edges can be obtained by fusion according to the following scheme. We have weight labeled by a spectral parameter z and a color c We fuse the vertices in order

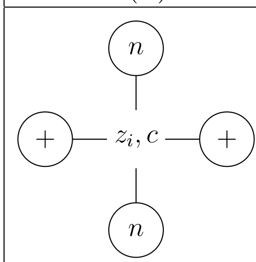
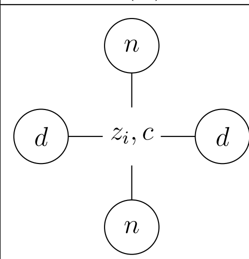
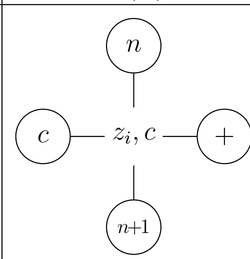
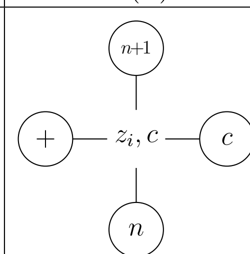
$c_N, c_{N-1}, \cdots, c_1$:



The vertical edges are also labeled by the colors c_i. *The vertical edge labeled c_i is only allowed to carry that color* and no others. For this reason, we call the vertex and color labeled c_i *monochrome.*

Remark 8. From this, we can see why we the last state is is forbidden if $c > d$. The reason is that if $c > d$, the horizontal edges between the c column, which is to the left of the d column, would have to carry both colors, and a horizontal edge is only allowed to carry one color.

Here are the weights of the monochrome vertices:

$A(n)$	$B(n)$	$C(n)$	$D(n)$
			
1	$1 \quad c < d$ $z_i \quad c = d$ $t^n \quad c > d$	$z_i(1 - t^{n+1})$	1

The auxiliary R-matrix depends on the color of the monochrome edge to the right of the vertex:

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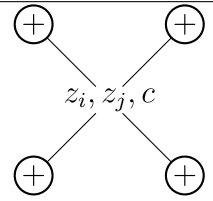
<td>
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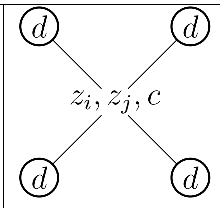
<td>
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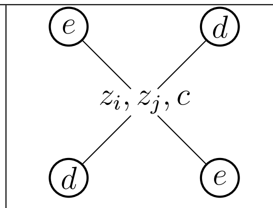
<td>
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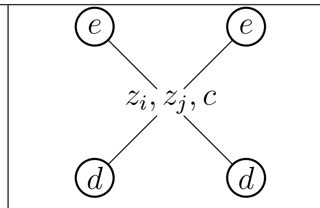
<td>
</td>
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<tr>
<td>$z_i - tz_j$</td>
<td>$z_i - tz_j$</td>
<td>$t(z_i - z_j)$ $e > d$
$z_i - z_j$ $d > e$</td>
<td>$(1-t)z_j$ $e > d > c$
$c > d > e$
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$(1-tz_i)$ $d > c > e$
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$e > d > c$</td>
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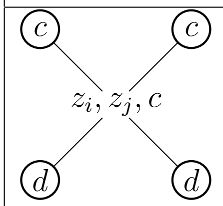
<td>
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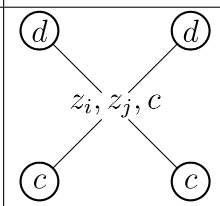
<td>
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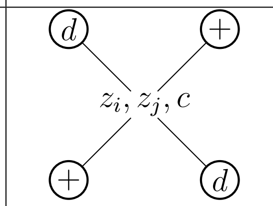
<td>
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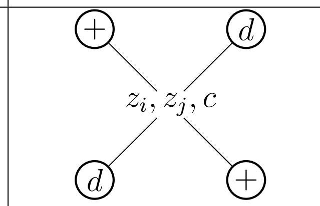
<td>
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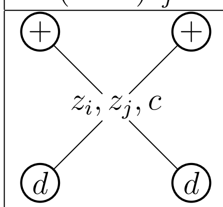
<td>
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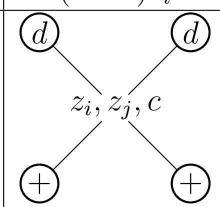
<td>
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<tr>
<td>$(1-t)z_i$</td>
<td>$(1-t)z_j$</td>
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We do not show the Yang-Baxter equation, but as in the last section, it changes when the R-matrix moves past the vertex. After moving past all the vertices, the R-matrix is restored to its original state. See [**26**] for further information.

3. Explanation in terms of Verma modules

The claims in this section are undoubtedly true but haven't been verified.

The Lie algebra $\mathfrak{g} = \mathfrak{gl}_{N+1}$ has a parabolic subalgebra $\mathfrak{p}$ that is the semidirect product of $\mathfrak{m} = \mathfrak{gl}_N \oplus \mathfrak{gl}_1$ with the nilpotent subalgebra $\mathfrak{u}_+$ supported on the last column. If $N = 3$:

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}, \qquad \mathfrak{m} = \left\{ \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \qquad \mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Let $\mathfrak{u}_-$ be the complementary nilpotent subalgebra that is the transpose of $\mathfrak{u}$ in matrix form. We then have $\mathfrak{gl}_{N+1} = \mathfrak{u}_- \oplus \mathfrak{p}$. By the PBW theorem,

$$U(\mathfrak{gl}_{N+1}) \cong U(\mathfrak{u}_-) \otimes_{\mathbb{C}} U(\mathfrak{p}).$$

Let us take any one-dimensional representation ψ of $\mathfrak{p}$, afforded by the module $\mathbb{C}_\psi$. The induced module $V_\psi = U(\mathfrak{gl}_{N+1}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\psi$ is then isomorphic to $U(\mathfrak{u}_-)$ as a vector space. Since $\mathfrak{u}_-$ is abelian, the enveloping algebra $U(\mathfrak{u}_-)$ is just the symmetric algebra $\mathrm{Sym}(\mathfrak{u}_-)$.

If $\beta \in \Phi$, we are regarding β as an element of $\mathfrak{h}^*$, where $\mathfrak{h}$ is the diagonal Cartan subalgebra. There is a unique (up to scalar) element $X_\beta \in \mathfrak{g}$ such that

$$[H, X_\beta] = \beta(H) X_\beta.$$

There are N negative roots $\beta_1, \cdots, \beta_N$ such that $X_{\beta_i} \in \mathfrak{u}_-$. We order these so that X_{β_i} has its nonzero entry in the $N+1-i$ column. Thus for $N = 3$:

$$X_{\beta_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_{\beta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{\beta_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $U(\mathfrak{u}_-) = \mathbb{C}[X_{\beta_1}, \cdots, X_{\beta_N}]$ is a polynomial ring.

Conjecture 1. *The quantized version of V_ψ for suitable ψ is the $U_q(\widehat{\mathfrak{gl}}_{N+1})$-module associated with the vertical edges in the model described in Section 2. The more general models of* [**10**]*, the module would be a Borel Verma module as in Lecture 14.*

For this it is likely important that the nilpotent subalgebra $\mathfrak{u}_-$ is abelian.

How should we view the monochrome edges and vertices? Only the fused edges correspond to $U_q(\widehat{\mathfrak{sl}}_{N+1})$ modules. However the constitutent unfused edges, each of which can carry only one color, resembles the $U_q(\widehat{\mathfrak{sl}}_2)$ vertex.

So there is an embedding of $\mathfrak{sl}_2 \longrightarrow \mathfrak{gl}_{N+1}$ along the positive root $-\beta_i$, namely

$$\langle X_{-\beta}, X_\beta \rangle \cong \mathfrak{sl}_2. \tag{42}$$

And the Verma module

$$V_\psi \cong U(\mathfrak{u}_-) \cong \bigotimes_{i=1}^{N} U(\mathbb{C} X_{\beta_i}).$$

Each factor $U(\mathbb{C}X_{\beta_i})$ is an $\mathfrak{sl}_2$ Verma module, for the copy (42) of $\mathfrak{sl}_2$. The last isomorphism shows that the $\mathfrak{sl}_{N+1}$ parabolic Verma module is a tensor product of these N $\mathfrak{sl}_2$ Verma modules, and this fact is reflected in the factorization of the edges into monochrome edges.

LECTURE 17

Heisenberg Spin Chains

A standard method in analysis, going back to Hilbert and Schmidt, and much earlier to Green, for studying operators is to find a larger commuting family of operators. A simple example is the Laplacian, an unbounded self-adjoint operator. This commutes with the integral operators $F \longmapsto F * \phi$, where $\phi \in C_c^\infty(\mathbb{R}^n)$, which are Hilbert-Schmidt operators, hence compact. This method is used extensively in the theory of automorphic forms, for example in the Selberg trace formula.

Baxter's approach to the six- and eight-vertex models was to embed the row transfer matrix into a larger commuting family of row transfer matrices. Another application of the same idea led to the solution of a problem in one-dimensional quantum mechanics, his analysis of the Heisenberg spin chains, a model of ferromagnetism [**42**]. Baxter was able to introduce the theory of elliptic functions by finding a commuting family of row transfer matrices from the eight-vertex model.

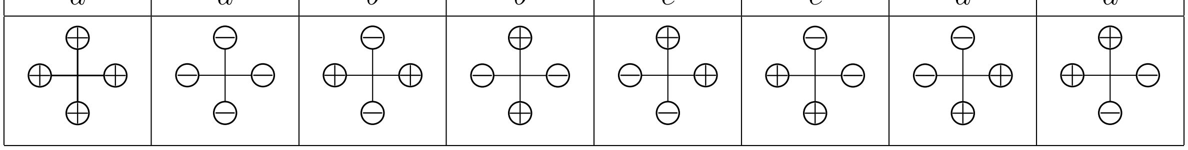
Baxter [**4, 5**] knew that two operators arising from physical problems were related to each other. (This fact was also observed by Sutherland [**89**].) The operators are:

- The row transfer matrices from the field-free 6 or 8 vertex models
- Hamiltonians for Heisenberg spin chains, called the XXZ and XYZ Hamiltonians.

Using the Yang-Baxter equation, the row transfer matrices can be organized into commuting families. This means that the row transfer matrix contains a parameter that can be differentiated, and roughly the Hamiltonian is the logarithmic derivative of the row transfer matrix. Equivalently, the row transfer matrix is an exponentiated Hamiltonian. As a consequence, the Hamiltonian also commutes with this family of row transfer matrices.

The field-free eight vertex model can be solved similarly to the six vertex model, using a parametrized Yang-Baxter equation. Let a, b, c, d be the Boltzmann weights, thus:

a	a	b	b	c	c	d	d



Define

$$\Delta = \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}, \qquad \Gamma = \frac{ab - cd}{ab + cd}. \tag{43}$$

If a', b', c', d' are another set of Boltzmann weights, and Δ', Γ' are defined like Δ, Γ, the condition for the Yang-Baxter equation to have a solution is

$$\Delta = \Delta', \qquad \Gamma = \Gamma'.$$

(See [**5**], Chapter 10.) Now with Γ and Δ fixed, the solutions $a : b : c : d$ to (43) form an elliptic curve, and indeed, the relevant Yang-Baxter equation is a parametrized Yang-Baxter equation with this curve as its parameter group. The relevant quantum group is an elliptic quantum group ([**36**]).

Baxter's work solving the eight-vertex model was carried out on a ship, where he took over the chart room for his calculations. Baxter [**5**] wrote in Chapter 10:

> Sutherland (1970) showed directly that the transfer matrix of any zero-field eight-vertex model commutes with an XYZ operator X. They therefore have the same eigenvectors. I was not aware of Sutherland's result when I solved the eight-vertex model (I did much of the work in the writing room of the P & O liner *Arcadia*, in the Atlantic and Indian Oceans. This was good for concentration, but not for communication). It should be obvious from Sections 10.4-10.6 that such commutation relations are closely linked with the solution of the problem.

This theory is outside the scope of these lectures, but we will consider the simpler case where $d = 0$, where the field-free six-vertex model is related to the XXZ Hamiltonian. As we know, the relevant quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$.

For the free-fermionic six vertex model (where the quantum group is $U_q(\widehat{\mathfrak{gl}}(1|1))$) a similar result was obtained by Brubaker and Schultz [**22**]. See also [**95**].

1. Heisenberg Spin Chains

In classical mechanics, *observables* are functions A on the *phase space*, which is a parameter space representing the state of a physical system, including the positions and momenta of all particles. Given a state of the system, every observable thus has a definite value.

In quantum mechanics, by contrast, it is possible for the system to be in a state where a given observable does not have a definite value. The state of the system is represented by a vector in a Hilbert space $\mathfrak{H}$, and the classical observable A is replaced by a Hermitian (self-adjoint) operator $\hat{A} : \mathfrak{H} \longrightarrow \mathfrak{H}$ (or an unbounded operator defined on a dense subspace). If $\Psi \in \mathfrak{H}$ represents the state of the system, the observable A has a definite value λ if $\hat{A}\Psi = \lambda\Psi$.

For simplicity let us assume that $\hat{A}$ has a discrete spectrum. By the spectral theorem, the state Ψ may be expanded as a "Fourier series"

$$\Psi = \sum a_i \Psi_i$$

where Ψ_i are eigenfunctions of $\hat{A}$. If we normalize Ψ so that $|\Psi| = 1$, then the "amplitudes" a_i have a probabilistic interpretation: if the observable f is measured, a definite value λ_i is returned, and the "wave function" Ψ collapses to the state Ψ_i. The probability of this happening is $|a_i|^2$. By the Plancherel theorem $\sum_i |a_i|^2 = 1$, and so this scheme gives a probability distribution on the spectrum of $\hat{A}$.

In quantum mechanics, two observables A and B can be measured simultaneously if and only if the corresponding operators $\hat{A}$ and $\hat{B}$ commute. In this case, the eigenfunctions Ψ_i can be chosen to be simultaneous eigenfunctions of $\hat{A}$ and $\hat{B}$.

A particular observable is *energy*, and the corresponding operator is the *Hamiltonian*. It determines the evolution of the system in time, through Schrödinger's equation.

An examples of a collection of observables that cannot be measured simultaneously are electron spin in different directions. A 2 dimensional Hilbert space $\mathfrak{H} = \mathbb{C}^2$ is sufficient to represent a particle such as the electron with spin $\frac{1}{2}$. If the spin is measured along the z axis, it will be found in one of two states, up or down. The spin operator is therefore represented by the matrix

$$\sigma^z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

On the other hand, if the spin is measured along the x or y axes, it will again be found in one of two possible states. The corresponding operators do not commute with σ^z, and with respect to the same basis, are represented by the matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The three matrices σ^x, σ^y, σ^z are called the *Pauli spin matrices*. They are both Hermitian and unitary. We have an alternative labeling

$$\sigma^1 = \sigma^x, \qquad \sigma^2 = \sigma^y, \qquad \sigma^3 = \sigma^z, \qquad \sigma^4 = I_2. \tag{44}$$

Heisenberg [**42**] proposed a quantum mechanical model of ferromagnetism. We consider a sequence of N magnetic atoms such as iron at adjacent sites. We will assume that the sites of the spin chain are arranged in a ring. Consequently the boundary conditions for the six-vertex model will also be periodic, as in Lecture 2.

Each atom is a magnetic dipole whose dipole moment is proportional to the spin. Since the spin module is 2-dimensional, it is represented by a vector in a 2-dimensional space. The Hilbert space of a single magnetic atom is $\mathbb{C}^2$. Therefore the Hilbert space $\mathfrak{H}$ of N atoms is $\otimes^N \mathbb{C}^2$. We let σ_j^x, σ_j^y and σ_j^z denote the Pauli matrices acting on the j-th site, and as the identity operator on all other sites.

To give the simplest formulation, we will assume that the chain is periodic, so $\sigma_{N+1}^x = \sigma_1^x$ etc. Adjacent dipoles tend to align in the same direction, which partly explains the form of the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{N} (J_x \sigma_j^x \otimes \sigma_{j+1}^x + J_y \sigma_j^y \otimes \sigma_{j+1}^y + J_z \sigma_j^z \otimes \sigma_{j+1}^z), \tag{45}$$

for suitable positive constants J_x, J_y, J_z. Due to the assumed periodicity, $\sigma_{N+1} = \sigma_N$. If $J_x = J_y$, this is called the XXZ Hamiltonian. It is an endomorphism of $\otimes^N \mathbb{C}^2$.

On the other hand, we can fix Boltzmann weights a, b, c, d and consider the row transfer matrix $T_{a,b,c,d}(\alpha, \beta)$ as follows. We will denote the standard basis of $\mathbb{C}^2$ as

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A basis of $\otimes^N \mathbb{C}^2$ consists of vectors v_α where $\alpha = (\alpha_1, \cdots, \alpha_N) \in \{+, -\}^N$, where

$$v_\alpha = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}.$$

We may thus regard both the Hamiltonian H and the row transfer matrices for the 8 vertex model as endomorphisms of the same Hilbert space, $\otimes^N \mathbb{C}^2$. Baxter proved:

- The XXZ Hamiltonian commutes with a family of 6-vertex model row transfer matrices.
- The XYZ Hamiltonian commutes with a family of 8-vertex model row transfer matrices.

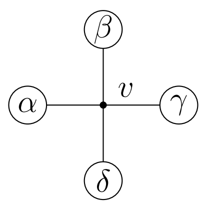
We will review the relationship of the XXZ Hamiltonian with the row transfer matrices for the field-free 6-vertex model. We will partly investigate the 8-vertex model, but we will specialize to the 6-vertex model before long, and prove the second statement. See Baxter [**4**] for the 8-vertex model case, which requires some elliptic and theta functions.

2. Preliminaries

We will make use of the Pauli spin matrices with respect to this basis, and if $\alpha, \beta \in \{+, -\}$, and if σ is one of the Paul spin matrices, we will denote by $\sigma_{\alpha,\beta}$ the corresponding matrix entry. Thus $\sigma^y_{-+} = i$ and $\sigma^y_{+-} = -i$. Let

$$p_1 = \frac{1}{2}(b+d), \qquad p_2 = \frac{1}{2}(b-d), \qquad p_3 = \frac{1}{2}(a-c), \qquad p_4 = \frac{1}{2}(a+c). \tag{46}$$

Let v be a vertex type. We will denote by $R^{\gamma\delta}_{\alpha\beta}(v)$ the Boltzmann weight



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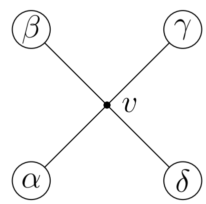


LEMMA 2.1. *Let $\alpha, \beta, \gamma, \delta \in \{+, -\}$. Then*

$$R^{\gamma\delta}_{\alpha\beta} = \sum_{k=1}^{4} p_k \sigma^k_{\beta\gamma} \sigma^k_{\alpha\delta}. \tag{47}$$

PROOF. This can be checked by case-by-case consideration. There are 16 choices for $\alpha, \beta, \gamma, \delta$, but only eight give a nonzero result. Let us consider for example $(\alpha, \beta, \gamma, \delta) = (+, -, +, -)$. Since σ^k_{-+} and σ^k_{+-} are nonzero only for $k = 1, 2$, there are two terms:

$$\frac{1}{2}(b+d)\sigma^1_{\beta\gamma}\sigma^1_{\alpha\delta} + \frac{1}{2}(b-d)\sigma^2_{\beta\gamma}\sigma^2_{\alpha\delta} = b.$$

The remaining cases are similar. □

There is a similar identity

$$R^{\gamma\delta}_{\alpha\beta} = \sum_{k=1}^{4} w_k \sigma^k_{\beta\delta} \sigma^k_{\gamma\alpha}$$

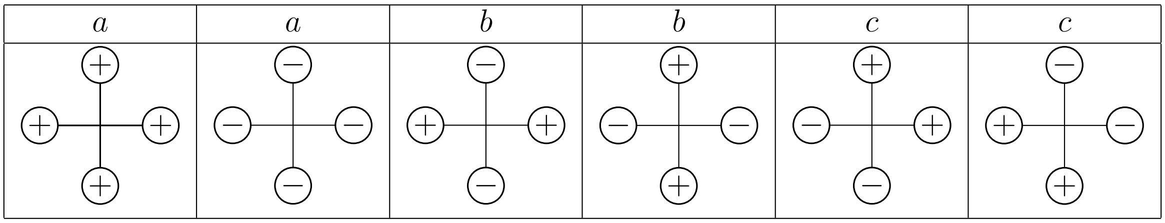
where

$$w_1 = \frac{1}{2}(c+d), \qquad w_2 = \frac{1}{2}(-c+d), \qquad w_3 = \frac{1}{2}(a-b), \qquad w_4 = \frac{1}{2}(a+b).$$

We won't need this but mention it for completeness.

3. Six-vertex model and the XXZ Hamiltonian

Now we specialize to the six-vertex model, referring to [**4**] for the general case. Thus now $d = 0$, and as a consequence of this simplification we will have $J_x = J_y$ in the Hamiltonian.



Let $T_{a,b,c}(\alpha, \beta)$ be the corresponding row transfer matrix.

We saw in Lecture 4 that if Δ is fixed, then we have a parametrized Yang-Baxter equation involving a, b, c such that

$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta.$$

Let q be such that $\Delta = \frac{1}{2}(q + q^{-1})$. We may parametrize the solutions by a map

$$R_\Delta : \mathbb{C}^\times \longrightarrow \{\text{field free Boltzmann weights } a, b, c\}$$

given by

$$R_\Delta(x) = (a, b, c) = \left(\frac{xq - (xq)^{-1}}{q - q^{-1}}, \frac{x - x^{-1}}{q - q^{-1}}, 1 \right).$$

These are the Boltzmann weights from Theorem 5.2 in Lecture 4, divided by the constant $\frac{1}{2}(q - q^{-1})$. We saw that this gives a parametrized Yang-Baxter equation. (Dividing by a constant does not affect this since both sides of the Yang-Baxter equation are divided by the same constant.)

Then by Theorem 1.1 of Lecture 2, the row transfer matrices $T_{a,b,c}(\alpha, \beta)$ form a commuting family. We choose fixed χ so that $e^{i\chi} = q$. Then we choose variable θ so that $e^{i\theta} = xq$. We slightly modify the notation, omitting Δ from the notation R_Δ because it is fixed, and regarding R as a function of θ instead of x. We will use the notation $R(\theta)^{\gamma\delta}_{\alpha\beta}$ as explained in Section 2.

Lemma 3.1. *When $\theta = \chi$, we have*

$$R(\chi)^{\gamma\delta}_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \delta \text{ and } \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that when $\theta = \chi$ we have $(a, b, c) = (1, 0, 1)$. So this follows from the definition of the Boltzmann weights. □

Let $T_\theta(\alpha, \beta)$ be the row transfer matrix $T_{a,b,c}(\alpha, \beta)$ with this parametrization.

Remark 9. At the special point $\theta = \chi$, since $b = 0$, we are in a 5-vertex model case in which the particles are allowed to move to the right but not straight down. In fact $T_\chi(\alpha, \beta)$ is the right shift operator, moving each particle one step to the right. Obviously T_χ is invertible, the inverse being the left shift operator.

We may differentiate the operator T_θ with respect to θ. The derivative T'_θ commutes with T_θ, and we may consider the logarithmic derivative at $\theta = \chi$.

$$\mathcal{L} = \frac{1}{2} T_\chi^{-1} T'_\chi.$$

It is at this point $\theta = \chi$ that there is a relationship between the XXZ Hamiltonian and the six-vertex model. Regard the p_i in (46) as functions of θ.

$$J_x = \frac{1}{2} p'_1(\chi), \qquad J_y = \frac{1}{2} p'_2(\chi), \qquad J_z = \frac{1}{2} p'_3(\chi). \tag{48}$$

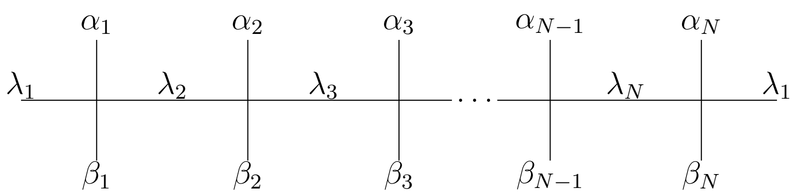
Since $d = 0$, we have $J_x = J_y$. Let H be the XXZ Hamiltonian (45).

THEOREM 3.2. *With these notations, we have*

$$\mathcal{L} = H + cI_{\otimes^N \mathbb{C}^2},$$

where c is an explicit constant. The operator H commutes with the 6-vertex row transfer matrices T_θ.

PROOF. We will label the interior edges of the single-layer grid whose partition function is T_θ by $\lambda_1, \cdots, \lambda_N$, thus:



Due to the periodic boundary conditions, $\lambda_{N+1} = \lambda_1$. Thus

$$T_\theta(\alpha, \beta) = \sum_\lambda \prod_{i=1}^N R(\theta)_{\lambda_i \alpha_i}^{\lambda_{i+1} \beta_i}.$$

Differentiating with respect to θ and setting $\theta = \chi$,

$$T'_\chi(\alpha, \beta) = \sum_{j=1}^N \sum_\lambda \left[\frac{d}{d\theta} R(\theta)_{\lambda_j \alpha_j}^{\lambda_{j+1} \beta_j} \right]_{\theta=\chi} \prod_{i \neq j} R(\chi)_{\lambda_i \alpha_i}^{\lambda_{i+1} \beta_i}.$$

By Lemma 3.1, if $i \neq j$ then $R(\chi)_{\lambda_i \alpha_j}^{\lambda_{i+1} \beta_i} = 1$ provided $\lambda_i = \beta_i$ and $\alpha_i = \lambda_{i+1}$, and is zero otherwise. Therefore the j-th term only contributes if $\beta_i = \alpha_{i-1}$ when $i \neq j, j+1$. Assuming this, since we are summing over λ, we may omit these factors and take $\lambda_j = \alpha_{j-1}$, $\lambda_{j+1} = \beta_{j+1}$ to obtain

$$T'_\chi(\alpha, \beta) = \sum_{j=1}^N \frac{d}{d\theta} R(\theta)_{\alpha_{j-1} \alpha_j}^{\beta_{j+1} \beta_j} |_{\theta=\chi} \ .$$

Now we substitute (47) to obtain

$$T'_\chi(\alpha, \beta) = \sum_{k=1}^4 p'_k(\chi) \sum_{j=1}^N \sigma^k_{\alpha_j \beta_{j+1}} \sigma^k_{\alpha_{j-1} \beta_j}.$$

But now we remember that it is not T'_χ that we are trying to compute, but $\frac{1}{2}T_\chi^{-1}T'_\chi$, and T_χ^{-1} is the left-shift operator by Remark 9. Thus

$$\mathcal{L} = \frac{1}{2}\sum_{k=1}^{4} p'_k(\chi)\sum_{j=1}^{N}\sigma^k_{\alpha_j\beta_j}\sigma^k_{\alpha_{j-1}\beta_{j-1}}.$$

The first three terms produce the XXZ Hamiltonian, with $J_x = J_y = p'_1(\chi)$ and $J_z = p'_3(\chi)$. The last term produces $cI_{\otimes^N\mathbb{C}^2}$ with the constant $c = \frac{1}{2}Np'_4(\chi)$. $\square$

LECTURE 18

Lie Superalgebras

1. Comparison of two similar models

This lecture will be a very quick introduction to Lie superalgebras. Our reason for wanting to introduce this topic is that some very interesting models are associated with the quantized enveloping algebras of Lie superalgebras. For more information about Lie superalgebras, we recommend Cheng and Wang [**29**] and Musson [**80**].

Let us compare two similar models, the Tokuyama model of Lectures 5 and 6, and the bosonic models of Lectures 8 and 16. The partition functions of the Tokuyama models are, we saw in Lectures 5 and 6, Schur polynomials times deformed Weyl denominators:

$$\mathbf{z}^{\rho} \prod_{\alpha \in \Phi^{+}} (1 - q\mathbf{z}^{-\alpha}) s_{\lambda}(\mathbf{z}).$$

The partition functions of the bosonic models are Hall-Littlewood polynomials [**26**]. Both models have colored variants that produce nonsymmetric polynomials ([**14, 26**]). It is possible to argue that these two symmetric polynomials are closely related "twins," a parallel that would extend to other Cartan types. We will not explain this point but see [**20, 13**].

Now let us compare the R-matrices. Here is the R-matrix for the bosonic models. The relevant quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$.

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$\oplus\quad\oplus$ $\oplus\quad\oplus$	$\ominus\quad\ominus$ $\ominus\quad\ominus$	$\ominus\quad\oplus$ $\oplus\quad\ominus$	$\oplus\quad\ominus$ $\ominus\quad\oplus$	$\oplus\quad\oplus$ $\ominus\quad\ominus$	$\ominus\quad\ominus$ $\oplus\quad\oplus$
$z - qw$	$z - qw$	$q(z - w)$	$z - w$	$(1 - q)z$	$(1 - q)w$

The R-matrix for the Tokuyama models is extremely similar, the only different entry being the $\mathtt{a}_1$ entry:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$\oplus\quad\oplus$ $\oplus\quad\oplus$	$\ominus\quad\ominus$ $\ominus\quad\ominus$	$\ominus\quad\oplus$ $\oplus\quad\ominus$	$\oplus\quad\ominus$ $\ominus\quad\oplus$	$\oplus\quad\oplus$ $\ominus\quad\ominus$	$\ominus\quad\ominus$ $\oplus\quad\oplus$
$w - qz$	$z - qw$	$q(z - w)$	$z - w$	$(1 - q)z$	$(1 - q)w$

It can be shown that the R-matrix is related to the standard representations of $U_q(\widehat{\mathfrak{gl}}(1|1))$, a superalgebra quantum group. See [**92, 61, 93**] for results on R-matrices of $U_q(\widehat{\mathfrak{gl}}(m|n))$.

The similarity between the R-matrices for $U_q(\widehat{\mathfrak{sl}}_{m+n})$ and $U_q(\widehat{\mathfrak{gl}}(m|n))$ extends to general m and n. The R-matrices correspond to colored models ([**15, 2**]).

Now the interesting thing is that this similarity between the Tokuyama and bosonic models also applies to the vertical edges. For the models in [**14**], which are related to $U_q(\widehat{\mathfrak{gl}}(n|1))$, the vertical edges have spinset of cardinality 2^n, in bijection with the set of subsets of n colors. The vertical edges can be understood in terms of fusion as in the bosonic models of [**26**]. The $U_q(\widehat{\mathfrak{sl}}_{n+1})$-modules in [**26**], we saw in Lecture 16, correspond to Verma modules, which are isomorphic to $\mathrm{Sym}(\mathbb{C}^n)$. Now $U(\mathfrak{gl}(n|1))$ and its quantized affinization $U_q(\widehat{\mathfrak{gl}}(n|1))$ have a kind of Verma module called a *Kac module* that is isomorphic to the exterior algebra $\bigwedge \mathbb{C}^n$ of cardinality 2^n, which is expected to be related to the vertical edges in these models. The purpose of this lecture will be to introduce Lie superalgebras and to define the Kac modules that we claim are to explain the vertical edges in the colored fermionic models.

2. Lie Superalgebras

Lie superalgebras are a generalization of a Lie algebra. They emerged in the 1970's from physical theories having symmetries that connect fermions and bosons [**31**]. In 1981 Perk and Schultz [**84**] found some new solvable lattice models which were explained by Yamane [**92**] as being related to the superalgebra quantum group $U_q(\mathfrak{gl}(m|n))$, whose R-matrices he computed. Later Brubaker, Bump and Buciumas [**12**] found supersymmetric lattice models whose partition functions are Whittaker functions on p-adic metaplectic groups. Such models were substantially generalized by Brubaker, Buciumas, Bump and Gustafsson [**15**]. Other supersymmetric models were found by Aggarwal, Borodin and Wheeler [**3**].

In retrospect, the Tokuyama models, which we have already looked at are supersymmetric, being associated with $U_q(\widehat{\mathfrak{gl}}(1|1))$. Colored variants associated with $U_q(\widehat{\mathfrak{gl}}(n|1))$. For these models, the quantum group is identified from the R-matrix. But the vertical edges are associated with *Kac modules*, which are Verma modules for superalgebras that are finite-dimensional.

A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, where V_0 is the *even part* and V_1 is the *odd part*. Elements of V_0 and V_1 are called *homogeneous*. It will be convenient to denote $|a| = i$ if $a \in V_i$, so $|a| = 0$ if a is even and $|a| = 1$ if a is odd. We define the super dimension of V to be $n|m$ where $n = \dim(V_0)$ and $m = \dim(V_1)$.

Other things such as associative algebras and Lie algebras have super variants. The common feature is that interchanges involve sign changes. The rule is that when an odd element "moves past" an even element, a minus sign is introduced.

For example, an associative superalgebra is just a $\mathbb{Z}_2$-graded associative algebra. No modification of the associative law is needed, because in the identity $a(bc) = (ab)c$, the elements occur in the same order. But what does it mean for an associative superalgebra to be *commutative?* The rule is that if a and b are homogeneous, then $ab = ba$ unless both are odd, in which case $ab = -ba$. We can write this more succinctly as $ab = (-1)^{|a|\cdot|b|}ba$. For example, the exterior algebra of a vector space, or the cohomology ring of a topological space are commutative superalgebras.

If V is an ordinary vector space, we will denote by $S(V)$ and $\bigwedge V$ the symmetric and exterior algebras. If V is a super vector space, our convention is that the symmetric and exterior algebras are

$$S(V) = S(V_0) \otimes \bigwedge V_1, \qquad \bigwedge V = \bigwedge V_0 \otimes S(V_1).$$

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space with a bilinear operation $[\ ,]$ such that (for x, y and z homogeneous)

$$[x, y] = -(-1)^{|x|\cdot|y|}[y, x] \tag{49}$$

and the Jacobi identity holds, in the form

$$(-1)^{|x|\cdot|z|}[[x, y], z] + (-1)^{|y|\cdot|x|}[[y, z], x] + (-1)^{|z|\cdot|y|}[[z, x], y] = 0.$$

The even part $\mathfrak{g}_0$ of the Lie superalgebra $\mathfrak{g}$ is an ordinary Lie algebra.

Example 2.1. Let V be a super vector space. Then $\mathfrak{gl}(V) = \mathrm{End}(V)$, with the following grading If $\phi \in \mathrm{End}(V)$. We can write

$$\phi = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix}$$

where $\phi_{ij} \in \mathrm{Hom}(V_j, V_i)$. We make $\mathfrak{gl}(V)$ into a super vector space in which $\phi_{00}, \phi_{01}, \phi_{10}$ are even, and ϕ_{11} is odd. As a particular case, let $\mathbb{C}^{m|n}$ denote the super vector space $V_0 \oplus V_1$ where the even part $V_0 = \mathbb{C}^m$ and the odd part $V_1 = \mathbb{C}^n$. With $V = \mathbb{C}^{m|n}$ we will denote $\mathfrak{gl}(V) = \mathfrak{gl}(m|n)$.

We are assuming that the ground field has characteristic not equal to 2 or 3. The enveloping algebra $U(\mathfrak{g})$ is the associative superalgebra generated by $\mathfrak{g}$ modulo the relations $[x, y] = xy - (-1)^{|x|\cdot|y|}yx$.

Lemma 2.2. *If $x \in \mathfrak{g}$ is odd, then $[x, x] = 0$, and $x^2 = 0$ in $U(\mathfrak{g})$.*

Proof. By (49) we have $0 = [x, x] = -[x, x]$ so $[x, x] = 0$ in $\mathfrak{g}$. Since x is odd, the relation $[x, x] = x^2 - (-1)^{|x|\cdot|x|}x^2 = 2x^2$, so $x^2 = 0$ in $\mathfrak{g}$. □

There is a PBW Theorem.

Theorem 2.3. *For simplicity let us assume that $\mathfrak{g}$ is finite-dimensional. Let us choose a basis $x_1, \cdots, x_d$ consisting of homogeneous elements. Then $U(\mathfrak{g})$ has a basis consisting of elements*

$$x_1^{k_1} \cdots x_d^{k_d}$$

where $k_i \in \mathbb{N}$ if x_i is even, and $k_i \in \{0, 1\}$ if x_i is odd.

Proof. See [**87**], Theorem 2.1, or [**80**] Chapter 6. □

Proposition 2.4. *Let $\mathfrak{g}$ be an abelian Lie superalgebra, so $[\mathfrak{g}, \mathfrak{g}] = 0$. Then*

$$U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong S(\mathfrak{g}_0) \otimes \bigwedge \mathfrak{g}_1 .$$

Proof. We leave this to the reader. □

Now let $\mathfrak{g}$ be a Lie superalgebra. The even part $\mathfrak{g}_0$ is a Lie algebra. We may choose a maximal abelian Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$, and decompose $\mathfrak{g}$ into root spaces as in Lecture 15:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{h}\} .$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, we call $\alpha \in \mathfrak{h}$ a root. The roots may be divided into even and odd roots, where a root is *even* if the *root space* $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$, and *odd* if $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$. The set Φ of roots can also be divided into positive and negative roots. Then we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}_-^{\mathrm{odd}} \oplus \mathfrak{g}_0 \oplus \mathfrak{u}_+^{\mathrm{odd}},$$

where $\mathfrak{u}_-^{\text{odd}}$ is the sum of the root spaces for the odd negative roots, and $\mathfrak{u}_+^{\text{odd}}$ is the sum of the root spaces for the odd positive roots.

For example, if $\mathfrak{g} = \mathfrak{gl}(2|2)$, the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ is $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$, and we may take $\mathfrak{h}$ to be the diagonal subalgebra. Then

$$\mathfrak{g}_0 = \left(\begin{array}{|cc|cc|} \hline * & * & 0 & 0 \\ * & * & 0 & 0 \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \\ \hline \end{array}\right), \qquad \mathfrak{u}_+^{\text{odd}} = \left(\begin{array}{|cc|cc|} \hline 0 & 0 & * & * \\ 0 & 0 & * & * \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}\right), \qquad \mathfrak{u}_-^{\text{odd}} = \left(\begin{array}{|cc|cc|} \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline * & * & 0 & 0 \\ * & * & 0 & 0 \\ \hline \end{array}\right).$$

Now by the PBW theorem we have

$$U(\mathfrak{g}) \cong U(\mathfrak{u}_-^{\text{odd}}) \otimes_{\mathbb{C}} U(\mathfrak{g}_0) \otimes_{\mathbb{C}} U(\mathfrak{u}_+^{\text{odd}})$$

where $\otimes = \otimes_{\mathbb{C}}$ is the tensor product of associative superalgebras, a modification of the usual tensor product of associative algebras in which (for homogeneous elements a, b, c, d)

$$(a \otimes b)(c \otimes d) = (-1)^{|b|\cdot|d|}(ac \otimes bd).$$

Assuming that $\mathfrak{u}_-^{\text{odd}}$ is abelian, which it is in the example of $\mathfrak{gl}(m|n)$, we have

$$U(\mathfrak{u}_-^{\text{odd}}) \cong \bigwedge \mathfrak{u}_-^{\text{odd}}, \tag{50}$$

the exterior algebra, of dimension $2^{\dim(\mathfrak{u}_-^{\text{odd}})}$.

Now we can explain the Kac modules that are intended as an explanation for the vertical edges in the colored fermionic models mentioned at the beginning of the lecture, including the (uncolored) Tokuyama model. Let V be a finite-dimensional $\mathfrak{g}_0$-module. We extend it to a $\mathfrak{p}$-module where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{u}_+^{\text{odd}}$ by letting $\mathfrak{u}_+^{\text{odd}}$ act trivially. Then the induced module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V \cong U(\mathfrak{u}_-^{\text{odd}}) \otimes_{\mathbb{C}} U(\mathfrak{p}) \otimes_{U(\mathfrak{p})} V \cong \bigwedge \mathfrak{u}_-^{\text{odd}} \otimes V$$

is called the *Kac module*. As we can see it is a kind of Verma module that happens to be finite-dimensional. Depending on V, it is usually irreducible, but not always [**55**]. Its character is easy to describe. See [**68**] for Kac modules of $U_q(\mathfrak{gl}(m|n))$.

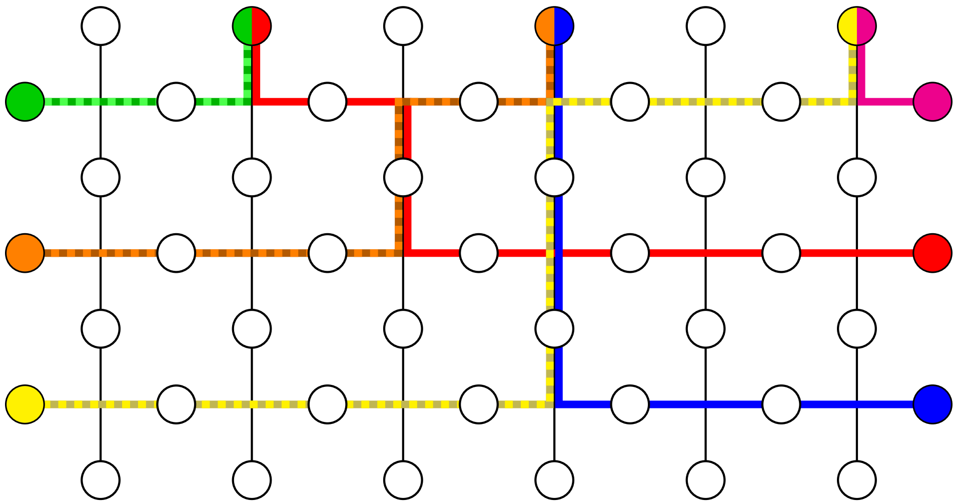
CONJECTURE 2. *For $\mathfrak{g} = \mathfrak{gl}(m|n)$. There exists a one-dimensional representation V of $\mathfrak{g}_0$ whose Kac module explains the vertical edges in the colored fermionic models of* [**15**].

3. Supersymmetric models

To illustrate these ideas, we describe some $U_q(\widehat{\mathfrak{gl}}(m|n))$ models from [**15**]. Other $U_q(\mathfrak{gl}(m|n))$ models may be found in [**2**]. We will omit full description of the Boltzmann weights but will discuss how these models make use of Kac modules for $U_q(\mathfrak{gl}(m|n))$. These models are generalizations of the Tokuyama model, and it is helpful to consider them even if one is only interested in simpler cases.

We require m colors $\mathcal{C} = \{c_1, \cdots, c_m\}$ and another palette of n "supercolors," which we will denote $\mathcal{D} = \{d_1, \cdots, d_n\}$. Colors move down and to the right, while supercolors move down and to the left. The spinset of the horizontal edges is $\mathcal{C} \cup \mathcal{D}$. Thus a horizontal edge can carry a color or supercolor (but not both).

The vertical edges carry color-supercolor pairs, such as (c_i, d_j). There are mn such pairs. A vertical edge may carry several of those, but the models are fermionic, so it may not carry multiple pairs. So the spinset of the vertical edge is the power set of $\mathcal{C} \cup \mathcal{D}$, and its cardinality is 2^{mn}. Here we illustrate a state of such a system with 3 colors and 3 supercolors. The paths of colors are represented by solid lines, and supercolor paths are represented by broken lines.



We omit the Boltzmann weights, which can be described by fusion (Lecture 16) of mn monochrome edges, each of which can carry only a single color-supercolor pair.

We note that the color-supercolor pairs are in bijection with the mn odd negative roots of $\mathfrak{g} = \mathfrak{gl}(m|n)$, so by (50), the Kac module of a one-dimensional representation of $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ can be identified with the exterior algebra of the free-vector space on the set $\mathcal{C} \times \mathcal{D}$ of such pairs. Its dimension is 2^{mn}. Hence it is natural to believe that the module associated with these vertical edges is such a Kac module.

LECTURE 19

The Fermionic Fock Space

1. Introduction

In Lecture 17 we saw that in the field-free six-vertex model there is a Hamiltonian H and also a commuting family of six-vertex model row transfer matrices T_θ acting on a Hilbert space, which in that case was $\mathcal{H} = \otimes^N \mathbb{C}^2$. The main theorem is that H commutes with T_θ, which was proved by showing that $H = (T_\theta^{-1} T'_\theta)|_{\theta=\chi} + cI_{\mathcal{H}}$ for a suitable constant c. This result was proved by Baxter, in the greater generality of the 8 vertex model.

For the free-fermionic six-vertex model, there is a similar result, due to Brubaker and Schultz [**22**]. In the proof (Lecture 20) we will follow [**16**], where a more general result is proved. (The models in [**16**] may be regarded as generalizations of the result in [**22**] to a colored model.) In this free-fermionic case there is a Hamiltonian operator H and a row transfer matrix T, and the result is now in the form $e^H = T$. But the conclusion is the same: the Hamiltonian H commutes with the row transfer matrix T.

The identity $e^H = T$ can be thought of as an expansion of T in terms of operators J_k which move particles right or left to lower or higher energy levels. If $k > 0$, then J_k moves the particle right to a lower energy level, and if $k < 0$ it moves the particle to the left. There are correspondingly two versions of both the Hamiltonian, and two versions of the row transfer matrix.

2. The Fermionic Fock space

The *fermionic Fock space* was invented by Dirac in the theory of the electron. The electron is described by the *Dirac equation*, which we will not discuss, except to mention that the energy levels are quantized, and there are solutions of arbitrary negative energy. This seems unphysical, since a particle could radiate an arbitrarily large amount of energy by falling to lower and lower energy levels.

But Dirac proposed a solution to this. Since the Dirac equation is linear, solutions can exist in superposition. The electron is a fermion, subject to the Pauli exclusion principle, meaning that no two electrons can occupy the same state. Dirac's proposal was that all sufficiently large negative energy level states are occupied, and all sufficiently large positive energy levels are unoccupied.

Mathematically, the states are vectors in a Hilbert space that is now called the *fermionic Fock space* $\mathfrak{F}$, which we will now describe. This is based on another Hilbert space that we will call V, with basis vectors u_i ($i \in \mathbb{Z}$). Each u_i represents a particle with a definite energy level equal to i. Let us fix $m \in \mathbb{Z}$ and consider a sequence $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ where $j_m > j_{m-1} > \cdots$ and $j_k = k$ for k sufficiently negative. Define the *charge m fermionic Fock space*, denoted $\mathfrak{F}_m$ to be the free vector space on formal symbols

$$|\mathbf{j}\rangle := |\mathbf{j}\rangle_m = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots , \qquad \mathbf{j} = (j_m, j_{m-1}, j_{m-2}, \cdots). \tag{51}$$

The Fock space $\mathfrak{F}$ resembles the exterior algebra $\bigwedge V$, except that the basis vectors are *infinite* wedges (called *semi-infinite monomials*).

We extend the notation $\xi_{\mathbf{j}}$ to sequences $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ where $j_k = k$ for k sufficiently negative, dropping the assumption that the sequence is strictly decreasing, by the usual rules for $\wedge$ in the exterior algebra. Thus $|\mathbf{j}\rangle = 0$ if $j_k = j_l$ for any distinct $k, l < m$. And interchanging two adjacent indices changes the sign of $|\mathbf{j}\rangle$.

We can visualize the vector $|\mathbf{j}\rangle$ by a *Maya diagram* in which sites numbered by integers are filled with stones. If the site n equals j_k for some k, the site is *occupied*, otherwise it is *unoccupied*. We put a black stone at the occupied sites, and a white stone at the unoccupied sites.

For example, if $\mathbf{j} = (4, 2, -1, -2, -3, -4, \cdots)$, so

$$|\mathbf{j}\rangle = u_4 \wedge u_2 \wedge u_{-1} \wedge u_{-2} \wedge u_{-3} \wedge u_{-4} \wedge \cdots$$

then the Maya diagram looks like this:

$$\begin{array}{ccccccccccccc} \cdots & 6 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & \cdots \\ \cdots & \circ & \circ & \bullet & \circ & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \cdots \end{array}$$

The main point is that every sufficiently negative site is occupied, and every sufficiently positive site is unoccupied. Although Maya diagrams are traditional (originating in soliton theory with M. Sato and his collaborators), because we want to relate this story to the six vertex model as we have been we prefer to use $-$ and $+$ for the occupied and unoccupied sites respectively, so the Maya diagram looks like this:

$$\begin{array}{ccccccccccccc} \cdots & 6 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & \cdots \\ \cdots & \oplus & \oplus & \ominus & \oplus & \ominus & \oplus & \oplus & \ominus & \ominus & \ominus & \ominus & \cdots \end{array}$$

For this state the charge $m = 1$.

If $j_k = k$ for all $k \leqslant m$ then we obtain the charge m *vacuum vector* for which we have an alternative notation

$$|\varnothing\rangle_m = u_m \wedge u_{m-1} \wedge \cdots .$$

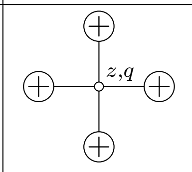
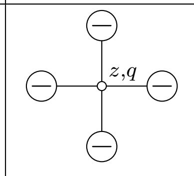
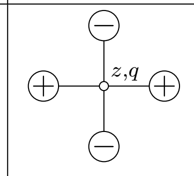
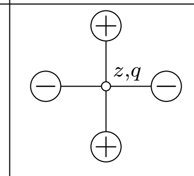
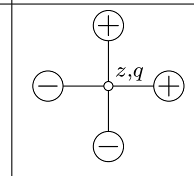
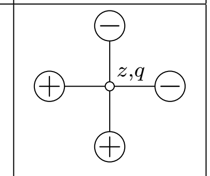
In general we may define the *energy* of $|\mathbf{j}\rangle_m$ to be $\sum_{k \leqslant m}(j_k - k)$. This is a finite sum. The vacuum is the unique semi-infinite monomial in $\mathfrak{F}_m$ of energy 0.

3. The Row Transfer matrix $T_\Delta(z;q)$

We will describe a kind of free-fermionic six-vertex model that we will call *Delta ice*. The grid will now be of infinite width, and the Boltzmann weights in each row will depend on a parameter $z \in \mathbb{C}^\times$.

Remark 10. The Δ here is different from Baxter's Δ, which is $(a_1a_2 + b_1b_2 - c_1c_2)/2a_1b_1$. Baxter's Δ is zero here, since all weights in this lecture are free-fermionic.

Now let $\mathbf{i} = (i_m, i_{m-1}, \cdots)$ and $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ be two sequences such that $i_m > i_{m-1} > \cdots$ and $j_m > j_{m-1} > \cdots$ and $i_i = j_k = k$ for k sufficiently negative. We will define a simple system consisting of a single row, and either no states or a single state. We consider a grid with only one row that is infinite in both directions. As boundary conditions, the spins of the vertical edges at the top will be given by the Maya diagram for $\xi_{\mathbf{i}}$, and for the vertical edges at the bottom, by the Maya diagram for $\xi_{\mathbf{j}}$. There is also a "boundary condition" for the horizontal edges, that there are only finitely many + spins. We use these Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Δ-ice						
	1	$-qz$	1	z	$(1-q)z$	1

Since as part of the boundary conditions there are only finitely many horizontal edges with + spins, all but finitely factors in the Boltzmann weight of a state are of type $\mathtt{b}_1$ (for vertices far to the left) or of type $\mathtt{a}_1$ (for vertices far to the right). Therefore the Boltzmann weight of a state is an infinite product with only finitely many terms not equal to 1, and so has a well-defined finite value.

Lemma 3.1. *The condition for the partition function to have a state (which is therefore unique) is that*

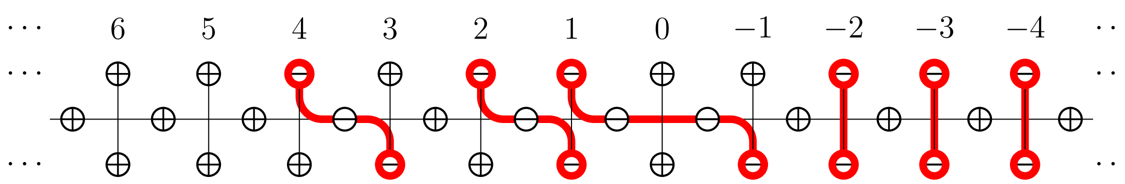
$$i_m \geqslant j_m \geqslant i_{m-1} \geqslant j_{m-2} \geqslant \cdots \,. \tag{52}$$

We express equation (52) by saying that the sequences $\mathbf{i}$ and $\mathbf{j}$ *interleave*.

Proof. This may be seen by consideration of the paths, which we recall from Lecture 2 Section 2 are obtained by joining edges with spin $-$. Because of our boundary condition, that there are only finitely many horizontal edges with spin $-$, each path must begin at the top and exit at the bottom for this system. For example, suppose that $m = 1$ and

$$\mathbf{i} = (4, 2, 1, -2, -3, -4, \cdots), \qquad \mathbf{j} = (3, 1, -1, -2, -3, -4, \cdots).$$

Then we have the following state.



Every path must start in the i_k column and end in the j_k column. Call this the k-th path. We must have $i_k \geqslant j_k$ since the paths move down and to the right. We also need $j_k \geqslant i_{k-1}$ since otherwise two paths will overlap between the i_{k-1} column and the j_k column. $\square$

We quickly review Dirac notation for operators. Let $\mathcal{H}$ be a Hilbert space. A vector in $v \in \mathcal{H}$ is denoted alternatively as $|v\rangle$, and called a *ket*. On the other hand, a vector w gives rise to a linear functional $v \to (v, w)$ using the inner product on $\mathcal{H}$, and we denote this linear functional as $\langle w|$. The notation works well in quantum mechanics due to the emphasis on Hermitian (self-adjoint) operators. If T is Hermitian, then $(Tv, w) = (v, Tw)$, which we denote $\langle w|T|v\rangle$. We can either think of this as either the linear functional $\langle w|$ applied to the vector $T|v\rangle$, or as the linear functional $\langle w|T$ applied to the vector v.

As a special case, the partition function of the monostatic system above will be denoted

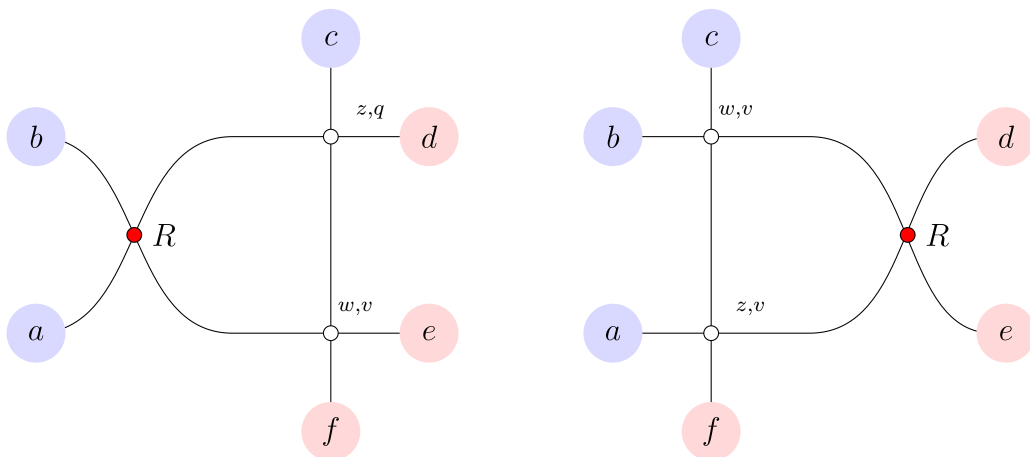
$$\langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle,$$

and we are now thinking of $T_\Delta(z;q)$ as being an operator on $\mathcal{H}$.

THEOREM 3.2. *The operators $T_\Delta(z;q)$ all commute. That is, if w and v are other parameters, we have*

$$T_\Delta(z;q)T_\Delta(w,v) = T_\Delta(w;v)T_\Delta(z,q).$$

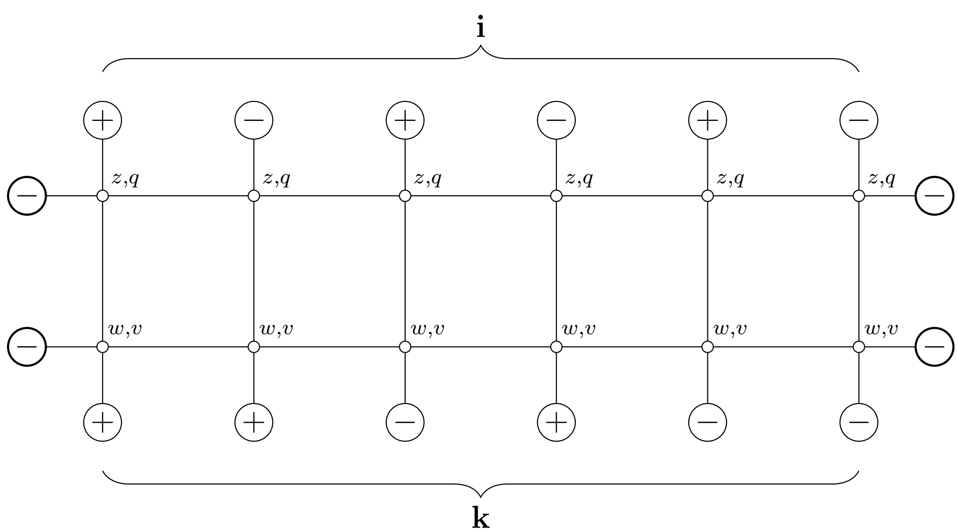
PROOF. We make use of the general free-fermionic Yang-Baxter equation from Lecture 7. By Theorem 1.1 of Lecture 7, there exists an R-matrix R depending on z, q, w, v such that we have a Yang-Baxter equation in the form



It is of course not hard to compute the Boltzmann weights but we do not need them for this proof. We only need that the $\mathtt{a}_2$ weight of R is nonzero. We fix $\mathbf{i}$ and $\mathbf{k}$ and will show that

$$\langle\mathbf{k}|T_\Delta(w;v)T_\Delta(z,q)|\mathbf{i}\rangle = \langle\mathbf{k}|T_\Delta(z;q)T_\Delta(w,v)|\mathbf{i}\rangle. \tag{53}$$

The left-hand side is the partition function of a 2-rowed infinite grid, but we may truncate this to a finite grid such that all sites of $|\mathbf{i}\rangle$ and $|\mathbf{k}\rangle$ to the right are occupied, and all sites to the left are unoccupied. This partition function looks like this:



All vertices outside this finite grid have type $\mathtt{a}_1$ or $\mathtt{b}_1$, and Boltzmann weight 1, so discarding them does not change the partition function. So the partition function of this system is

$$\langle\mathbf{k}|T_\Delta(w;v)T_\Delta(z;q)|\mathbf{i}\rangle.$$

Now we attach the R-matrix, which multiplies the Boltzmann weight by $a_2(R)$. We apply the train argument, and discard the R-matrix on the right, which divides the Boltzmann weight by the same constant $a_2(R)$. The resulting system has the rows switched, proving (53). Since this is true for all $\mathbf{i}$ and $\mathbf{k}$, the row transfer matrices are proved to commute. $\square$

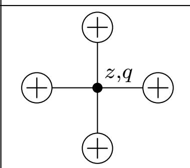
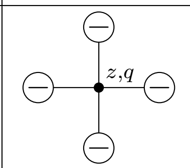
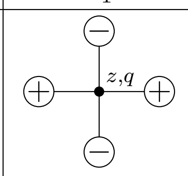
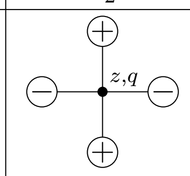
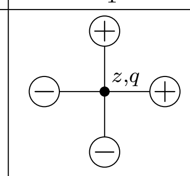
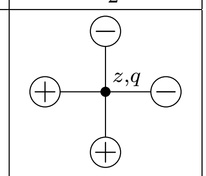
We can define $T_\Delta(z;q)$ as an operator on $\mathfrak{F}$ by

$$T_\Delta(z;q)|\mathbf{i}\rangle = \sum_{\mathbf{j}} \langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle\, |\mathbf{j}\rangle. \tag{54}$$

The sum on the right is finite, so this defines an element of $\mathfrak{F}$. However $T_\Delta(z;q)$ is not a bounded operator. That is, if we make $\mathfrak{F}$ into a Hilbert space where the semi-infinite monomials $|\mathbf{i}\rangle$ are an orthonormal basis, since the number of terms on the right side of (54) can be arbitrarily large, the map $T_\Delta(z;q)$ defined on basis elements does extend to an operator with bounded operator norm.

4. The Row Transfer Matrix $T_\Gamma(z;q)$

There is another type of six-vertex model that is in a sense dual to the models in Section 3. For these we use the following Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Γ-ice						
	z^{-1}	1	$-qz^{-1}$	1	$1-q$	z^{-1}

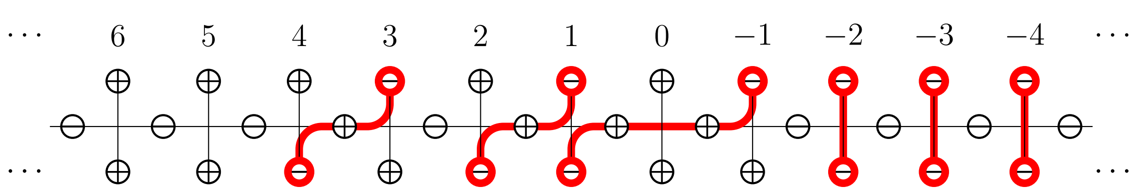
Remark 11. These are the same as the weights Tokuyama models introduced in Lecture 5, Section 2, divided by z. Since every weight is divided by the same constant, we could use these weights in the Tokuyama model, and the partition functions would be essentially unchanged, altered only be a constant monomial. However our boundary conditions will be different from the Tokuyama models.

Now we change the boundary conditions. We will requre all but finitely many horizontal spins to be $-$. This guarantees that the row transfer matrix will be an essentially finite product, since all but finitely many spins will be of type $\mathtt{a}_2$ or $\mathtt{b}_2$.

We can define $\langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle$ as before, but now the condition for this to be nonzero is changed: now we require

$$j_m \geqslant i_m \geqslant j_{m-1} \geqslant i_{m-1} \geqslant \cdots\ . \tag{55}$$

Here is a sample state with $\mathbf{i} = (3, 1, -1, -2, -3, \cdots)$ and $j = (4, 2, 1, -2, -3, \cdots)$. We modify the rule for describing the paths: now the paths follow the $-$ spins on vertical edges, and $+$ spins on the horizontal edges. This means that the paths move down and to the left, so the row transfer matrix is energy raising, in accordance with (55).



We can try to define $T_\Gamma(z;q)$ as an operator,

$$T_\Gamma(z;q)|\mathbf{i}\rangle = \sum_{\mathbf{j}} \langle \mathbf{j}|T_\Gamma(z;q)|\mathbf{i}\rangle \, |\mathbf{j}\rangle.$$

However (in contrast with Δ-ice) the sum on the right-hand side is no longer finite.

5. The Heisenberg Lie Algebra

We now come to a representation of the *Heisenberg Lie algebra* $\mathfrak{s}$ with generators

$$\{j_k | k \in \mathbb{Z}\} \qquad \text{and} \qquad \mathbf{1},$$

with $\mathbf{1}$ central, and

$$[j_k, j_l] = \begin{cases} k & \text{if } k = -l, \\ 0 & \text{otherwise.} \end{cases}$$

The center of $\mathfrak{s}$ is spanned by $\mathbf{1}$ and j_0. This representation is at the heart of the *boson-fermion correspondence*. This is a relationship between the fermionic Fock space and the bosonic Fock space which originated in mathematical physics, and has important applications to representation theory and algebraic combinatorics ([**38, 57, 69**]).

We remind the reader that we have defined

$$u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots$$

even if we do not have $j_m \geqslant j_{m-1} \geqslant \cdots$. It is only necessary that $j_k = k$ for k sufficiently negative. However this monomial might be zero (if some index is repeated) or the negative of a basis element if putting the vectors in order produces an odd number of sign changes. If $j_m \geqslant j_{m-1} \geqslant \cdots$ we will denote this vector as $|\mathbf{j}\rangle$. Otherwise we will avoid this notation.

Let $k \in \mathbb{Z}$. For the time being assume that $k \neq 0$. We define an operator J_k on V by $J_k(u_n) = u_{n-k}$. Then we transport J_k to acting on $\mathfrak{F}$ by the Leibnitz rule, so that

$$J_k|\mathbf{j}\rangle = (u_{j_m-k} \wedge u_{j_{m-1}} \wedge \cdots) + (u_{j_m} \wedge u_{j_{m-1}-k} \wedge \cdots) + \cdots \, .$$

In other words, to apply J_k, we pick one occupied location, and move the particle at that location k steps lower or higher (depending on the sign of k) to an unoccupied location. We also define J_0 to have eigenvalue m on $\mathfrak{F}_m$.

THEOREM 5.1. *The operators J_k on $\mathfrak{F}_m$ satisfy*

$$[J_k, J_l] = \begin{cases} k \cdot 1_{\mathfrak{F}_m} & \textit{if } k = -l, \\ 0 & \textit{otherwise.} \end{cases}$$

Hence $j_k \mapsto J_k$ defines a representation of the Heisenberg Lie algebra.

PROOF. Let us first show that

$$J_k J_{-k}|\mathbf{j}\rangle - J_{-k} J_k |\mathbf{j}\rangle = k|\mathbf{j}\rangle. \tag{56}$$

We may assume $k > 0$ since the statements for k and $-k$ are trivially equivalent.

First suppose that $|\mathbf{j}\rangle = |\varnothing\rangle_m$ is the vacuum. Then $J_k|\varnothing\rangle_m = 0$. On the other hand, $J_{-k}|\varnothing\rangle$ is a sum of k terms, and applying J_k to each of these produces a copy of $|\varnothing\rangle_m$. Now we prove (56) for general $\mathbf{j}$. If $|\mathbf{j}\rangle = |\mathbf{j}\rangle_m$ is not the vacuum may write $|\mathbf{j}\rangle_m = u_j \wedge \eta$ where $j = j_m$ and

$$\eta = u_{j_{m-1}} \wedge u_{j_{m-1}} \wedge \cdots$$

has strictly smaller energy than $|\mathbf{j}\rangle_m$. By induction on enery we may assume that (56) is true for η. Now we have $J_{-k} = u_{j+k} \wedge \eta + u_j \wedge J_{-k}\eta$ and so

$$J_k J_{-k}|\mathbf{j}\rangle_m = u_j \wedge \eta + u_{j+k} \wedge J_k\eta + u_{j-k} \wedge J_{-k}\eta + u_j \wedge J_k J_{-k}\eta.$$

Similarly

$$J_{-k} J_k|\mathbf{j}\rangle_m = u_j \wedge \eta + u_{j-k} \wedge J_{-k}\eta + u_{j+k} \wedge J_k\eta + u_j \wedge J_{-k} J_k\eta.$$

Subtracting,

$$J_k J_{-k}|\mathbf{j}\rangle_m - J_{-k} J_k|\mathbf{j}\rangle_m = u_j \wedge (J_k J_{-k}\eta - J_{-k} J_k\eta) = u_j \wedge k\eta = k|\mathbf{j}\rangle_m,$$

where we have used our induction hypothesis.

We leave it to the reader to show that J_k and J_l commute unless $k = -l$. □

6. Row Transfer Matrices as Vertex Operators

We emphasize that the J_k with $k > 0$ all commute, and the J_{-k} with $-k < 0$ all commute, so we have two large commuting families of "operators" on $\mathfrak{F}$ or $\mathfrak{F}_m$. The J_{-k} are not operators in the usual sense, since each turns each basis vector into an infinite sum of basis vectors, which is not in $\mathfrak{F}$. Still, the two-point functions

$$\langle \mathbf{i}|J_k|\mathbf{j}\rangle$$

do make sense for all k, and as long as we couch our results in terms of these, there are no difficulties.

Now let us introduce two "Hamiltonians"

$$H_+(z;q) = \sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^k J_k, \qquad H_-(z;q) = \sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^{-k} J_{-k}$$

THEOREM 6.1 ([**22**]). *We have*

$$e^{H_+(z;q)} = T_\Delta(z;q), \qquad e^{H_-(z;q)} = T_\Gamma(z;q), \tag{57}$$

The operator $H_+(z;q)$ commutes with $T_\Delta(w;v)$ for all w,v, and the operator $H_-(z;q)$ commutes with $T_\Gamma(w;v)$ for all w,v.

PROOF. We will take this up in Lecture 20. For now we point out that the identities (57) imply the commutativity statements, since for example the operators $T_\Delta(w;v)$ and the operator $H_+(z;q)$ are all seen to be expressible in terms of the J_k with $k > 0$, which commute with each other. We also obtain a new proof of the commutativity statement in Theorem 3.2 from this observation. □

"Operators" such as $e^{H_+(z;q)}$ and $e^{H_-(z;q)}$, particularly in combinations such as:

$$e^{H_-(z;q)} e^{H_+(z;q)} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^{-k} J_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^k J_k\right) \tag{58}$$

are called *vertex operators*. Here "operators" is in quotation marks since there is a nontrivial problem in making sense of this. Similar expressions appear in conformal field theory and in soliton theory. A purely algebraic and rigorous axiomatization of the underlying mathematics may be found in the theory of *vertex algebras*. In this context, expressions such as (58) appear in *lattice vertex algebras* ([**37**] Chapter 5 or [**54**] Section 5.4). See also [**58**] and [**52**].

LECTURE 20

Fermionic Operators

This lecture contains the proof of Theorem 6.1 of Lecture 19, expressing the row transfer matrix $T_\Delta(z;q)$ as the exponential of the Hamiltonian

$$H_+(z;q) = \sum_{k=1} \frac{1}{k}(1-q^k)z^k J_k.$$

There is a corresponding result for T_Γ and H_+ but we will omit that. (It can be deduced from the T_Δ case by taking adjoints, as at the end of Section 4 in [**16**].)

1. Fermionic operators

We introduce *fermionic creation operators* ψ_n^* ($n \in \mathbb{Z}$) on $\mathfrak{F}$ that create particles by

$$\psi_n^*(\eta) = u_n \wedge \eta.$$

If η is a basis vector of $\mathfrak{F}_m$, say

$$\eta = |\mathbf{j}\rangle = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots ,$$

then $\psi^*(\eta) = 0$ if n is among the indices $j_m, j_{m-1}, \cdots$. Otherwise, $\psi_n^*(\eta)$ can be calculated by moving u_n to its proper place among the indices. This can involve interchanging some u_j, which can introduce sign changes and so $\psi_n^*(\eta)$ is either zero or $\pm|\mathbf{j}'\rangle$, where $\mathbf{j}'$ is obtained by sorting $\{n, j_m, j_{m-1}, \cdots\}$ into descending order. We see that $\psi_n^* : \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+1}$.

Dual to the creation operators ψ_n^* are their adjoints $\psi_n : \mathfrak{F}_{m+1} \longrightarrow \mathfrak{F}_m$. The operator ψ_n deletes u_n from the semi-infinite monomial if $n \in \{j_m, j_{m-1}, \cdots\}$, which can result in a sign change. If $n \notin \{j_m, j_{m-1}, \cdots\}$ then $\psi_n|\mathbf{j}\rangle = 0$.

Lemma 1.1. *We have*

$$[J_k, \psi_j^*] = \psi_{j-k}^*.$$

Proof. From the Leibnitz rule, if $\eta \in \mathfrak{F}$, then

$$J_k\psi_j^*\eta = J_k(u_j \wedge \eta) = J_k(u_j) \wedge \eta + u_j \wedge J_k(\eta) = u_{j-k} \wedge \eta + \psi_j^*(J_k\eta).$$

Rearranging,

$$[J_k, \psi_j^*]\eta = u_{j-k} \wedge \eta = \psi_{j-k}^*(\eta).$$

□

Now let us introduce the *fermion field*

$$\psi(x) = \sum_{j\in\mathbb{Z}} \psi_j^* x^j.$$

For our purposes this is just a formal expression that we can use to do a calculation. (The "field" terminology comes from quantum field theory.)

PROPOSITION 1.2. *We have*

$$[H_+(z;q),\psi^*(x)] = \log\left(\frac{1-qxz}{1-xz}\right)\psi^*(x). \tag{59}$$

PROOF. Note that by Lemma 1.1 we have

$$[J_k,\psi^*(x)] = \sum_j x^j[J_k,\psi_j^*] = \sum_j x^j[J_k,\psi_j^*] = \sum_j x^j\psi_{j-k}^* = x^k\psi^*(x).$$

Now the left-hand side of (59) equals

$$\sum_k \frac{1}{k}(1-q^k)z^k[J_k,\psi^*(x)] = \sum \frac{1}{k}(1-q^k)(xz)^k\psi^*(x) = -\log(1-xz)+\log(1-qxz)$$

from the identity

$$-\log(1-t) = \sum_{k=1}^{\infty}\frac{t^k}{k}.$$

□

LEMMA 1.3. *Suppose that $xa - ax = ca$, where $c \in \mathbb{C}^\times$. Then*

$$e^x a e^{-x} = e^c a.$$

PROOF. This is a special case of the Baker-Campbell-Hausdorff formula. We treat this as a formal identity, disregarding convergence. We need the following identity, for $k \geqslant 0$:

$$\sum_j \binom{k}{j}(-1)^j x^{k-j}ax^j = c^k a \tag{60}$$

To avoid some bookkeeping we sum over all $j \in \mathbb{Z}$ but regard $\binom{k}{j}$ as zero unless $0 \leqslant j \leqslant k$, so most terms are zero. Assuming this true for $k-1$, we may establish (60) by induction, writing $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$. The left-hand side equals

$$x\cdot\left[\sum_j \binom{k-1}{j-1}(-1)^j x^{k-1-j}ax^j\right] - \left[\sum_j \binom{k-1}{j-1}(-1)^{j-1}x^{k-j}ax^{j-1}\right]\cdot x.$$

Both terms in brackets equal $c^{k-1}a$ by induction, so we obtain $c^{k-1}[x,a] = c^k a$. This proves (60).

Now expand the exponentials and collect terms of degree k to write

$$e^x a e^{-x} = \sum_k \frac{1}{k!}\sum_j \binom{k}{j}(-1)^j x^{k-j}ax^j = \sum_k \frac{1}{k!}c^k a = e^c a,$$

as required. □

PROPOSITION 1.4. *Let $H = H_+(z;q)$. We have*

$$e^H\psi^*(x)e^{-H} = \frac{1-qxz}{1-xz}\psi^*(x). \tag{61}$$

PROOF. This follows from our Proposition 1.2 by exponentiating (using Lemma 1.3). □

Now the key point is to show that the row transfer matrices $T_\Delta(z;q)$ satisfy the same identity in Proposition 4. Let us introduce the operator $\rho_k(z) : \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+1}$ defined by:

$$\rho_k(z) = \psi_k^* - z\psi_{k-1}^*$$

LEMMA 1.5. *Granted the invertibility of e^H, the identity (61) is equivalent to*

$$e^H \rho_k(z) = \rho_k(qz)e^H. \tag{62}$$

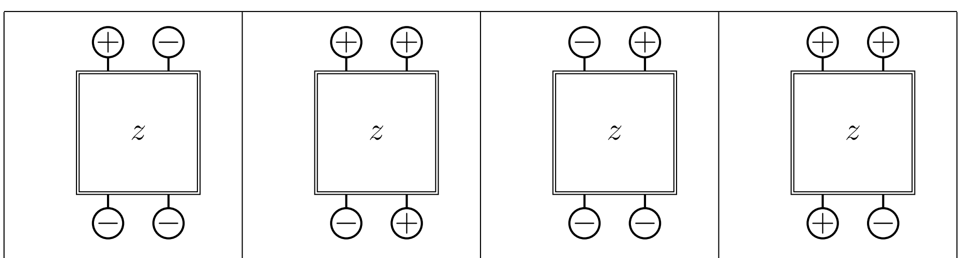
for $k \in \mathbb{Z}$.

PROOF. We rewrite (61) in the form

$$(1 - xz)e^H \psi^*(x) = (1 - qxz)\psi^*(x)e^H.$$

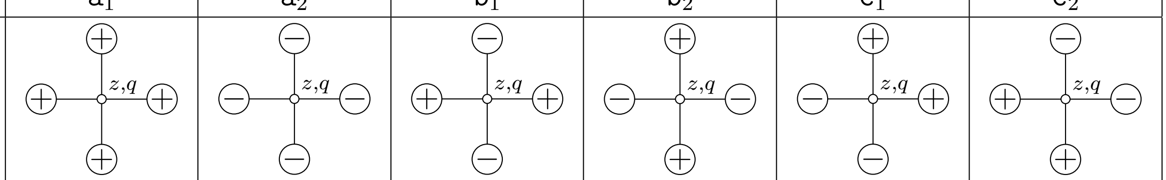
This is a formal identity that can be expanded in powers of x. Comparing the coefficient of x^k gives exactly the identity (62). □

Our goal is to show that the row $T = T_\Delta(z;q)$ satisfies the same identity $T\rho_k(z) = \rho_k(qz)T$ as e^H. Let us represent ρ_k graphically as a "gate" that can be attached to the lattice model. Remembering that ψ_k creates a particle in the k-th column, and that $+$ denotes the absence of a particle, $-$ its presence, we see that we have the following Boltzmann weights:



1	1	$-z$	$-z$

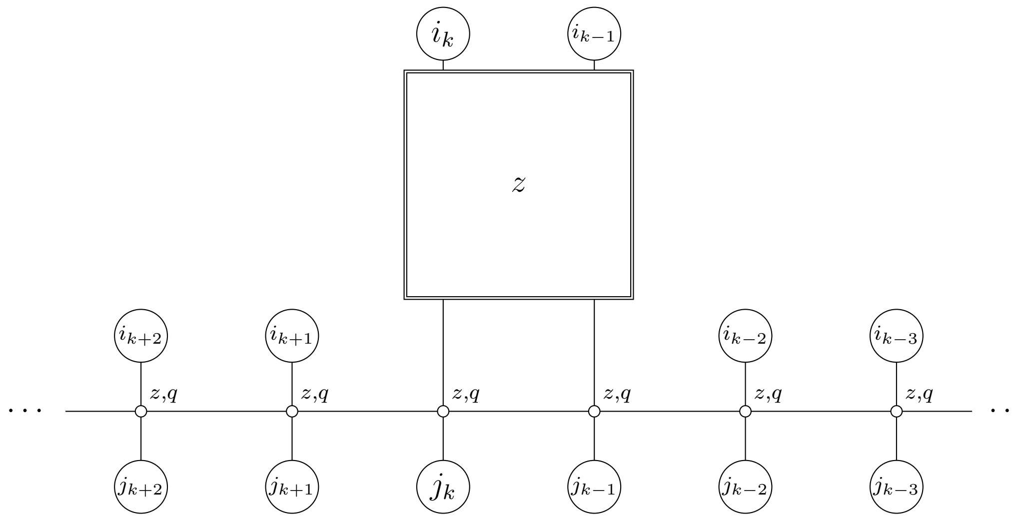
For reference, here are the Delta Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Δ-ice						
	1	$-qz$	1	z	$(1-q)z$	1

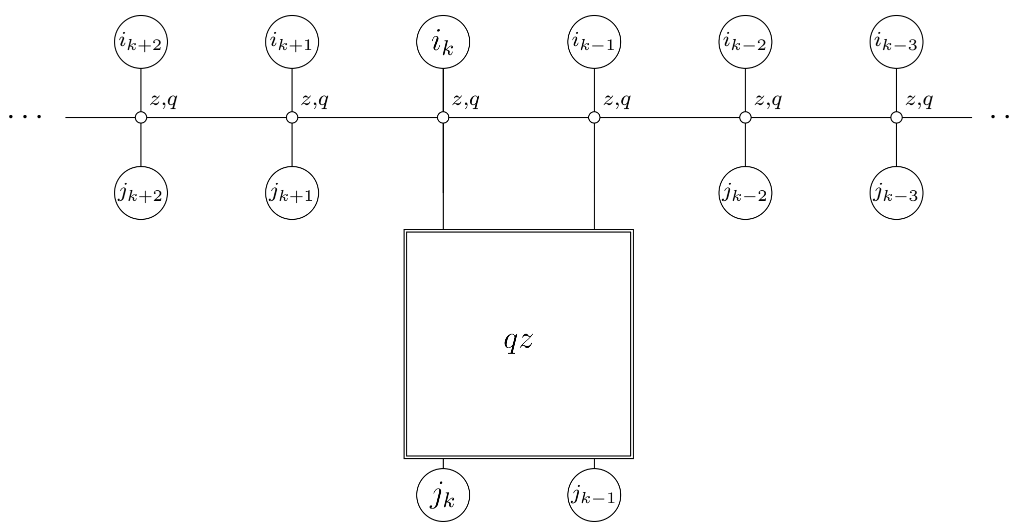
PROPOSITION 1.6. *The row transfer matrix*

$$T\rho_k(z) = \rho_k(qz)T.$$

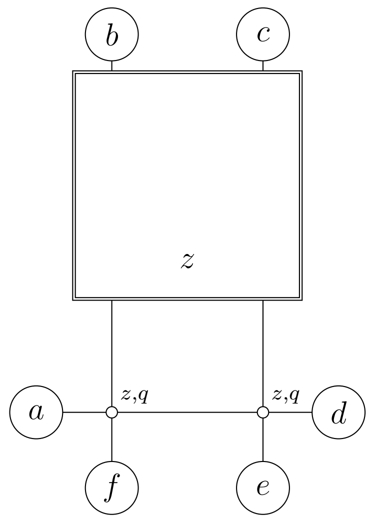
PROOF. Graphically this means that we must show the equivalence of the two following partition functions:



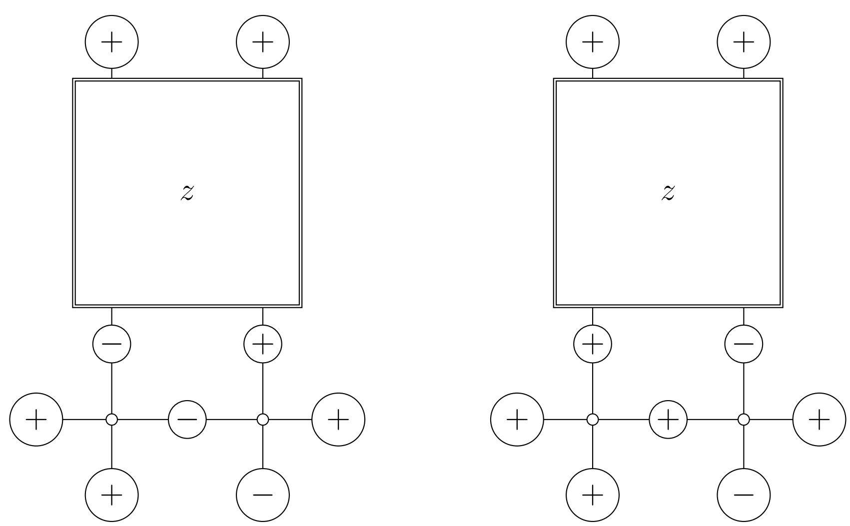
and



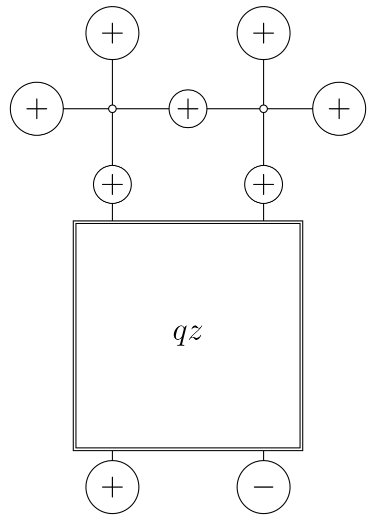
We can clip out the middle part and just prove the equivalence of these systems:



This can be thought of as a kind of a Yang-Baxter equation, but of the sort mentioned in Lecture 16 Section 1, where the R-matrix changes as it moves past the vertices. This verification is now subject to case by case verification. Let us check just one case. Suppose that the boundary values are $(a, b, c, d, e, f) = (+, +, +, +, -, +)$. On the left-hand side there are two admissible states:



Their Boltzmann weights are, respectively $(1-q)z$ and $-z$, for a total of $-qz$. On the right-hand side there is only one admissible state:



The Boltzmann weight is $-qz$. Since $(1-q)z+(-z)=-qz$, the required identity is satisfied in this case, and the remaining cases are similar. $\square$

2. Proof of Theorem 6.1 of Lecture 19

We will only prove that $e^{H_+(z;q)} = T_\Delta(z;q)$. The identity $e^{H_-(z;q)} = T_\Gamma(z;q)$ can be deduced using adjointness considerations, as in [**16**].

As in the last section, we abbreviate $H = H_+(z;q)$ and $T = T_\Delta(z;q)$. We have proved that both operators e^H and T both satisfy the same identities

$$e^H \rho_k(z) = \rho_k(qz) e^H, \qquad T\rho_k(z) = \rho_k(qz)T.$$

We need to show that there is enough information in this fact to deduce that $T|\mathbf{j}\rangle = e^H|\mathbf{j}\rangle$ for every semi-infinite monomial $|\mathbf{j}\rangle \in \mathfrak{F}$.

Recall that the *energy* of $|\mathbf{j}\rangle$, with $\mathbf{j} = (j_m, j_{m-1}, \cdots) \in \mathfrak{F}_m$ is $\sum_k (j_k - k)$. This is actually a finite sum. The unique basis vector in $\mathfrak{F}_m$ of energy 0 is the vacuum

$$|\varnothing\rangle_m = u_m \wedge u_{m-1} \wedge \cdots .$$

The identity

$$e^{H_-(z;q)}|\varnothing\rangle_m = T_\Delta(z;q)|\varnothing\rangle_m$$

is clear since both sides are $|\varnothing\rangle_m$.

So assume that $|\mathbf{j}\rangle_m$ is not the vacuum. Then it has positive energy. This means $j_m > m$. We will show

$$e^{H_-(z;q)}|\mathbf{j}\rangle_m = T_\Delta(z;q)|\mathbf{j}\rangle_m. \tag{63}$$

We are assuming inductively that the identity is known for states of lower energy.

Let $|\mathbf{j}'\rangle = u_{j_{m-1}} \wedge u_{j_{m-2}} \wedge \cdots \in \mathfrak{F}_m$, so $|\mathbf{j}\rangle_m = \psi^*_{j_m}|\mathbf{j}'\rangle_{m-1}$. We have

$$|\mathbf{j}\rangle_m = \rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1} + z\xi, \tag{64}$$

where

$$\xi = u_{j_m-1} \wedge |\mathbf{j}'\rangle.$$

Now *both* terms on the right-hand side of (64) have lower energy than $|\mathbf{j}\rangle_m$. It is possible that $\xi = 0$ (if $j_{m-1} = j_m - 1$) but if $\xi \neq 0$ it has lower energy than $|\mathbf{j}\rangle_m$. So by our induction hypothesis

$$e^H|\mathbf{j}'\rangle_{m-1} = T|\mathbf{j}'\rangle_{m-1}, \qquad e^H\xi = \xi.$$

Now we have

$$e^H|\mathbf{j}\rangle_m = e^H\rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1} + ze^H\xi = \rho_{j_m}(qz)e^H|\mathbf{j}'\rangle_{m-1} + ze^H\xi,$$

$$T|\mathbf{j}\rangle_m = T\rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1} + zT\xi = \rho_{j_m}(qz)T|\mathbf{j}'\rangle_{m-1} + zT\xi,$$

and using (2) we obtain (63). So the theorem is proved.

3. Delta Ice and U-Turn models

Delta ice, which we introduced in Lecture 19, plays well with Gamma ice, and they often appear together. The distinction between them is in the horizontal edges, not the vertical. This situation persists in the colored case.

The Yang-Baxter equation can be used to prove:

- The row transfer matrices $T_\Gamma(z)$ commute with each other for varying z.
- The row transfer matrices $T_\Delta(z)$ commute with each other for varying z.

(There are versions of these statements for both infinite and finite grids.)

But the row transfer matrices $T_\Gamma(z)$ and $T_\Delta(w)$ do not commute, though

$$T_\Gamma(z)T_\Delta(w) = \text{const} \times T_\Delta(w)T_\Gamma(z),$$

for a computable constant. This can be proved using the Yang-Baxer equation. For the infinite grids, it can also be deduced from the $T_\Gamma(z) = e^{H_-(z;q)}$ and $T_\Delta(w) = e^{H_+(z;q)}$ using the technique of the first section of this lecture.

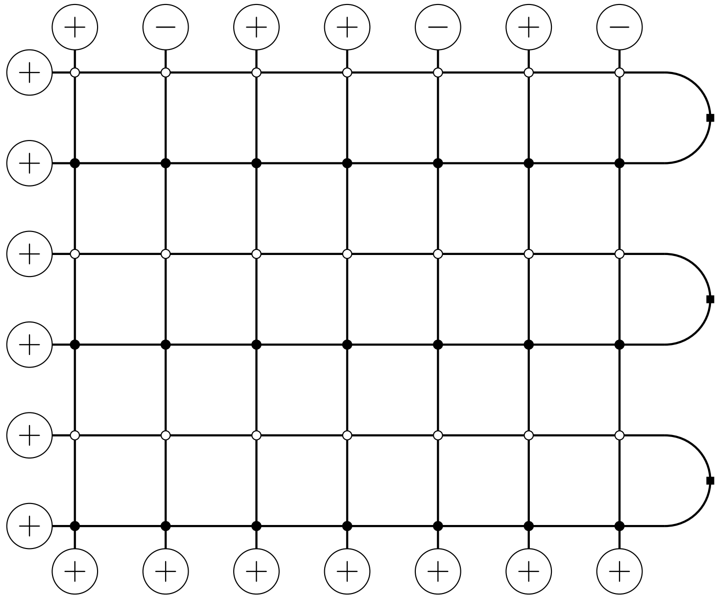
Gamma ice and Delta ice appeared in [**19**]. (Use the arxiv version of this paper.) We considered Gamma ice in Lectures 5 and 6, and computed the partition function as

$$s_\lambda(\mathbf{z})\prod_{i<j} x_i - qx_j.$$

There is a similar Tokuyama result for Delta ice.

Ivanov [**48**] gave a Tokuyama result for characters of symplectic groups. The lattice models had been considered previously by Hamel and King [**41**], but we prefer Ivanov's treatment since Hamel and King do not use the Yang-Baxter equation, but instead combinatorial arguments based on *jeu de taquin*. (They also preceded [**19**] in reinterpreting the formula of Tokuyama [**91**] in terms of lattice models.)

The models look like this, with alternating rows of Gamma and Delta ice:



Boltzmann weights at the "cap" vertices on the right edge must be specified, resulting in what is sometimes called "U-turn models." We are changing the Boltzmann weights from Ivanov by switching + and − on the Delta ice. With this convention, the "paths" switch from + and − when they cross a cap. Paths move right on the Delta rows (marked ∘) along the − horizontal edges, and left along the Gamma rows (marked •) eventually exiting on the left.

Changing just the cap weights results in another interesting model [**18**] related to metaplectic Whittaker functions. Both the models of [**48, 18**] were vastly generalized in [**39**], which can now be understood as colored variants of the original model. U-turn models are also employed in [**79, 23, 94**]

U-turn models and other exotic variations of the standard grid go back to [**67**]. In [**90, 21**], many of these exotic variations of the grid are used in interesting models that are deformations of the Weyl character formula. However they are different from the results of [**41**], [**19**] and [**48**]. In those papers, models are exhibited whose partition functions are of the form

$$\prod_{\alpha\in\Phi^+}(1-q\mathbf{z}^{-\lambda})\chi_\lambda(\mathbf{z}),$$

where χ_λ is either a character of $\mathrm{GL}(n,\mathbb{C})$ (that is, a Schur function) or in Ivanov's case of $\mathrm{Sp}(2n,\mathbb{C})$. The character itself is undeformed. We will call such a result a *Tokuyama formula*. These are significant because of the similarity to the Casselman-Shalika formula [**28**]. The models of Tabony, Brubaker and Schultz are not Tokuyama results since the character itself is also deformed. Finding a Tokuyama formula for orthogonal groups is an open problem.

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