Lecture 9

1 Some Lie Theory

If G is a (reductive) Lie group, we may associate with G a *Weyl group* W, a *maximal torus* T, a *root system* Φ and a *weight lattice* Λ. Everything in this section generalizes to that setup. But we will specialize to the case $G = \mathrm{GL}(n, \mathbb{C})$.

Let $G = \mathrm{GL}(n, \mathbb{C})$, and let $T = (\mathbb{C}^\times)^n$, which we will embed in G via

$$\mathbf{z} = (z_1, \cdots, z_n) \longmapsto \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}.$$

The *weight lattice* $\Lambda = \mathbb{Z}^n$. If $\mathbf{z} \in T$ and $\mu = (\mu_1, \cdots, \mu_n) \in \Lambda$, we will denote

$$\mathbf{z}^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}.$$

Let $\mathcal{O}(T)$ be the ring of regular functions on T. This as the space spanned by the functions $\mathbf{z}^\mu$ with $\mu \in \Lambda$. Occasionally we may want to embed Λ in the vector space $\mathbb{R} \otimes \Lambda \cong \mathbb{R}^n$.

Let $\mathbf{e}_i$ be the standard basis of $\Lambda = \mathbb{Z}^n$. The *root system* $\Phi \subseteq \Lambda$ consists of the $n(n-1)$ vectors $\mathbf{e}_i - \mathbf{e}_j$ with $i \neq j$. Then $\Phi = \Phi^+ \cup \Phi^-$ (disjoint) where Φ^+ consists of the $\frac{1}{2}n(n-1)$ vectors $\mathbf{e}_i - \mathbf{e}_j$ with $i < j$, and Φ^- is the complement. The elements of Φ^+ and Φ^- are called *positive* and *negative roots*. The particular positive roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ with $1 \leqslant i \leqslant n-1$ are called *simple roots*. Every positive root may be written as a sum of simple roots.

The *Weyl group* W of $\mathrm{GL}(n, \mathbb{C})$ is the symmetric group S_n. It acts on T, Λ and Φ by permuting the coordinates.

Let W be a group with a fixed set I of generators, $I = \{s_1, \cdots, s_r\}$. Then W is called a *Coxeter group* if the following relations are satisfied. First, the *quadratic relations*

$$s_i^2 = 1 \tag{1}$$

and the *braid relations*

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \tag{2}$$

where there are $n_{i,j}$ entries on both sides, where $n_{i,j}$ is the order of $s_i s_j$; and furthermore that these relations are a presentation of W. This means that if Γ is any group with generators t_i satisfying the quadratic and braid relations, then there is a homomorphism $W \longrightarrow \Gamma$ such that $s_i \longmapsto t_i$.

For the symmetric group $W = S_n$ we take $I = \{s_1, \cdots, s_n\}$ where s_i is the transposition $(i, i+1)$. The element s_i is called a *simple reflection*.

Theorem 1.1. *The Weyl group is a Coxeter group.*

Proof. This is true for the Weyl group of any Lie group, though we are specializing to the case of the symmetric group. See [1] Theorem 25.1 or [2] Theorem 19.1. □

There is a close relationship between the Weyl group and the root system. In particular, the simple reflections are related to the simple roots by the following property.

Lemma 1.2. *The reflection s_i sends α_i to its negative, and permutes other positive roots. In other words s_i maps $\Phi^+ - \{\alpha_i\}$ to itself.*

Proof. This simple but important property is easily checked for the symmetric group. □

Definition 1. A *Weyl vector* is a vector $\rho \in \Lambda$ or $\mathbb{R} \otimes \Lambda$ such that $\rho - s_i(\rho) = \alpha_i$ for simple roots α_i and corresponding simple reflections α_i.

Example 1.3. We could take ρ to be half the sum of the positive roots. Then the fact that $\rho - s_i(\rho) = \alpha_i$ follows easily from Lemma 1.2. However for $W = S_n$ and $\Lambda = \mathbb{Z}^n$ we prefer to take

$$\rho = (n-1, n-2, \cdots, 0). \tag{3}$$

Defining ρ to be half the sum of the positive roots would give $\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2}\right)$, and if n is even, this vector has denominators that we can avoid by the choice (3).

2 Matsumoto's Theorem

The Weyl group has a *length function* $\ell : W \longrightarrow \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Two possible definitions can be given which are equivalent.

Definition 2. The length $\ell(w)$ is the smallest length k of a word $w = s_{i_1} \cdots s_{i_k}$ expressing w as a product of simple reflections. Alternatively, $\ell(w)$ is the cardinality of the set

$$\{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}.$$

For the equivalence of the two definitions see [1] Proposition 20.5.

An expression $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$ is called *reduced*. There may be many reduced expressions for w. For example if $W = S_4$ and $w_0 = (1, 4)(2, 3)$ is the longest word, there are 14 reduced expressions for w_0.

Matsumoto's theorem, found independently by H. Matsumoto and Tits in the 1960's is an extremely useful fact. Roughly it says if $\ell(w) = k$ and if

$$w = s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$$

are two reduced expressions, then the equivalence of the two relations can be proved using only the braid relations (2) and not the quadratic relations (1). To give an example, if $W = S_4$ then $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$ and $w_0 = s_3 s_2 s_3 s_1 s_2 s_3$ are two reduced expression. The braid relations are:

$$s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3, \qquad s_1 s_3 = s_3 s_1.$$

Matsumoto's theorem asserts that we can prove $s_1s_2s_1s_3s_2s_1 = s_3s_2s_3s_1s_2s_3$ using the braid relations and not the quadratic relations. Let us write 121321 instead of $s_1s_2s_1s_3s_2s_1$. Using the braid relations:

$$121321 = 212321 = 213231 = 231213 = 232123 = 323123.$$

To formulate Matsumoto's theorem rigorously, we introduce the *braid group* $B(W)$ of a Coxeter group W. This is the group with generators u_i (in bijection with the s_i) that satsify the braid relations but not the quadratic relations.

Theorem 2.1 (Matsumoto [3]). *If $s_{i_1} \cdots s_{i_k}$ and $s_{j_1} \cdots s_{j_k}$ are reduced expressions for the same element of W, then the corresponding elements $u_{i_1} \cdots u_{i_k}$ and $u_{j_1} \cdots u_{j_k}$ are equal in the braid group $B(W)$.*

Proof. For a proof using some geometric ideas, see [1], Theorem 25.2. $\square$

3 The ground state

We return to the open models. Recall that the order of colors at the top, from right to left (in descending column number) is:

$$\begin{array}{cc}
\text{column} & \text{color} \\
\lambda_1 + n - 1 & c_1 \\
\lambda_2 + n - 1 & c_2 \\
\vdots & \vdots \\
\lambda_n & c_n
\end{array}$$

where $c_1 > c_2 > \cdots > c_n$. The order of colors at the right is determined by a permutation $\mathbf{d}$ of

$$\mathbf{c}_0 = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

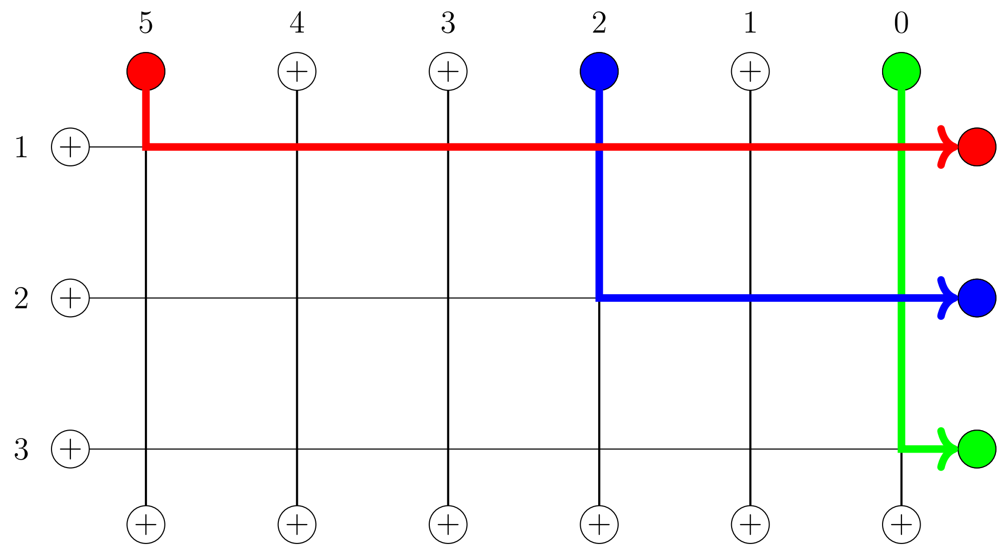
the vector of colors.

We call a system *monostatic* if it has but one state.

Proposition 3.1. *If $\mathbf{d} = \mathbf{c}_0$ the open model is monostatic. The partition function is*

$$Z_\lambda(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^{\lambda+\rho}.$$

Proof. A glance at an example will convincingly persuade you that this is true.



The red path must follow the given course, given that the path can only move down and to the right. Then the blue path, since the top row is blocked to it, can only follow the indicated path, and so on. So this system has only one state. We leave the reader to check that the Boltzmann weight of that state is $\mathbf{z}^{\lambda+\rho}$. ☐

We proved in the last Lecture that if $d_i > d_{i+1}$ then

$$Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}) \tag{4}$$

where δ_i° is the divided difference operator

$$\delta_i^\circ f(\mathbf{z}) = \frac{z_{i+1}f(\mathbf{z}) - z_i f(s_i\mathbf{z})}{z_i - z_{i+1}}.$$

We may take f to be any function in $\mathcal{O}(T)$.

Let us divide the numerator and denominator by z_{i+1}. Remembering that $z_i/z_{i+1} = \mathbf{z}^{\alpha_i}$, we may write

$$\delta_i^\circ f = \frac{f - \mathbf{z}^{\alpha_i}s_i(f)}{\mathbf{z}^{\alpha_i} - 1}.$$

Thus we may write

$$\delta_i^\circ = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - \mathbf{z}^{\alpha_i}s_i).$$

Let us consider where this operator actually lives. Let $\mathcal{M}(T)$ be the field of fractions of $\mathcal{O}(T)$, which is an integral domain. Then $\mathcal{M}(T)$ acts as operators on itself (by multiplication). Also W acts on $\mathcal{M}(T)$ by the formula $(wf)(\mathbf{z}) = f(w^{-1}\mathbf{z})$. So δ_i° lives in the ring

$$\mathcal{R} = \bigoplus_{w\in W} \mathcal{M}(T)w.$$

Taking this point of view if $f \in \mathcal{M}(T)$ then $wfw^{-1} = w(f)$. The operator δ_i° is actually special since although its coefficients have denominators, δ_i° maps $\mathcal{O}(T)$ into itself. (See Lemma 3.1 of Lecture 8.) So δ_i° actually lives in the subring $\mathcal{R}_\mathcal{O}$ of $\mathcal{R}$ that preserves $\mathcal{O}(T)$.

Let us normalize the partition function $Z_\lambda(\mathbf{z};\mathbf{d})$ as follows:

$$Y_\lambda(\mathbf{z};\mathbf{d}) = \mathbf{z}^{-\rho} Z_y(\mathbf{z};\mathbf{d}).$$

Then (4) can be rewritten as follows.

Proposition 3.2. *If $d_i > d_{i+1}$ then*

$$Y_\lambda(\mathbf{z}; s_i\mathbf{d}) = \partial_i^\circ Y_\lambda(\mathbf{z};\mathbf{d})$$

where

$$\partial_i^\circ f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i\mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}.$$

Proof. Let us check that $\partial_i^\circ = \mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho$. Indeed, since $s_i(\rho) = \rho - \alpha_i$ we $\mathbf{z}^{-\rho} s_i \mathbf{z}^\rho = \mathbf{z}^{-\alpha_i} s_i$ in the ring $\mathcal{R}$. Thus

$$\mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho = \mathbf{z}^{-\rho}(\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - \mathbf{z}^{\alpha_i} s_i)\mathbf{z}^\rho = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - s_i) = \partial_i^\circ.$$

Now

$$Y_\lambda(\mathbf{z}; s_i\mathbf{d}) = \mathbf{z}^{-\rho} Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \mathbf{z}^{-\rho}\delta_i^\circ Z_\lambda(\mathbf{z};\mathbf{d}) = \mathbf{z}^{-\rho}\delta_i^\circ \mathbf{z}^\rho Y_\lambda(\mathbf{z};\mathbf{d}),$$

as required. □

4 Looking ahead

The open models illustrate a scenario that is very common with the colored models: There are monostatic systems, whose partition functions are very simple, and there are Demazure recursion relations between the models. The beauty of the open models is that the recursions are in terms of operators ∂_i° that may be shown to satisfy the braid relations for the symmetric group, but not the quadratic relation since you may check that $(\partial_i^\circ)^2 = -\partial_i^\circ$. They generate a *degenerate Hecke algebra*. We will look at the implications of this next time.

References

[1] D. Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[2] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[3] H. Matsumoto. Générateurs et relations des groupes de Weyl généralisés. *C. R. Acad. Sci. Paris*, 258:3419–3422, 1964.