Lecture 8

We will continue looking at some examples that introduce new concepts. In this Chapter we will look (briefly) at some bosonic models before turning to a new major topic, colored models. In this lecture we will look at perhaps the simplest colored models, and take the theory up to the point were we see Demazure operators emerging from the Yang-Baxter equation. In Lecture 9 we will take this theory further. The appearance of Demazure operators gives us a point of contact with representation theory, since they generate a Hecke algebra.

1 Bosonic Models

The paths in lattice models can be thought of as the trajectories of particles. In the six-vertex model as we have been treating it, these move downwards and to the right.

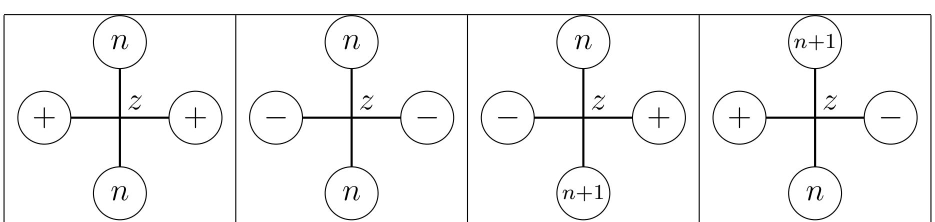
In physics, there is a distinction between particles which are called *bosons* and particles called *fermions*. The distinction is that no two fermions are allowed to occupy the same state: this is called the *Pauli exclusion principle*. Bosons, on the other hand, are allowed to occupy the same state.

The spinset for all edges in the six-vertex model just consists of $\{+, -\}$ where we interpret $+$ to be the absence of a particle, and $-$ to be the presence. An alternative spinset consist of the nonnegative integers $\{0, 1, 2, \cdots\}$ where the integer value indicates the number of identical particles.

We consider a simple type of model, invented by Kulish [6], with the partition functions computed by Korff [5]. We will call these models the bosonic *Hall-Littlewood models.*

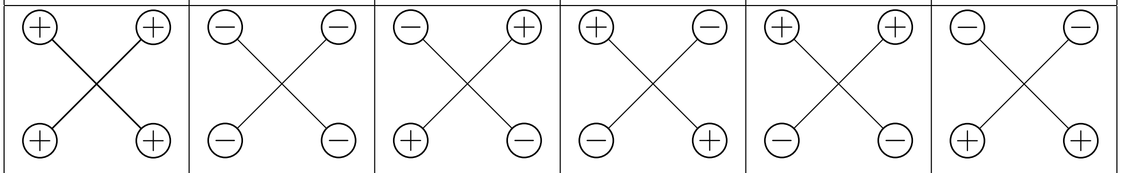
The horizontal edges will have the fermionic spinset $\{+, -\}$, but the vertical edges will have the bosonic spinset $\{0, 1, 2, \cdots\}$.

Paths are still relevant but now a single vertical edge can carry more than one path. The fermionic horizontal edges can only carry a single path. We thus arrive at the following vertex types, for which we have assigned Boltzmann weights:



1	z	$z(1-t^{n+1})$	1

The R-matrix is:



$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$z-tw$	$z-tw$	$t(z-w)$	$z-w$	$(1-t)z$	$(1-t)w$

For comparison, the R-matrix for the Tokuyama model is identical *except* for the $\mathtt{a}_1$ weight. The parameter t is analogous to the parameter we have been calling q in other models we have been looking at.

Now for the boundary conditions, these are similar to the Tokuyama models, with one exception. We choose a partition λ. As in the Tokuyama models, the columns are labeled 0 to N from right to left, for sufficiently large N, and the rows are labeled 1 to n from top to bottom.

Now we put the following spins on boundary edges, same as the Tokuyama models, with one exception. We put $+$ on the left (horizontal) boundary edges, $-$ on the right boundary edges, 0 on the bottom (vertical boundary edges). For the top vertical edge in column j, we put spin k, where k is the number of parts λ_i equal to j. This last choice differs from the Tokuyama models where we put the spins in the columns $\lambda_i + n - i$. That had the effect of preventing two spins from landing on the same edge. Since this model is bosonic, it is unnecessary to do that. The i-th row of the pattern is labeled by $z = z_i$, and we use the above Boltzmann weights.

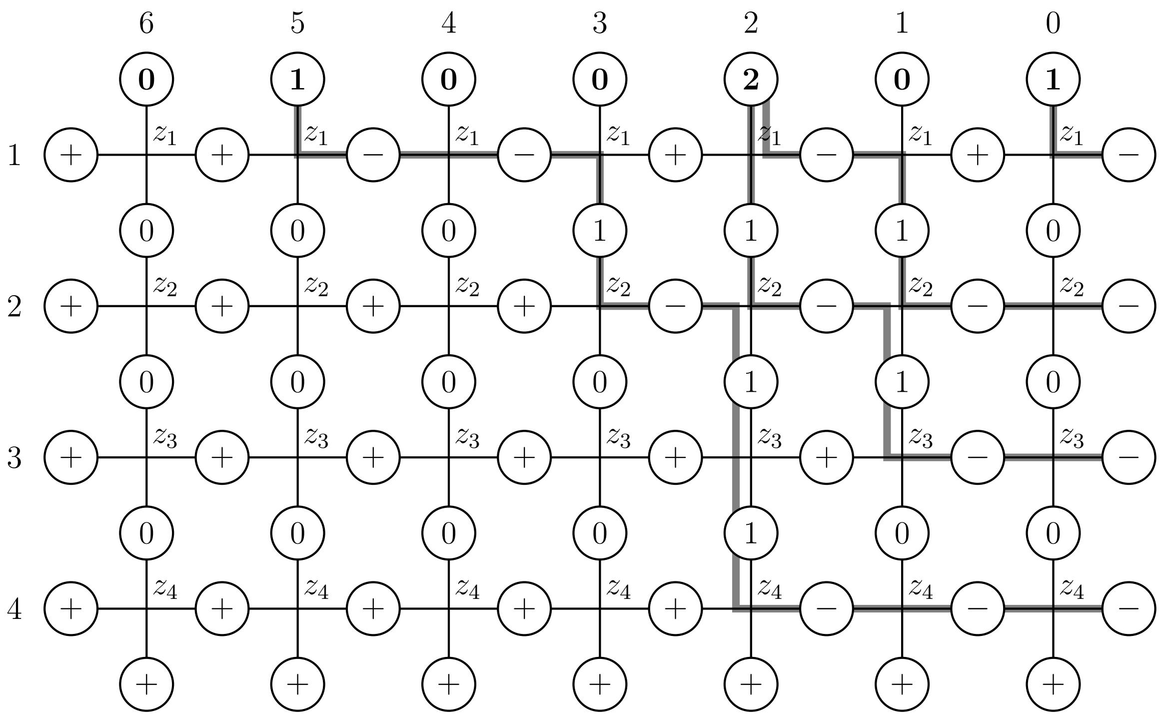
Theorem 1.1 (Korff [5])**.** *The partition function of this model equals the Hall-Littlewood symmetric polynomial* $P_\lambda(z_1, \cdots, z_n; t)$*.*

We will not digress now to define the Hall-Littlewood polynomials, but see Macdonald [7] Chapter 3 for their definitions and properties. We will point out that the information that we get from the Yang-Baxter equation is precisely a symmetric function, and in contrast with the Tokuyama models, that information does not seem to be enough to evaluate the partition function.

Similarly to the Tokuyama case, we parametrize the states by Gelfand-Tsetlin patterns of size n with top row λ. For example, suppose that:

$$\lambda = \left\{ \begin{array}{ccccccc} 5 & & 2 & & 2 & & 0 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{array} \right\}$$

The entries of this Gelfand-Tsetlin pattern are precisely the vertical edges that carry paths, and we easily arrive at the following state:



The paths are seen to double up in Column 2.

The quantum group underlying thes bosonic models is $U_q(\widehat{\mathfrak{sl}}_2)$. The vertical edges correspond to "Verma modules", which are infinite-dimensional representations of $\mathfrak{sl}_2$ (or its affine quantization) that do not lift to representations of $\mathrm{SL}_2(\mathbb{C})$.

2 The Symmetric group

A *Coxeter group* is a group W with generators $s_1, \cdots, s_r$ subject to relations $s_i^2 = 1$, and *braid relations* which have the form

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

where for some $n_{i,j}$ depending on i and j there are exactly $n_{i,j}$ terms on both sides. For example if $n_{i,j} = 3$ then

$$s_i s_j s_i = s_j s_i s_j$$

and if $n_{i,j} = 2$, then s_i and s_j commute. It is assumed in the definition of the Coxeter group that these relations are a presentation of W.

In discussing the colored models we will start to need some properties of the $\mathrm{GL}(r, \mathbb{C})$ Weyl group, which is the *symmetric group* S_r. Let $s_1, \cdots, s_{r-1}$ be the *simple reflections*, so s_i is the transposition $(i, i+1)$.

Theorem 2.1. *The group S_r is a Coxeter group with generators s_i. This means that the s_i generate S_r and satisfy the quadratic relations*

$$s_i^2 = 1$$

and the braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \textit{ if } |i-j| > 1,$$

and that moreover these relations give a presentation of S_r*.*

You can find proofs of this in many places such as my Lie groups book (second edition) Theorem 25.1.

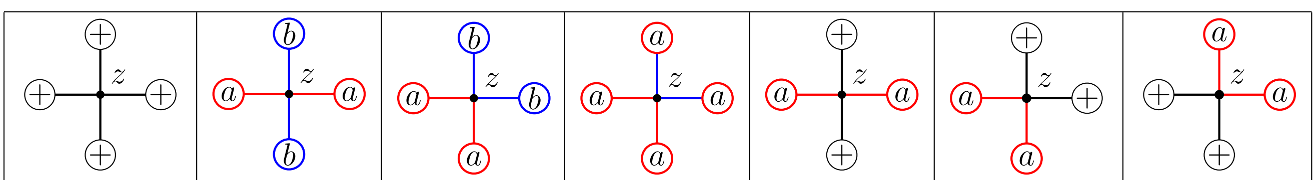
3 Open Colored Models

Borodin and Wheeler [2] instigated the current mania for colored models. Their models were bosonic, but fermionic models are also possible. We will look at the two very simplest examples, which we call the *open* and *closed* models.

Formulated in terms of paths, the idea behind the colored models is very simple: instead of one type of path there will be m types, where m is some sufficiently large number. These different types are called *colors*. It is important that they have an order, so let $c_1 > c_2 > \cdots > c_m$ be the m colors. Actually the largest number of colors that we can make use of is the number of rows of the grid, so we can take m to be the number r of rows in the grid. If there are more than m colors, there is no harm in taking $m = r$.

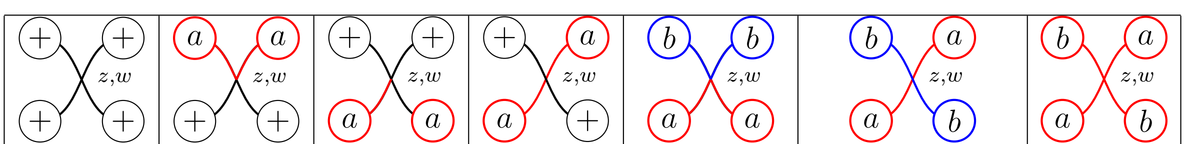
There is a feeling that if we have an uncolored model (e.g. six-vertex model) that we should be able to find a colorized version. The relationship between colored model and the uncolored model often shed light on the uncolored model

We could start with the Tokuyama model for this, but to get the simplest possible theory, we will start with the crystal limit 5-vertex model, which we encountered in Lecture 6. There are *two* theories which we will call *open* and *closed*. The open model was studied in [3], and we will look at it in this section.



1	$z \quad a>b$ $0 \quad a<b$	$0 \quad a>b$ $z \quad a<b$	z	z	z	1

Here is the R-matrix.



w	w	z	$z-w$	$w \quad a>b$ $z \quad a<b$	$z-w \quad a>b$ $0 \quad a<b$	z

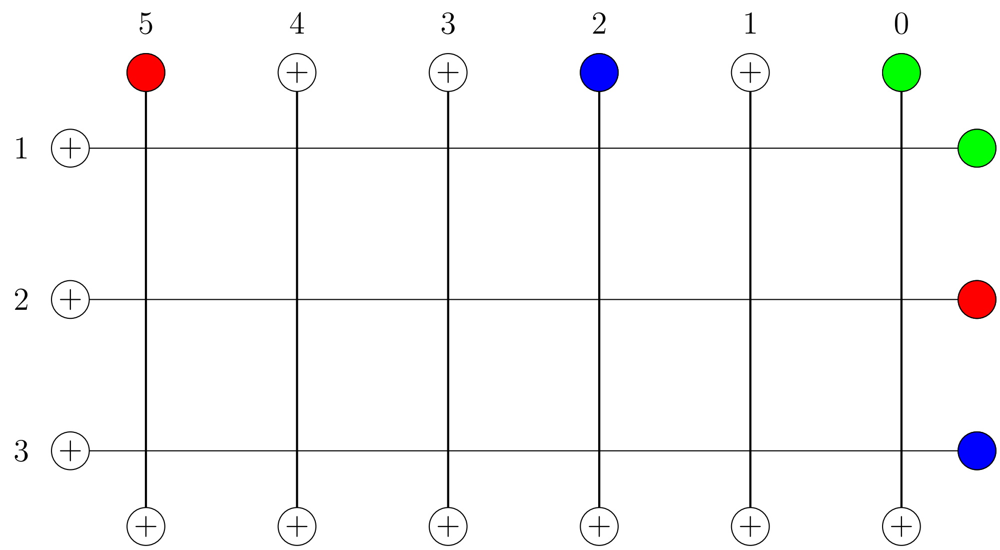
Let us specify boundary conditions. Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition. The model will be similar to the Tokuyama model with columns labeled $0, \cdots, N$ from right to left and rows labeled 1 to N from top to bottom. We put + spins on the top and bottom edges. For the top edge, we will put color c_i in column $\lambda_i + n - i$.

Note that this implies that the colors on the top edge are in decreasing order from right to left. For the right edge, we choose a *flag* $\mathbf{d} = (d_1, \cdots, d_r)$ where $d_1, \cdots, d_r$ are the colors $c_1, \cdots, c_r$ that we use on the top edge, in some order. We will denote by $\mathbf{c}_0 = (c_1, \cdots, c_r)$ where, we remind the reader, we have ordered $c_1 > \cdots > c_r$. Then there is a permutation $w \in W$ such that $\mathbf{d} = w\mathbf{c}_0$.

Here is an example. The three colors are ordered

$$\text{red } ({\color{red}\bullet}) > \text{blue } ({\color{blue}\bullet}) > \text{green } ({\color{green}\bullet})$$

So if $w = (123) = s_1 s_2$ then $\mathbf{c} = s_1 s_2 \mathbf{c}_0 = ({\color{green}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet})$. Then if λ is the partition $(3, 1, 0)$, so $\lambda + \rho = (5, 2, 0)$. Here are the boundary conditions as we have described them:



Let $Z_\lambda(\mathbf{z}; \mathbf{d})$ or $Z_\lambda(\mathbf{z}; w)$ denote the partition function.

Now let us define an operator δ_i° on functions $f(\mathbf{z})$ by:

$$\delta_i^\circ f(\mathbf{z}) = \frac{z_{i+1} f(\mathbf{z}) - z_i f(s_i \mathbf{z})}{z_i - z_{i+1}}.$$

This is a *divided difference operator*, of a type used by Demazure [4] and Bernstein-Gelfand-Gelfand [1] in algebraic geometry. They are also important in algebraic combinatorics.

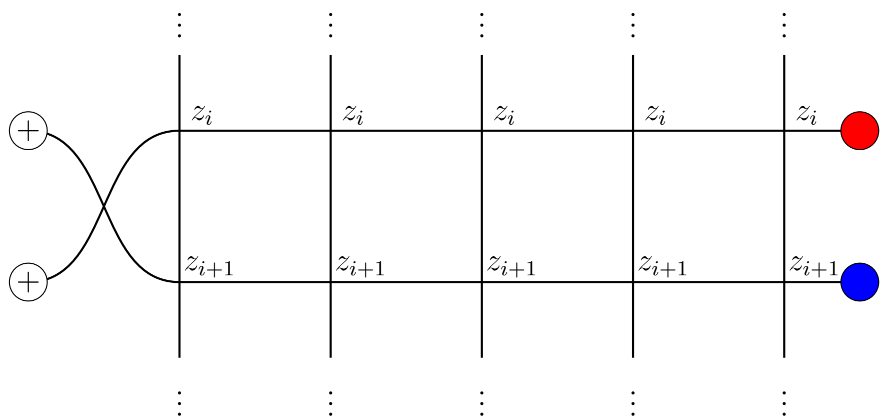
Lemma 3.1. *If f is holomorphic as a function of $\mathbf{z}$, so is $\delta_i^\circ f$.*

Proof. We need to show that the numerator is divisible by the denominator. The numerator vanishes where the denominator does, because if $z_i = z_{i+1}$ then $\mathbf{z} = s_i \mathbf{z}$. The vanishing of the numerator $z_{i+1} f(\mathbf{z}) - z_i f(s_i \mathbf{z})$ along the hyperplane $z_i = z_{i+1}$ implies that the denominator divides the numerator. □

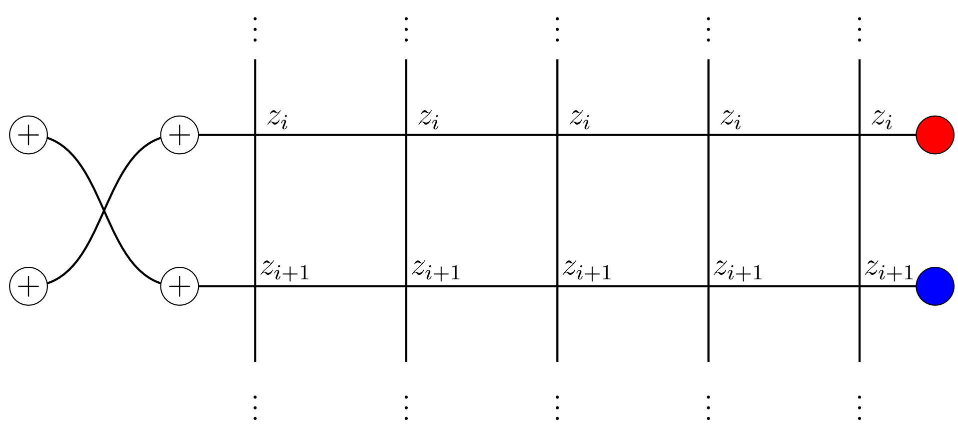
Proposition 3.2. *Suppose that $d_i > d_{i+1}$. Then the partition function $Z_\lambda(\mathbf{z}; s_i \mathbf{d})$ satisfies*

$$Z_\lambda(\mathbf{z}; s_i \mathbf{d}) = \delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}).$$

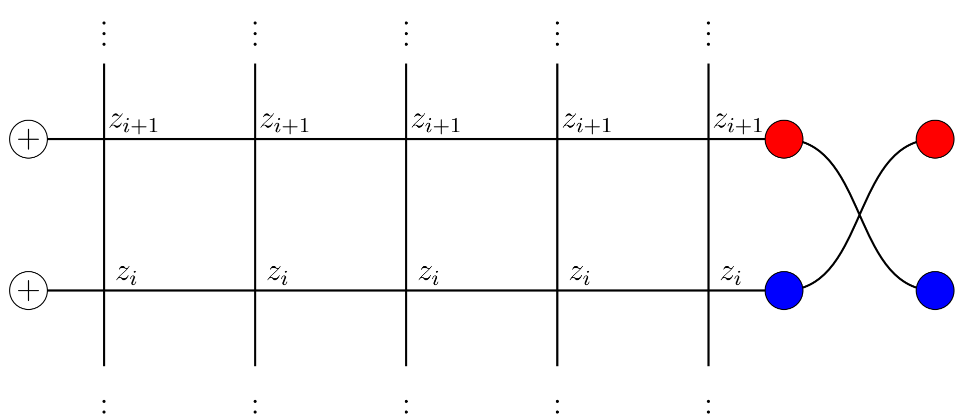
Proof. Let us attach the R-matrix to the left:



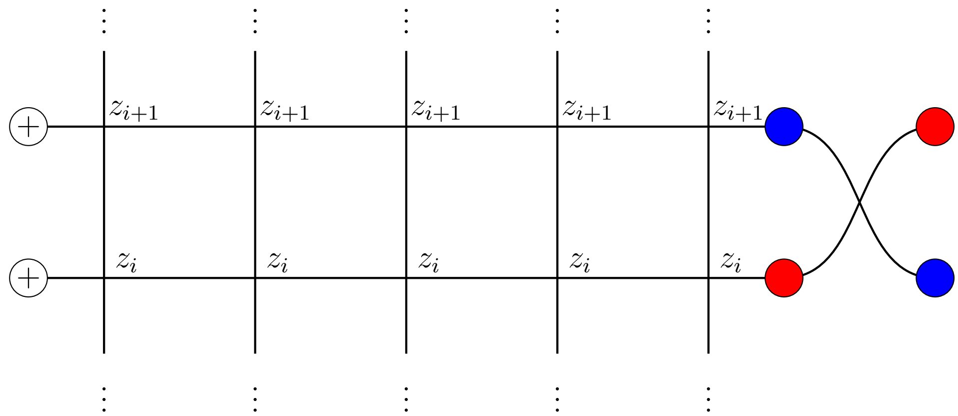
Given the spins $+$, $+$ on the left edge the spins on the R-matrix can only be all $+$, so we may assume that the configuration is as follow:



the partition function of this system is $Z_\lambda(\mathbf{z};\mathbf{d})$ times the value z_{i+1} of the R-matrix. Running the train argument, it turns out there are two possible configurations on the right-hand side, namely



and



Consulting the Boltzmann weights for the R-matrix, the partition functions for these configurations are

$$z_i Z_\lambda(s_i\mathbf{z};\mathbf{d})$$

and (since the colors get switched for the second one):

$$(z_i - z_{i+1}) Z_\lambda(s_i\mathbf{z}; s_i\mathbf{d}).$$

Hence we obtain the identity

$$z_{i+1} Z_\lambda(\mathbf{z};\mathbf{d}) = z_i Z_\lambda(s_i\mathbf{z};\mathbf{d}) + (z_i - z_{i+1}) Z_\lambda(s_i\mathbf{z}; s_i\mathbf{z}).$$

We want to interchange z_i and z_{i+1}, so replace $\mathbf{z}$ by $s_i\mathbf{z}$. Then

$$z_i Z_\lambda(s_i\mathbf{z};\mathbf{d}) = z_{i+1} Z_\lambda(\mathbf{z};\mathbf{d}) + (z_{i+1} - z_i) Z_\lambda(\mathbf{z}; s_i\mathbf{d}).$$

Reorganizing this gives

$$Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \frac{z_{i+1} Z_\lambda(\mathbf{z};\mathbf{d}) - z_i Z_\lambda(s_i\mathbf{z};\mathbf{d})}{z_i - z_{i+1}}$$

as required. □

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