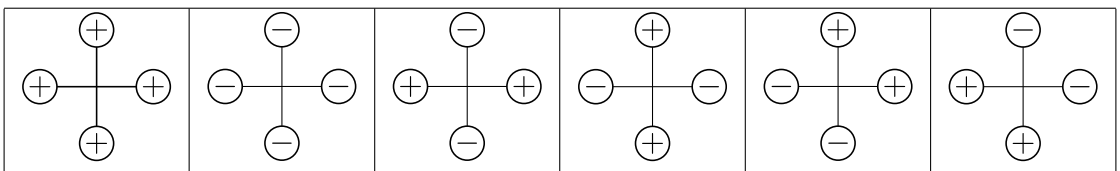
Lecture 7

1 The general free-fermionic six-vertex model

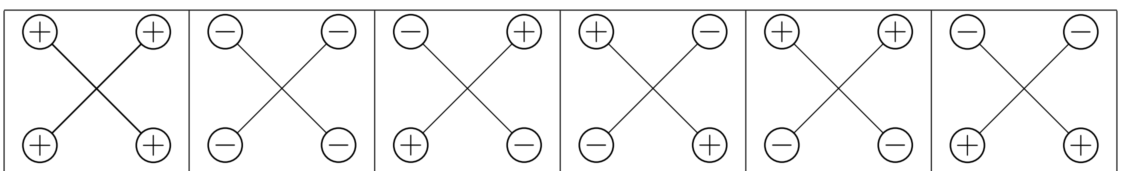
In this section we will consider a very remarkable parametrized Yang-Baxter equation with a nonabelian parameter group $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ which includes the Tokuyama weights as a special case. More typical parameter groups are usually abelian: $\mathbb{C}^\times$, $\mathbb{C}$ or an elliptic curve.

The Tokuyama weights are examples of free-fermionic weights. Label the the Boltzmann weights at a vertex R as follows:



$a_1(R)$	$a_2(R)$	$b_1(R)$	$b_2(R)$	$c_1(R)$	$c_2(R)$

or alternatively:



$a_1(R)$	$a_2(R)$	$b_1(R)$	$b_2(R)$	$c_1(R)$	$c_2(R)$

We call the weights *free-fermionic* if

$$a_1(R)a_2(R) + b_1(R)b_2(R) = c_1(R)c_2(R)$$

and $c_1(R)$, $c_2(R)$ are both nonvanishing.

It turns out that *all* free-fermionic weights fit into a parametrized Yang-Baxter equation with parameter group $\Gamma = \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$. This parametrized Yang-Baxter equation was discovered by Korepin (see [3] page 126, and rediscovered by Brubaker, Bump and Friedberg [1]). Let

$$\rho : \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \longrightarrow \{\text{free-fermionic vertices}\}$$

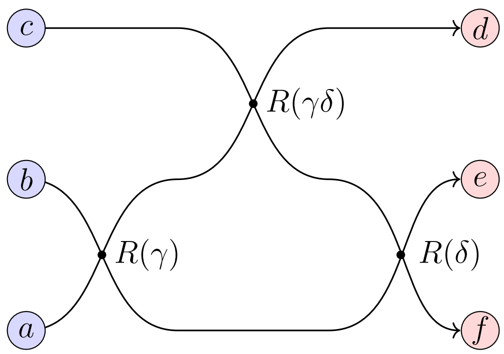
be the map that sends the matrix

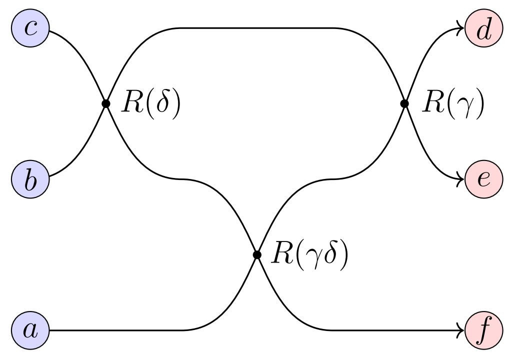
$$\gamma = \begin{pmatrix} c_1 & & & \\ & a_1 & b_2 & \\ & -b_1 & a_2 & \\ & & & c_2 \end{pmatrix}$$

to the vertex with those Boltzmann weights.

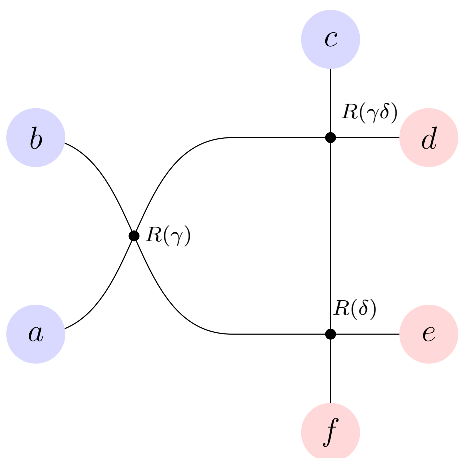
Theorem 1.1. *The map R is a parametrized Yang-Baxter equation with parameter group* $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$.

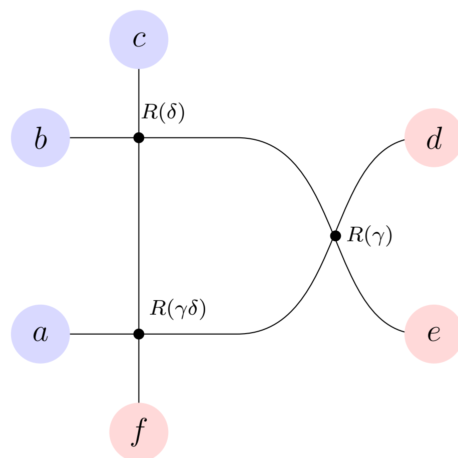
The parametrized Yang-Baxter equation can be either of the two forms from Lecture 4: we ask that for all a, b, c, d, e, f the two following partition functions are equal:





Alternatively:





Proof. Let $R = \rho(\gamma)$, $T = R(\delta)$ and $S = R(\gamma\delta)$, where the product $\gamma\delta$ is just matrix multiplication. Thus

$$\gamma = \begin{pmatrix} c_1(R) & & & \\ & a_1(R) & b_2(R) & \\ & -b_1(R) & a_2(R) & \\ & & & c_2(R) \end{pmatrix}, \qquad \delta = \begin{pmatrix} c_1(T) & & & \\ & a_1(T) & b_2(T) & \\ & -b_1(T) & a_2(T) & \\ & & & c_2(T) \end{pmatrix},$$

Multiplying the matrices γ and δ and remembering that $S = R(\gamma\delta)$, the S Boltzmann weights are:

$$c_1(S) = c_1(R)c_1(T), \qquad c_2(S) = c_2(R)c_2(T).$$

$$a_1(S) = a_1(R)a_1(T) - b_2(R)b_1(T), \qquad a_2(S) = -b_1(R)b_2(T) + a_2(R)a_2(T),$$

$$b_1(S) = b_1(R)a_1(T) + a_2(R)b_1(T), \qquad b_2(S) = a_1(R)b_2(T) + b_2(R)a_2(T).$$

With these values, and taking the values

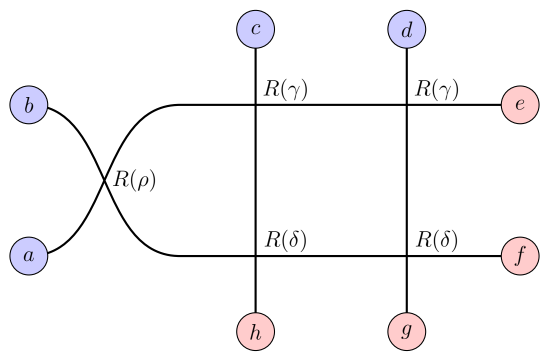
$$c_2(R) = (a_1(R)a_2(R) + b_1(R)b_2(R))/c_1(R),$$
$$c_2(T) = (a_1(T)a_2(T) + b_1(T)b_2(T))/c_1(T),$$

it is straightforward to check all cases of the Yang-Baxter equation. This is done in a computer program called `free-fermionic1.sage`, posted on the class web page. □

2 Column parameters

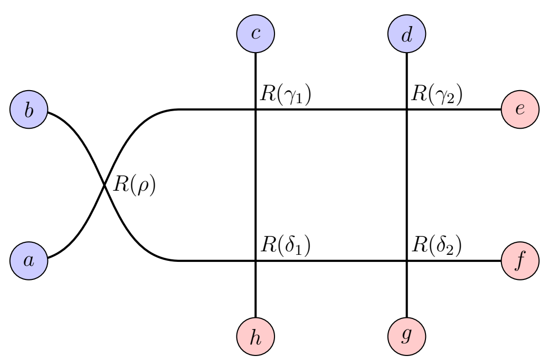
In the Tokuyama models, the Boltzmann weights depended on the rows but not the columns. We may use the parametrized Yang-Baxter equation to predict another model in which the weights do depend on the columns. But let us ask for free-fermionic models that do show column dependence. Such models exist and give for example factorial Schur functions **??**.

However we want to predict their existence by thinking about the $\mathrm{GL}(2) \times \mathrm{GL}(1)$ parametriczed free-fermionic Yang-Baxter equation. Let us postulate a grid, with free-fermionic vertices. We start with two rows of free-fermionic vertices $S = R(\gamma)$ and $T = R(\delta)$. If $R = R(\rho)$ where $\rho\delta = \gamma$, then by Theorem 1.1 we may have the Yang-Baxter equation that we need and can do the train argument. So we want $\rho = \gamma\delta^{-1}$:

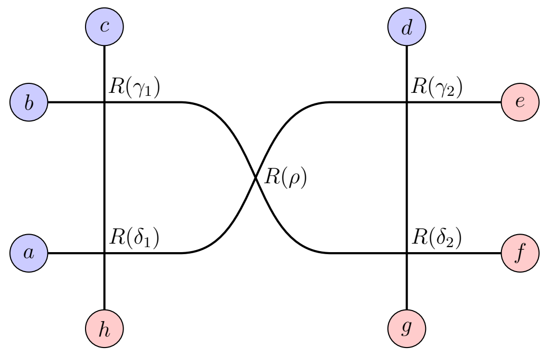


Note that γ and δ could be *arbitrary* free-fermionic vertex types.

Now let us show that Theorem 1.1 allows us to modify the weights S and T so that they are dependent on the columns, not just the rows. This procedure will not affect the R-matrix $R = R(\rho)$. Let $\gamma_1, \gamma_2, \cdots$ be the elements of the parameter group $G = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ such that $R(\gamma_i)$ and $R(\delta_i)$ are to be the row weights in the modified system. We want to be able to attach ths *same* matrix $R = R(\rho)$ for the train argument.



Now we need $\gamma_1 = \rho\delta_1$ so $\rho = \gamma_1\delta_1^{-1}$. Assuming this we can do the Yang-Baxter equation:



But now to do the next step, we need $\delta_2\rho = \gamma_2$. To complete the train argument, we clearly need $\delta_j^{-1}\gamma_j = \rho$ for all j.

To summarize, a necessary and sufficient condition to be able to do the train argument is that $\delta_j^{-1}\gamma_j = \rho$ for all j, and we already have γ and δ such that $\delta^{-1}\gamma = \rho$.

Now to arrange this, let us fix an element $\alpha_j \in G$ for every column j, and we define $\gamma_j = \gamma\alpha_j$ and $\delta_j = \delta\alpha_j$. Then the conditions are satisfied.

Let us do an example. We want γ to correspond to Tokuyama weights at the vertex S with parameter z. This means that the weights of γ are given by the following table:

$a_1(S)$	$a_2(S)$	$b_1(S)$	$b_2(S)$	$c_1(S)$	$c_2(S)$
1	z	$-q$	z	$(1-q)z$	1

Therefore

$$\gamma = \begin{pmatrix} c_1(S) & & & \\ & a_1(S) & b_2(S) & \\ & -b_1(S) & a_2(S) & \\ & & & c_2(S) \end{pmatrix} = \begin{pmatrix} (1-q)z & & & \\ & 1 & z & \\ & q & z & \\ & & & 1 \end{pmatrix}$$

Similarly let δ correspond to Tokuyama weights with parameter w, so

$$\delta = \begin{pmatrix} (1-q)w & & & \\ & 1 & w & \\ & q & w & \\ & & & 1 \end{pmatrix}.$$

Now to choose the perturbing matrices α_j, let $a_1, a_2, \cdots$ be an arbitrary sequence of integers and take

$$\alpha_j = \begin{pmatrix} 1 & & & \\ & 1 & a_j & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Note that this matrix satisfies the free-fermionic condition. Now

$$\gamma\alpha_j = \begin{pmatrix} (1-q)z & & & \\ & 1 & z & \\ & q & z & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & a_j & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} (1-q)z & & & \\ & 1 & z+a_j & \\ & q & z+qa_j & \\ & & & 1 \end{pmatrix}$$

This leads to the following modification of the Tokuyama weights:

$a_1(S)$	$a_2(S)$	$b_1(S)$	$b_2(S)$	$c_1(S)$	$c_2(S)$
1	$z + qa_j$	$-q$	$z + a_j$	$(1-q)z$	1

The partition functions of the modified models replace the Schur functions of the Tokuyama model with *factorial Schur functions*. See [2] for more information, and [4] for more general free-fermionic models that generalize Schur functions.

References

[1] B. Brubaker, D. Bump, and S. Friedberg. Schur polynomials and the Yang-Baxter equation. *Comm. Math. Phys.*, 308(2):281–301, 2011, https://arxiv.org/abs/0912.0911.

[2] D. Bump, P. J. McNamara, and M. Nakasuji. Factorial Schur functions and the Yang-Baxter equation. *Comment. Math. Univ. St. Pauli*, 63(1-2):23–45, 2014, arXiv:arXiv:1108.3087.

[3] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin. *Quantum inverse scattering method and correlation functions.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1993.

[4] S. Naprienko. Free fermionic Schur functions, 2023, arXiv:2301.12110.