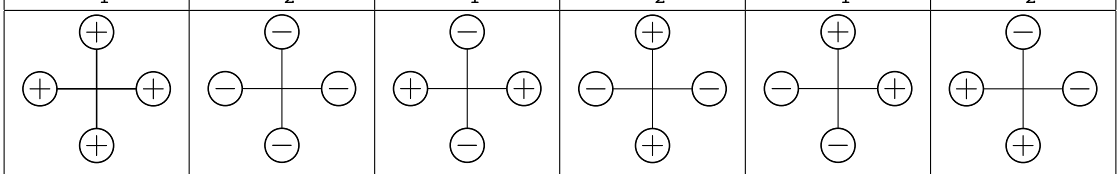
Lecture 6

1 Our story so far

We continue from Lecture 5, which we briefly review. For reference, here are the Boltzmann weights for Tokuyama ice from Lecture 5:



$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	z_i	$-q$	z_i	$z_i(1-q)$	1

We also described boundary conditions depending on a partition λ. The resulting system was denoted $\mathfrak{S}_\lambda(\digamma;q)$. It's partition function was denoted $Z_\lambda(\mathbf{z};q)$ where $\mathbf{z}=(z_1,\cdots,z_n)$ are the row parameters. We then proved that there is a symmetric polynomial $S_\lambda(\mathbf{z})$, independent of q such that

$$Z_\lambda(\mathbf{z};q)=\prod_{i<j}(z_i-qz_j)S_\lambda(\mathbf{z}).$$

In this lecture we will prove that S_λ agrees with the Schur function, using either the original Jacobi definition $\det(z_j^{\lambda_i+n-i})/\det(z_j^{n-i})$ when $q=1$, or the combinatorial definition as a sum over semistandard Young tableaux (SSYT) when $q=0$.

Our first result does not depend on q.

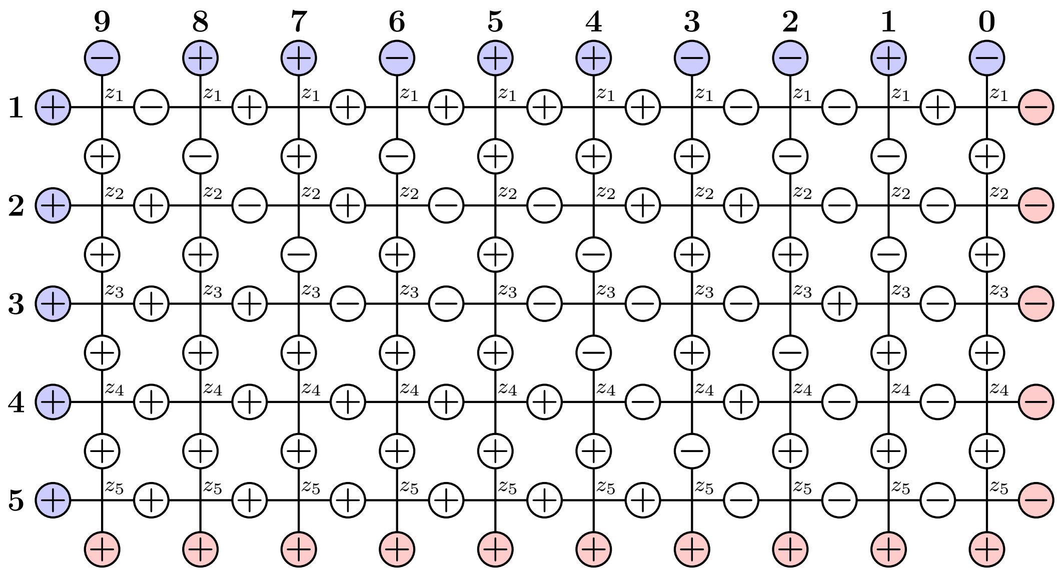
Proposition 1.1. *Let $\mathfrak{s}$ be a state of the system, and let*

$$G=\left\{\begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{21} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{n1} & & & \end{array}\right\}$$

be the corresponding strict Gelfand-Tsetlin pattern. Let $A_i=\sum_j a_{ij}$ be the row sums. Then the Boltzmann weight $\beta(\mathfrak{s})$ equals a polynomial in q times the monomial $\mathbf{z}^\mu$ where

$$\mu=(A_1-A_2,A_2-A_3,\cdots,A_n).$$

Before we prove this, let us work out an example. We will take $n = 5$ and $\lambda = (5, 3, 1, 1) = (5, 3, 1, 1, 0)$. After adding $\rho = (4, 3, 2, 1, 0)$ we get $\lambda + \rho = (9, 6, 3, 2, 0)$, and these are the columns at the top where we put $-$ spins in the boundary conditions. Consider the following state.

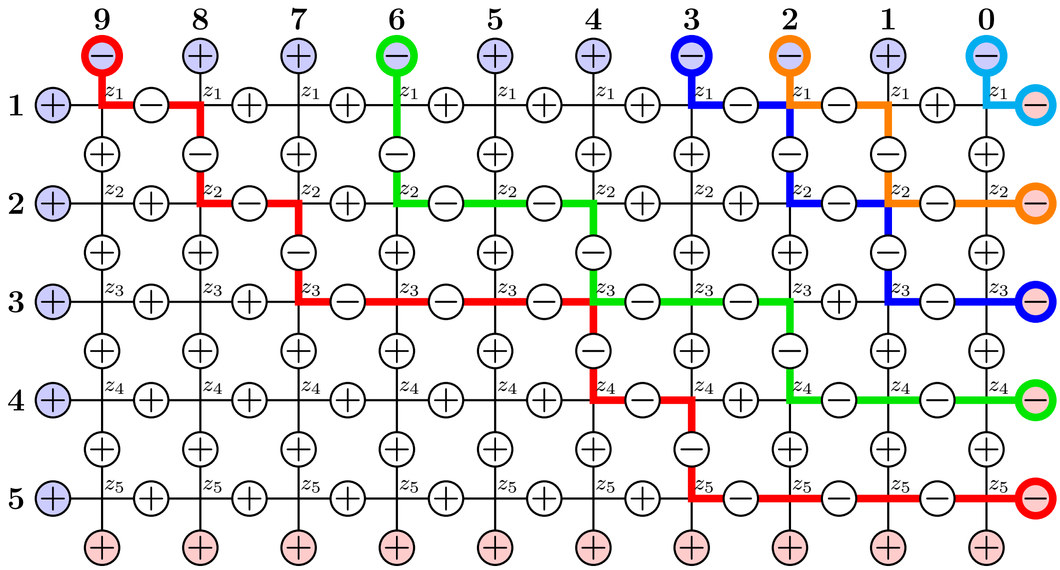


We recall from Lecture 3 that the entries in the corresponding Gelfand-Tsetlin patterns are the columns where a vertical edge with a $-$ spin occurs. Thus:

$$G = \left\{ \begin{array}{ccccccccc} 9 & & 6 & & 3 & & 2 & & 0 \\ & 8 & & 6 & & 2 & & 1 & \\ & & 7 & & 4 & & 1 & & \\ & & & 4 & & 2 & & & \\ & & & & 3 & & & & \end{array} \right\}$$

Conversely, given a strict Gelfand-Tsetlin pattern with top row $\lambda+\rho$, we may put $-$ spins on the vertical edges with entries in the pattern, and $+$ spins in the remaining edges. Then the spins on the horizontal edges are determined by the requirement that the number of $-$ spins adjacent to every vertex must be even, leading to a unique admissible state of the six-vertex model.

Then, we recall from Lecture 2 that we may find paths running through the edges with $-$ spins. Let us see how this works for the above example. There will be six paths, each beginning with an "input" boundary edge (colored blue) and terminating at an "output" edge (colored red). We show the paths as follows, using color to distinguish the six paths.



Proof of Proposition 1.1. To prove the Proposition, we note from the Boltzmann weights that $\beta(\mathfrak{s})$ is a polynomial in q times a monomial $\mathbf{z}^\mu$ for some μ. There is a contribution of z_i from every pattern of type $\mathtt{a}_2$, $\mathtt{b}_2$ or $\mathtt{c}_1$. These are precisely the vertices with a $-$ spin to the left of the vertex. Therefore the number of z_i in the product of local Boltzmann weights equals the number μ_i of $-$ spins in the i-th row, not counting the right boundary edge.

Thus in th example, $\mu = (3, 4, 5, 6, 3)$. This is consistent with the statement of the Proposition with where the row sums of the Gelfand-Tsetlin pattern are $(A_1, A_2, A_3, A_4, A_5) = (20, 17, 12, 6, 3)$.

We must show that $\mu_i = A_i - A_{i+1}$ (or just A_i if $i = n$). To count the number of $-$ spins on the horizontal edges in the i-th row, not counting the right boundary edge, we enumerate them by the paths. We note that one path enters from the top in the column $a_{i,j}$ and exits at the column $a_{i+1,j}$. There are $a_{i,j} - a_{i+1,j}$ $-$ spins on this edge.

The argument requires minor modification for the last path, which exits on the right and contributes a_{n+1-i}. We do not need to consider this an exception if we extend the Gelfand pattern by zero and define $a_{i+1,n+1-i} = 0$. With this convention, $A_{n+1} = 0$.

Summing the contributions of all paths,

$$\mu_i = \sum_{j=1}^{n+1-i} a_{i,j} - a_{i+1,j} = A_i - A_{i+1},$$

as required. $\square$

2 Tokuyama Ice: $q = 1$

If either $q = 0$ or $q = 1$, one of the six vertex types in the Tokuyama model disappears. In these two cases, there are only five allowed states of spins adjacent to a vertex, and we will call the resulting models *five-vertex models*. In the case $q = 1$, the Boltzmann weights are:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$\begin{smallmatrix} & \oplus & \\ \oplus & + & \oplus \\ & \oplus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \ominus & + & \ominus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \oplus & + & \oplus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \oplus & \\ \ominus & + & \ominus \\ & \oplus & \end{smallmatrix}$	$\begin{smallmatrix} & \oplus & \\ \ominus & + & \oplus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \oplus & + & \ominus \\ & \oplus & \end{smallmatrix}$
1	z_i	-1	z_i	0	1

We see that there can no longer be any $\mathtt{c}_1$ patterns. This has a profound effect on the paths and on the Gelfand-Tsetlin patterns.

Lemma 2.1. *If G is the Gelfand-Tsetlin pattern of a state having no $\mathtt{c}_1$ patterns, then every row of the Gelfand-Tsetlin pattern is a subset of the row above, obtained by deleting one entry.*

Proof. If the $(i+1)$-st row is *not* obtained from the i-th row by deleting a single entry, then there is an element $a_{i+1,j}$ that is not in the i-th row. Since $a_{i,j} \geqslant a_{i+1,j} \geqslant a_{i,j+1}$ by the definition of a Gelfand-Tsetlin pattern we must have $a_{i,j} > a_{i+1,j} > a_{i,j+1}$. This implies that there is a $\mathtt{c}_1$ pattern at in the i-th row at column $a_{i+1,j}$, which is a contradiction. $\square$

Recall that the "Weyl group" W is the symmetric group S_n.

Proposition 2.2. *When $q = 1$, we have*

$$Z(\mathbf{z}; 1) = \sum_{w \in W} \operatorname{sgn}(w) \mathbf{z}^{w(\lambda+\rho)}. \tag{1}$$

Proof. There are $n!$ states $\mathfrak{s}$ that omit $\mathtt{c}_1$ patterns, namely those in which each row is obtained from the previous one by dropping a single entry. By Proposition 1.1, the Boltzmann weight $\beta(\mathfrak{s})$ is $\pm\mathbf{z}^\mu$, where $\mu_i = A_i - A_{i+1}$. By the Lemma, this value $A_i - A_{i+1}$ is some element of the i-th row, hence of the top row $\lambda+\rho$. (The sign $-$ is the number of $\mathtt{b}_1$ patterns.) We may therefore write $\mu = w(\lambda+\rho)$ for some permutation $w \in W$, and $\beta(\mathfrak{s}) = \pm\mathbf{z}^{w(\lambda+\rho)}$, where the sign must be determined.

We have proved in Lecture 5 that

$$S_\lambda(\mathbf{z}) = \frac{Z(\mathbf{z}; 1)}{\prod_{i<j}(z_i - z_j)} \tag{2}$$

is symmetric. The denominator is alternating, that is, it changes sign when an odd permutation is applied. Therefore the numerator $Z(\mathbf{z}; 1)$ is also alternating. Now there is one state which has no $\mathtt{b}_1$ patterns: this is the state in which the entry in the i-th row of the Gelfand-Tsetlin pattern G that is dropped is always the first one. For this state, $\beta(\mathfrak{s}) = \mathbf{z}^{\lambda+\rho}$. Therefore $Z(\mathbf{z}; 1)$ is of the form $\sum_{w\in W} \pm\mathbf{z}^{w(\lambda+\rho)}$, is known to be alternating, and one of the terms is $\mathbf{z}^{\lambda+\rho}$. Hence the signs of the other terms are determined. This proves (1). $\square$

Now we recognize the numerator and denominator in the ratio (2)

$$S_\lambda(\mathbf{z}) = \frac{\sum_{w\in W} \pm\mathbf{z}^{w(\lambda+\rho)}}{\prod_{i<j}(z_i - z_j)} = \frac{\det(z_j^{\lambda_i+n-i})}{\det(z_j^{n-i})},$$

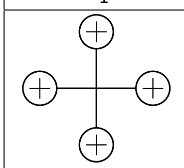
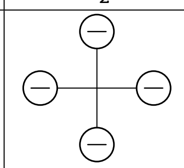
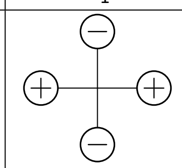
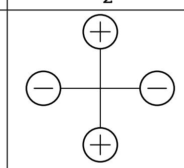
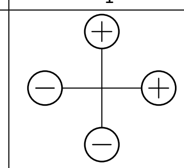
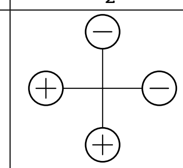
using the Vandermonde identity. This equals the Schur polynomial $s_\lambda(\mathbf{z})$ by the first definition.

3 The Crystal Limit

Before we consider the case $q = 0$, a word about how important this case is. Before the 1980's, an analogy between the representation theory of $\mathrm{GL}(n, \mathbb{C})$ and the theory of semistandard Young tableaux (SSYT) emerged in work of Robinson, Littlewood, Schensted, Knuth, Lascoux and Schützenberger. For example, if λ is a partition, then λ indexes two particular things, an irreducible representation $\pi_\lambda^{\mathrm{GL}(n)}$ of $\mathrm{GL}(n, \mathbb{C})$, and the set $\mathcal{B}_\lambda$ of semistandard Young tableaux. The cardinality of $\mathcal{B}_\lambda$ equals the dimension of $\pi_\lambda^{\mathrm{GL}(n)}$, and this is the beginning of a fruitful parallel. Ultimately Kashiwara, in the theory of crystal bases (crystals) gave an explanation for this: the representation $\pi_\lambda^{\mathrm{GL}(n)}$ can be thought of as being in a family of modules of the quantum groups $U_q(\mathfrak{gl}_n)$. These are somewhat complicated objects, but in the "crystal limit" $q \longrightarrow 0$ much of the complexity disappears, and the combinatorial theory remains. The quantum group $U_q(\mathfrak{gl}_n)$ does not, itself, have a limit when $q = 0$, but some of its operations do survive, giving $\mathcal{B}_\lambda$ some extra structure, that of a crystal. We will therefore refer to the case $q \longrightarrow 0$ as the "crystal limit."

4 The case $q \longrightarrow 0$

When $q = 0$, we have the following Boltzmann weights:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
					
1	z_i	0	z_i	z_i	1

Now we see that the pattern $\mathtt{b}_1$ no longer appears. This means that every path that comes down to a vertex from the top must bend to the right.

Lemma 4.1. *Let $\mathfrak{s}$ be a state of the system $\mathfrak{S}_\lambda(\mathbf{z}; q)$, and let*

$$G = \left\{ \begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{21} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & \iddots & & \\ & & & a_{n1} & & & \end{array} \right\}$$

be the corresponding strict Gelfand-Tsetlin pattern. Then a necessary and sufficient condition that $\mathfrak{s}$ contains no $\mathtt{b}_1$ patterns is that for every i, j we have $a_{i,j} > a_{i+1,j}$.

Proof. In terms of the paths, one path descends from above to the vertex in the i-th row in column $a_{i,j}$ and leaves downwards in the column $a_{i+1,j}$. Thus if $a_{i,j} = a_{i+1,j}$, that means precisely that the vertex in row i and column $a_{i,j}$ produces a $\mathtt{b}_1$ pattern. $\square$

We will call a Gelfand-Tsetlin pattern *left-strict* if its entries satisfy $a_{i,j} > a_{i+1,j} \geqslant a_{i,j+1}$. (The second inequality is part of the definition of a Gelfand-Tsetlin pattern, so the significant assumption is that $a_{i,j} > a_{i+1,j}$.) We see that the states of the five-vertex model $\mathfrak{S}_\lambda(\mathbf{z}; 0)$ are in bijection with the left-strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$.

Let us denote by ρ_k the vector $(k-1, k-2, \cdots, 0)$ in $\mathbb{Z}^k$, so that $\rho = \rho_n$ in our previous notation. We can make a Gelfand-Tsetlin pattern with rows $\rho_n, \rho_{n-1}, \cdots, \rho_1$ thus:

$$P = \left\{ \begin{array}{ccccccccc} n-1 & & n-2 & & & & \cdots & & 0 \\ & n-2 & & & \cdots & & & 0 & \\ & & & \ddots & & \iddots & & & \\ & & & & 0 & & & & \end{array} \right\}.$$

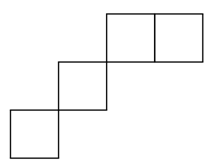
Lemma 4.2. *The map $G \longrightarrow G - P$ is a bijection between left-strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$ and Gelfand-Tsetlin patterns with top row λ.*

Proof. This is easy to check. $\square$

Lemma 4.3. *Let G be a Gelfand-Tsetlin pattern and let T be the corresponding semistandard Young tableau as defined in Section 2. Let $\lambda_1, \cdots, \lambda_n$ be the rows of G and let $A_i = |\lambda_i|$ denote the corresponding row sums. Then*

$$\mathrm{wt}(T) = (A_n, A_{n-1} - A_n, \cdots, A_2 - A_3, A_1 - A_2).$$

Proof. Let λ, μ be two partitions with Young diagrams λ, μ. If the Young diagram $\mathrm{YD}(\mu)$ is contained in $\mathrm{YD}(\lambda)$, then the pair λ, μ, denoted λ/μ is called a *skew shape*. Its Young diagram is the set-theoretic difference $\mathrm{YD}(\lambda) - \mathrm{YD}(\mu)$. For example $(5,3,2)/(3,2,1)$ is a skew shape and its diagram is



We may use the skew shape terminology to reformulate the relationship between a Gelfand-Tsetlin pattern G and its associated tableau T, first discribed in Lecture 2. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the rows of G. We also let $\lambda_{n+1} = ()$ be the empty partition. Then $\lambda_{n+1-i}/\lambda_{n+2-i}$ is a skew shape, which is the union of all the boxes in the tableau T that contain the entry i.

By definition, $wt(T) = (\mu_1, \mu_2, \cdots, \mu_n)$ where μ_i is the number of boxes that contain the entry i. The these comprise the skew tableau with shape $\lambda_{n+1-i}/\lambda_{n+2-i}$, and since $|\lambda_i| = A_i$, we obtain the advertised formula for $\mathrm{wt}(T)$. $\square$

Example 4.4. To illustrate Lemma 6, suppose $n = 3$ and

$$G = \begin{array}{ccccc} 5 & & 3 & & 1 \\ & 4 & & 1 & \\ & & 3 & & \end{array}.$$

Then the corresponding tableau is

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

Thus $\mathrm{wt}(T) = (3, 5-3, 8-5) = (3, 2, 3)$. The three skew shapes corresponding to $1, 2, 3$ are

$$(3)/\varnothing, \qquad (4,1)/3, \qquad (5,3,1)/(4,1),$$

that is:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array},$$

$$\begin{array}{cccc} & & & \boxed{2} \\ \boxed{2} & & & \end{array}$$

$$\begin{array}{ccccc} & & & & \boxed{3} \\ & \boxed{3} & \boxed{3} & & \\ \boxed{3} & & & & \end{array}$$

We've left the letters 1,2,3 in the skew tableau to remind us that these skew shapes came from the original semistandard Young tableau by keeping only the boxes with a given label.

We let w_0 be the "long element" of the Weyl group $W = S_n$, which is the permutation that maps k to $n+1-k$ of $\{1, 2, 3, \cdots, n\}$. If $\mathbf{z} = (z_1, \cdots, z_n) \in (\mathbb{C}^\times)^n$ and $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$, then $w_0\mathbf{z} = (z_n, \cdots, z_1)$ and $w_0\mu = (\mu_n, \cdots, \mu_1)$. Obviously $\mathbf{z}^{w_0\mu} = (w_0\mathbf{z})^\mu$.

Proposition 4.5. *Let $\mathfrak{s}$ be an admissible state of the system $\mathfrak{S}_\lambda(\mathbf{z}; 0)$. Since $\mathfrak{s}$ has no $\mathtt{b}_1$ patterns, the corresponding Gelfand-Tsetlin pattern G is column strict. Let $G^\circ = G - P$, which is a Gelfand-Tsetlin pattern with top row λ. Let T be the semistandard Young tableau associated with G° as in Lecture 2. Then $\beta(\mathfrak{s}) = \mathbf{z}^\rho \cdot (w_0\mathbf{z})^{\mathrm{wt}(T)}$.*

Proof. Since the Boltzmann weights of every vertex can only be 1 or z_i for some i, it is obvious that $\beta(\mathfrak{s})$ is a monomial $\mathbf{z}^\mu$ and we need to compute μ. This is accomplished by Proposition 1.1. Writing $G = P + G^\circ$ the contribution of P is obviously $\mathbf{z}^\rho$, and we must discuss the contribution of G^o but by Lemma 4.3 and Proposition 1.1, this is $\mathbf{z}^{w_0 \ \mathrm{wt}(T)} = (w_0\mathbf{z})^{\mathrm{wt}(T)}$. $\square$

Theorem 4.6. *The polynomial $S_\lambda = s_\lambda$ where s_λ is the Schur function defined by the second combinatorial definition.*

Proof. To summarize what we have done so far, culminating in Proposition 4.5, we have seen that every state $\mathfrak{s}$ of $\mathfrak{S}_\lambda(\mathbf{z};0)$ has no $\mathtt{b}_1$ patterns. Such states are parametrized by left-strict Gelfand-Tsetlin patterns with top row $\lambda+\rho$. Each such pattern G can be written as $G^\circ + P$ where G° is a Gelfand-Tsetlin pattern with top row λ. If T is tableau corresponding to G° then $\beta(\mathfrak{s}) = \mathbf{z}^\rho \cdot (w_0\mathbf{z})^{\mathrm{wt}(T)}$. Summing over all states and using the combinatorial definition of the Schur function we obtain

$$Z_\lambda(\mathbf{z};0) = \mathbf{z}^\rho\, s_\lambda(w_0\mathbf{z}).$$

On the other hand, we have shown for all q that

$$Z_\lambda(\mathbf{z};q) = \left(\prod_{i<j} z_i - qz_j\right) S_\lambda(\mathbf{z}).$$

When $q=0$, the product becomes $z_1^{n-1}z_2^{n-2}\cdots = \mathbf{z}^\rho$. Comparing gives

$$S_\lambda(\mathbf{z}) = s_\lambda(w_0\mathbf{z}).$$

We may replace $\mathbf{z}$ by $w_0\mathbf{z}$ and remember that we proved (using the Yang-Baxter equation) that S_λ is symmetric, so $S_\lambda = s_\lambda$. $\square$

Comparing the evaluations of $S_\lambda(\mathbf{z})$ when $q=1$ and $q=0$, we have now proved the equivalence of the two definitions of the Schur function.