Lecture 5

1 Schur Polynomials

Schur polynomials are symmetric polynomials very important in representation theory and combinatorics. Some useful references are [8, 10, 3, 4]. They have direct generalizations that are introduced in [7]. See [5, 2, 1, 9] for treatments using the *free-fermionic* six-vertex model. We recall that the Boltzmann weights are free-fermionic if $a_1(v)a_2(v) + b_1(v)b_2(v) - c_1(v)c_2(v) = 0$ at every vertex. This Lecture and the next are based on [2].

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of length $\leqslant n$. If $r < n$ we pad λ with 0's so that $\lambda = (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0)$ has exactly n parts. This is customary in dealing with partitions. We will give two definitions of the Schur polynomial s_λ. It will not be obvious that the two definitions are equivalent. We will use a lattice model to prove this.

1.1 First Definition

Define

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})}. \tag{1}$$

For example, if $n = 3$,

$$s_\lambda(x_1, x_2, x_3) = \frac{\begin{vmatrix} x_1^{\lambda_1+2} & x_2^{\lambda_1+2} & x_3^{\lambda_1+2} \\ x_1^{\lambda_2+1} & x_2^{\lambda_2+1} & x_3^{\lambda_2+1} \\ x_1^{\lambda_3} & x_2^{\lambda_3} & x_2^{\lambda_3} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}}. \tag{2}$$

Wikipedia attributes this definition to Jacobi, who defined Schur functions prior to Schur. The denominator is the Vandermonde determinant:

$$\det(x_j^{n-i}) = \prod_{i<j}(x_i - x_j).$$

It will be useful to introduce the vector $\rho = (n-1, n-2, \cdots, 0)$ so that the exponents are $\lambda_i + \rho_i$ and write the numerator as $\det(x_j^{(\lambda+\rho)_i})$.

Lemma 1.1. *The function s_λ is a symmetric polynomial. It is homogeneous of degree $|\lambda| = \sum \lambda_i$.*

Proof. The polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ is a unique factorization domain. Let us note that the numerator is divisible by every factor $x_i - x_j$ with $i < j$ of the Vandermonde denominator. Indeed, the numerator vanishes when $x_i = x_j$ since two columns of the determinant $\det(x_j^{\lambda_i+n-i})$ are then equal. Thus the numerator is divisible by each factor and therefore by their product since they are coprime. Therefore s_λ is a polynomial. It is symmetric since interchanging x_i and x_j multiplies the numerator and the denominator by -1. The homogeneity is also clear since the numerator and denominator are both homogeneous polynomials. □

Remark 1. The partition λ may be thought of as a dominant weight for the Lie group $\mathrm{GL}(n, \mathbb{C})$. The definition (2) is essentially the Weyl character formula. This formula gives the value character χ_λ of an irreducible representation with highest weight λ for an arbitrary Lie group at a point $\mathbf{z}$ in a fixed maximal torus as

$$\frac{\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\rho)}}.$$

In this formula ρ would be usually be the "Weyl vector" which is half the sum of the positive roots, $\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2}\right)$ for $\mathrm{GL}(n, \mathbb{C})$. We've taken $\rho = (n-1, n-2, \cdots, 0)$, but this change just multiplies the numerator and denominator by the same constant. The Weyl group for $\mathrm{GL}(n, \mathbb{C})$ is the symmetric group S_n, so for $\mathrm{GL}(n, \mathbb{C})$, the alternating sum is just the determinant in (2). It follows that s_λ is essentially the character χ_λ of an irreducible representation π_λ of $\mathrm{GL}(n, \mathbb{C})$. More precisely, if $g \in \mathrm{GL}(n, \mathbb{C})$ has eigenvalues $\alpha_1, \cdots, \alpha_n$ then $\chi_\lambda(g) = s_\lambda(\alpha_1, \cdots, \alpha_n)$.

1.2 Second Definition

The *Young diagram* $\mathrm{YD}(\lambda)$ of a partition λ a collection of boxes with λ_1 in the first row, λ_2 in the second row, etc. A *semistandard Young tableau* T (SSYT) of shape λ in the alphabet $\{1, 2, \cdots, n\}$ is a filling of $\mathrm{YD}(\lambda)$ with integers $1, \cdots, n$ such that the rows are weakly increasing, and the columns are strictly decreasing. The *weight* $\mathrm{wt}(T)$ is $(\mu_1, \cdots, \mu_n)$ where μ_i is the number of i's in T.

Example 1.2. *Let $\lambda = (5, 2, 2)$ and $n = 5$. Then*

$T =$

1	1	2	2	5
2	2			
3	5			

is a SSYT of shape λ. Its weight is $(2, 4, 1, 0, 2)$.

If $\mathbf{z} = (z_1, \cdots, z_n) \in (\mathbb{C}^\times)^n$ and $\mu \in \mathbb{Z}^n$ let $\mathbf{z}^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$. The second definition of the Schur function is due to D.E. Littlewood (1938). It is this formula:

$$s_\lambda(z_1, \cdots, z_n) = \sum_T \mathbf{z}^{\mathrm{wt}(T)} \tag{3}$$

It is not obvious that this is symmetric. On the other hand, the Schur polynomial has another important property, *positivity*, that is not obvious from the first definition. This is the fact that as a polynomial, the coefficients are nonnegative.

We will use a lattice model to show that (3) is symmetric and equivalent to the first definition.

2 Tokuyama models

There is a formula due to Tokuyama [11] for the Schur function, or more precisely for

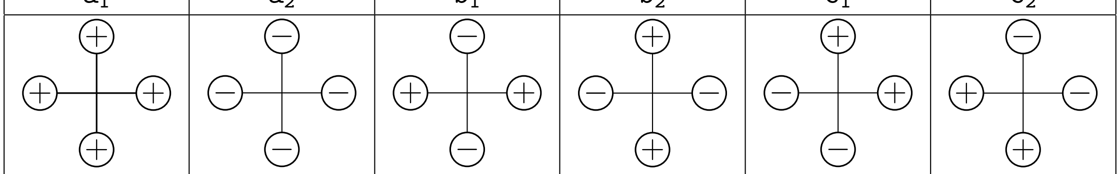
$$\left\{\prod_{i<j}(z_i - qz_j)\right\} s_\lambda(z_1, \cdots, z_n)$$

as a sum over strict Gelfand-Tsetlin patterns. If $q = 1$, the product is the Vandermonde determinant in the denominator of the first definition, and Tokuyama's formula reduces to the first definition of the Schur polynomial. On the other hand, if $q = 0$, Tokuyama's formula reduces to the combinatorial definition.

The models we will describe are similar to models in Hamel and King [5]. However they did not use the Yang-Baxter equation. The Yang-Baxter equation we need is associated with the quantum group $U_q(\widehat{\mathfrak{gl}}(1|1))$, related to the Lie superalgebra $\mathfrak{gl}(1|1)$. The modules associated with the horizontal edges are related to two-dimensional evaluation modules, but the module associated to the vertical edges are associated to two-dimensional Kac modules. This is not a parametrized Yang-Baxter equation as we formulated it in Lecture 4. However this solution can be embedded in a parametrized Yang-Baxter equation found by Korepin (see [6], page 126) and Brubaker, Bump and Friedberg [2], with parameter group $\mathrm{GL}(2) \times \mathrm{GL}(1)$ that parametrizes *all* free-fermionic vertices.

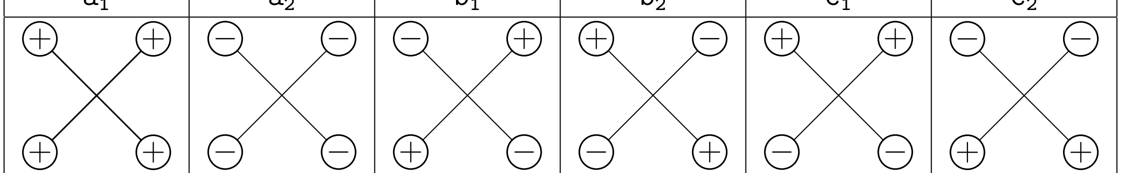
We take the following weights, labeled by a complex number z:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	z_i	$-q$	z_i	$z_i(1-q)$	1

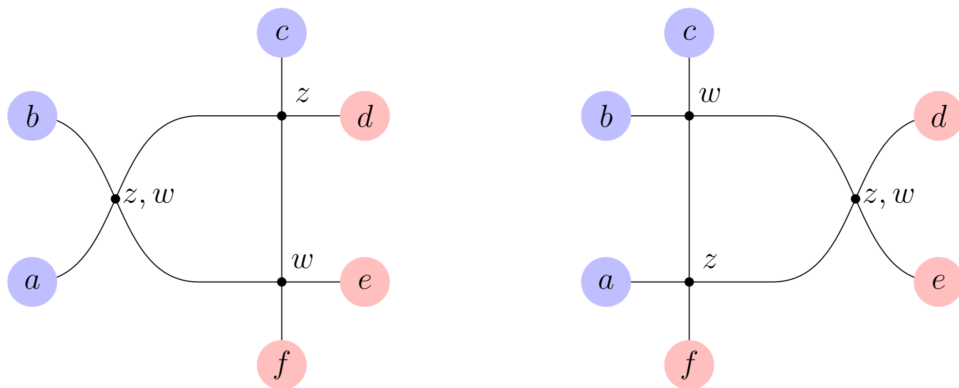


we also take the following R-matrix, labeled by two complex numbers z, w:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
$z_j - qz_i$	$z_i - qz_j$	$q(z_i - z_j)$	$z_i - z_j$	$(1-q)z_i$	$(1-q)z_j$

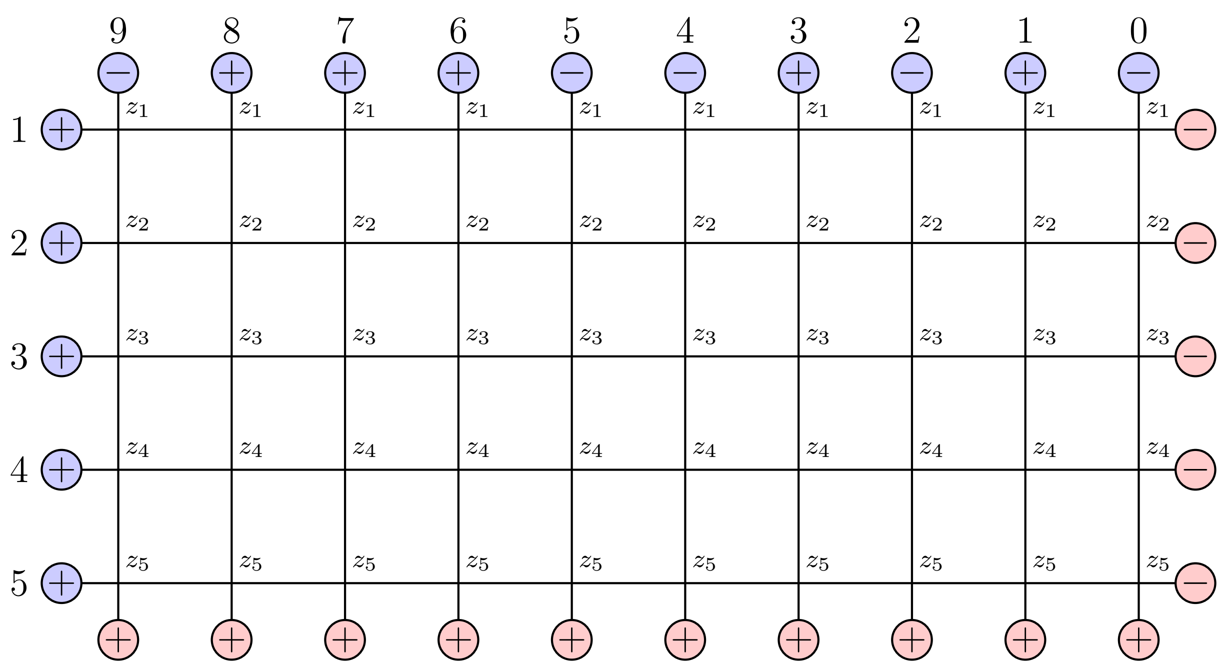


Theorem 2.1. *The Yang-Baxter equation equation is satisfied in that he following two systems are equivalent for all choices of $a, b, c, d, e, f \in \{+, -\}$:*



Now let us explain the models we want to use, called “Gamma Ice” in [2].

Fix a partition λ. We begin with grid with rows labeled from top to bottom by $z_1, \cdots, z_n$ and columns labeled $0, \cdots, N$ with $N \geqslant \lambda_1$, ordered *from right to left*. The partition function will turn out to be independent of N. We will use the weights described above, so every vertex in the same row has the same label z_i. We must describe the spins on the boundary edges. On the left and bottom edges we put $+$, on the right we put $-$, and on the vertical j we put $-$ if j is an entry in $\lambda + \rho = (\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_n)$, or $+$ if j is not an element of this vector. For example if $n = 5$ and $\lambda = (5, 2, 2)$, we pad λ with zeros to get $(5, 2, 2, 0, 0)$ and then $\lambda + \rho = (9, 5, 4, 2, 0)$. Therefore we arrive at the following boundary conditions:



We have defined a system that we will denote $\mathfrak{S}_\lambda(\mathbf{z}; q)$. Let $Z_\lambda(z_1, \cdots, z_n; q) = Z_\lambda(\mathbf{z}; q)$ be the corresponding partition function.

Theorem 2.2. *The partition function*

$$Z_\lambda(\mathbf{z};q) = \prod_{i<j}(z_i - qz_j) S_\lambda(\mathbf{z})$$

where $S_\lambda(\mathbf{z}) = S_\lambda(z_1, \cdots, z_n)$ is a symmetric polynomial that is independent of q.

We will give part of the proof in the next section using the train argument and the Yang-Baxter equation. We will then show in Lecture 6 that it implies the equivalence of the two definitions of the Schur function.

3 Proof of the Theorem, part I

We can break the proof into three steps.

Proposition 3.1. *The quotient*

$$S_\lambda(\mathbf{z};q) = \frac{Z_\lambda(\mathbf{z};q)}{\prod_{i<j}(z_i - qz_j)} \tag{4}$$

is symmetric, that is, invariant under permutations of the z_i.

Proof. We multiply (4) by:

$$\prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} (z_i - qz_j) \ .$$

This is a symmetric polynomial of degree $n(n-1)$ consisting of the $\frac{1}{2}n(n-1)$ factors in the denominator of (4) and $\frac{1}{2}n(n-1)$ others, so we see that it is enough to show that

$$Z_\lambda(\mathbf{z};q) \prod_{i<j}(z_j - qz_i)$$

is symmetric. Let $1 \leqslant k < n$ and let s_k be the "simple reflection" in the symmetric group which interchanges k and $k+1$. These generate the symmetric group, so it is sufficient to show that the last expression is invariant under s_k.

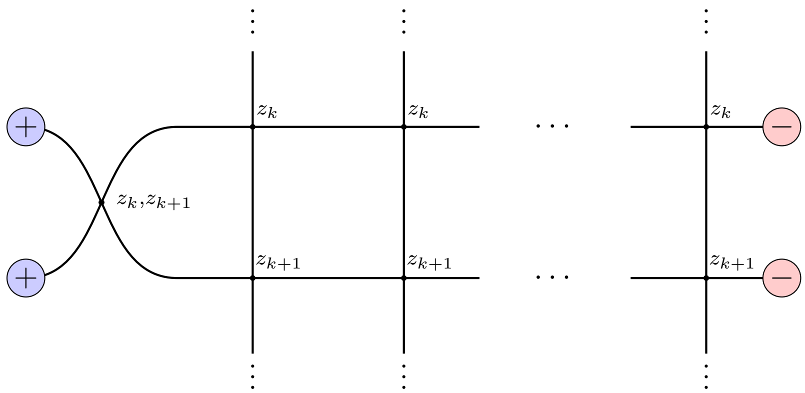
We can pull one factor out and write this as

$$Z_\lambda(\mathbf{z};q)(z_{k+1} - qz_k) \left[\prod_{\substack{i<j \\ (i,j) \neq (k,k+1)}} (z_j - qz_i). \right]$$

The permutation s_k just permutes the $\frac{1}{2}n(n-1) - 1$ factors in brackets. So we may drop these and it is sufficient to show that

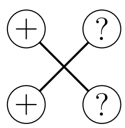
$$Z_\lambda(\mathbf{z};q)(z_{k+1} - qz_k) = Z_\lambda(s_k\mathbf{z};q)(z_k - qz_{k+1}). \tag{5}$$

To see this, let us consider the following system. We attach the R-matrix with coordinates z_k, z_{k+1} to the left at the $k, k+1$ rows:



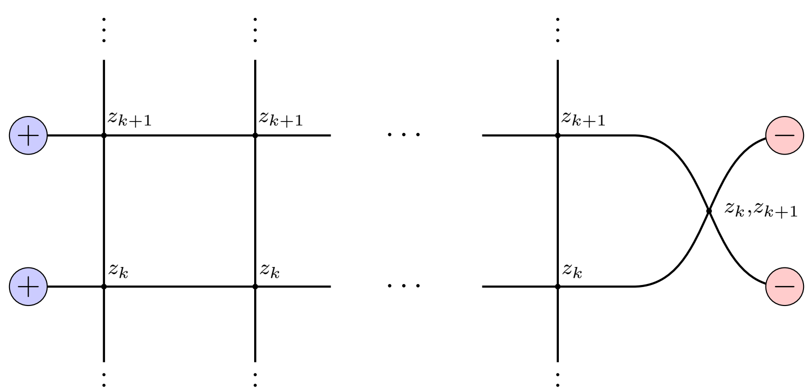
(6)

We note that from the Boltzmann weights if the "input" spins are $+,+$ there is only one possibility for the output spins, which must also be $+,+$:



spins marked ? can only = + !

The Boltzmann weight of the R-matrix is $z_{k+1} - qz_k$, and so the partition function of the system (6) is the left-hand side of (5). Using the train argument, this equals the partition function of



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and by the same reasoning, this equals the right-hand side of (5). This proves (5) and the symmetry of $S_\lambda(\mathbf{z}; q)$ is established. □

$$Z_\lambda(\mathbf{z}; q) = \prod_{i<j} (z_i - qz_j) S_\lambda(\mathbf{z})$$

Proposition 3.2. *$S_\lambda(\mathbf{z}; q)$ is a polynomial in $z_1, \cdots, z_n$ and q.*

Proof. It is clear that $Z_\lambda(\mathbf{z};q)$ is a polynomial, since every Boltzmann weight is a polynomial. Rewrite (4) as

$$S_\lambda(\mathbf{z};q) = \frac{\prod_{i>j}(z_i - qz_j) Z_\lambda(\mathbf{z};q)}{\prod_{i\neq j}(z_i - qz_j)}. \tag{7}$$

Both the numerator and the denominator on the right-hand side here are symmetric. In the polynomial ring $\mathbb{C}[z_1, \cdots, z_n, q]$, which is a unique factorization domain, the denominator is a product of coprime polynomials, and it is sufficient to show that it is divisible by each. If $i > j$ then it is obvious that the numerator in (7) is divisible by $z_i - qz_j$ since it is included as a factor in the product defining the numerator. Because it is symmetric, it is divisible by all factors $z_i - qz_j$ because the symmetric group permutes these transitively. Thus the quotient $S_\lambda(\mathbf{z};q)$ is a polynomial. □

Lemma 3.3. *Let $\mathfrak{s}$ be a state of the model. The total number of patterns of types $\mathtt{a}_2, \mathtt{b}_1$ and $\mathtt{c}_1$ in the state is $\frac{1}{2}n(n-1)$.*

Proof. A vertex is of type $\mathtt{a}_2, \mathtt{b}_1$ or $\mathtt{c}_1$ if and only if it has a $-$ in the vertical edge below the vertex. We recall the Gelfand-Tsetlin pattern associated to the state in Lemma 1.3 of Lecture 3. There is a $-$ spin on the vertical edge below the vertex in row i and column j if and only if j is one of the entries in the $(i+1)$-th row of the Gelfand-Tsetlin pattern. There are thus $n-1$ patterns of type $\mathtt{a}_2, \mathtt{b}_1$ or $\mathtt{c}_1$ in the first row, $n-2$ in the second row, and so forth, and $\frac{1}{2}n(n-1)$ altogether. □

Proposition 3.4. *$S_\lambda(\mathbf{z};q)$ is independent of q.*

Proof. The numerator and denominator in (4) are both polynomials in $z_1, \cdots, z_n, q$ and the denominator has degree $\frac{1}{2}n(n-1)$ in q. We claim that the numerator has a too. Reviewing the Boltzmann weights, only patterns of types $\mathtt{b}_1$ and $\mathtt{c}_1$ can contribute a power of q. The number of such patterns is at most $\frac{1}{2}n(n-1)$ by Lemma 3.3.

Since the degree in q of the numerator of (4) is at most $\frac{1}{2}n(n-1)$, and the degree of the denominator is exactly $\frac{1}{2}n(n-1)$. Since the quotient is known to be a polynomial, it has degree 0 in q, hence is independent of q. □

Since $S_\lambda(\mathbf{z};q)$ is independent of q, we may suppress q from the notation and write $S_\lambda(\mathbf{z};q) = S_\lambda(\mathbf{z})$. We have proved that it is a symmetric polynomial. In the next lecture we will show that if $q=0$, this agrees with the combinatorial definition of $s_\lambda(\mathbf{z})$, and if $q=1$, it agrees with the Jacobi definition.

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