Lecture 4

We will start by surveying the origin of solutions to the Yang-Baxter equation in the notions of *braided monoidal categories* and (sketchily) *quantum groups*. Then we will look again at the notion of a *parametrized Yang-Baxter equation*, in which the vertex types are indexed by a group or groupoid. We will give one example, coming from the field-free Yang-Baxter equation in Lecture 1, obtaining a clearer picture.

1 Review of Lecture 3

Let U, V and W be vector spaces. Suppose that we are given three linear transformations:

$$R : U \otimes V \longrightarrow V \otimes U,$$
$$S : U \otimes W \longrightarrow W \otimes U,$$
$$T : V \otimes W \longrightarrow W \otimes V.$$

We will consider two homomorphisms $U \otimes V \otimes W \longrightarrow W \otimes V \otimes U$. The first is the composition

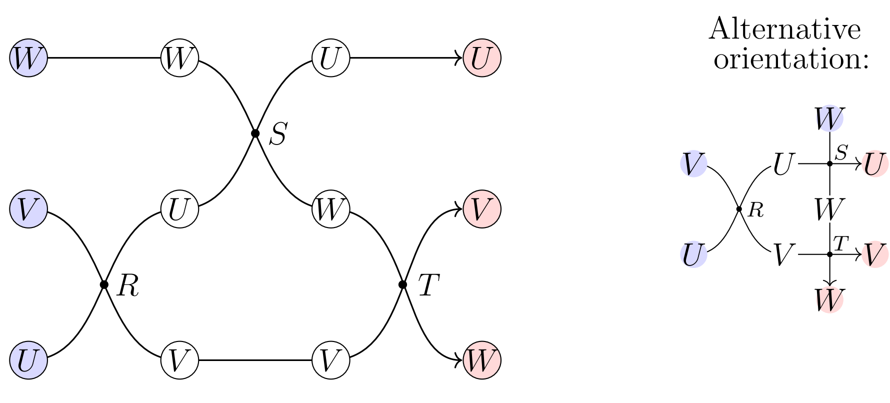
$$U \otimes V \otimes W \xrightarrow{R_{12}} V \otimes U \otimes W \xrightarrow{S_{23}} V \otimes W \otimes U \xrightarrow{T_{12}} W \otimes V \otimes U$$

where the notation is that R_{ij} means R applied to the i and j components of a tensor. Thus $R_{12} = R \otimes I_W$, $S_{23} = I_V \otimes S$ and $T_{12} = T \otimes I_U$. (The subscript notation is popular in Hopf algebra and quantum group literature.) The other homomoprhism is

$$U \otimes V \otimes W \xrightarrow{T_{23}} U \otimes W \otimes V \xrightarrow{S_{12}} W \otimes U \otimes V \xrightarrow{R_{23}} W \otimes V \otimes U.$$

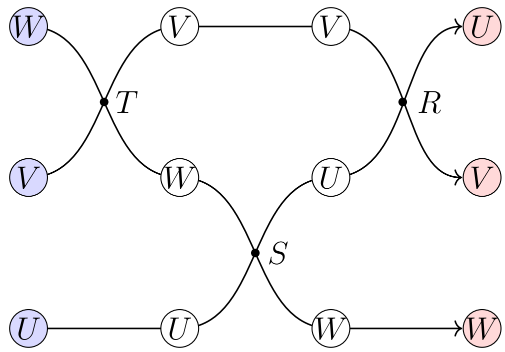
We can diagram the homomorphisms graphically as follows.

$$U \otimes V \otimes W \xrightarrow{R_{12}} V \otimes U \otimes W \xrightarrow{S_{23}} V \otimes W \otimes U \xrightarrow{T_{12}} W \otimes V \otimes U$$

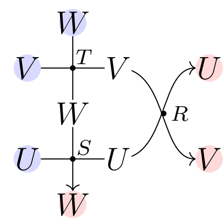


and

$$U \otimes V \otimes W \xrightarrow{T_{23}} U \otimes W \otimes V \xrightarrow{S_{12}} W \otimes U \otimes V \xrightarrow{R_{23}} W \otimes V \otimes U$$



Alternative orientation:



To understand these pictures, they diagram homomorphisms from inputs $U \otimes V \otimes W$ (in blue, read from bottom to top) to $W \otimes V \otimes U$ (in red, read from bottom to top). The alternative orientations are supplied since those will often occur in practice.

The vector space version of the Yang-Baxter equation is that these two homomorphisms $U \otimes V \otimes W \to W \otimes V \otimes U$ are equal, in other words we want a commutative diagram:

$$\begin{tikzcd}
& U \otimes V \otimes W \arrow[dl, "R_{12}"'] \arrow[dr, "T_{23}"] & \\
V \otimes U \otimes W \arrow[d, "S_{23}"'] & & U \otimes W \otimes V \arrow[d, "S_{12}"] \\
V \otimes W \otimes U \arrow[dr, "T_{12}"'] & & W \otimes U \otimes V \arrow[dl, "R_{23}"] \\
& W \otimes V \otimes U &
\end{tikzcd}$$

It turns out that the "natural habitat" for the Yang-Baxter equation is a *braided monoidal category*. In the next sectionm we digress to introduce this notion.

2 Braided Monoidal Categories

This section is **optional** and can be skipped or postponed on first reading.

The axioms for a braided monoidal category are due to Joyal and Street [2] in the 1980's. It is surprising that such an important concept was not formulated until so late. But there weren't many obvious examples of braided monoidal categories until quantum groups. But it turns out that the modules of a quantum group form a braided category, giving a tremendous fount of examples of the Yang-Baxter equation. We digress to introduce this notion.

$\boxed{\text{Wikipedia link}}$

A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes$ satisfying certain natural axioms. There is a *unit object* I with natural isomorphisms

$$A \otimes I \cong I \otimes A \cong A$$

for A any object in the category, and for three objects A, B, C a natural isomorphism

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

satisfying Maclane's *pentagon axiom*

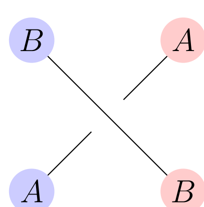
$$\begin{tikzcd}
& & (A \otimes B) \otimes (C \otimes D) \arrow[drr] & & \\
((A \otimes B) \otimes C) \otimes D \arrow[urr] \arrow[dr] & & & & A \otimes (B \otimes (C \otimes D) \\
& (A \otimes (B \otimes C)) \otimes D \arrow[rr] & & A \otimes ((B \otimes C) \otimes D) \arrow[ur] &
\end{tikzcd}$$

Maclane's coherence theorem asserts that all similar identities (perhaps involving many tensors) can be deduced from this one.

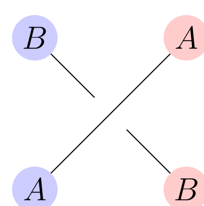
Let $\mathcal{C}$ be a monoidal category. We recall that if A, B, C are objects in $\mathcal{C}$ then there are natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. We will not distinguish between these objects and just denote either as $A \otimes B \otimes C$.

In a braided category there are explicit *braid isomorphisms* $c_{A,B} : A \otimes B \to B \otimes A$ but now we must be careful. For example the composition $c_{B,A}c_{A,B}$ is not assumed to be the identity. So $c_{A,B}$ and $c_{B,A}^{-1}$ are *distinct isomorphisms* $A \to B$.

We will notate the morphism $c_{A,B}$ by an over crossing and $c_{B,A}$ by an under crossing.



$c_{A,B} : A \otimes B \to B \otimes A$



$c_{B,A}^{-1} : A \otimes B \to B \otimes A$

We review the important notion of a natural transformation. We used this implicitly when we defined a monoidal category in Lecture 1, where we said that the isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

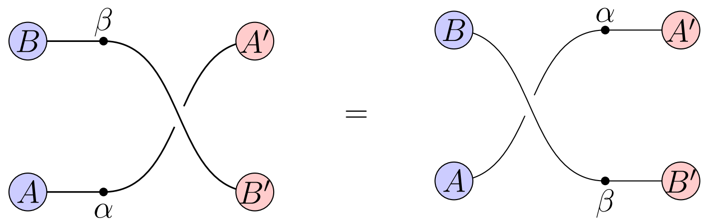
are required to be *natural*.

This means, explicitly, the following. Since $\otimes$ is a bifunctor, if $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ are morphisms then we have on the left and right of the following diagram.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\cong"] \arrow[d, "(\alpha\otimes\beta)\otimes\gamma"] & A \otimes (B \otimes C) \arrow[d, "\alpha\otimes(\beta\otimes\gamma)"] \\
(A' \otimes B') \otimes C' \arrow[r, "\cong"] & A' \otimes (B' \otimes C')
\end{tikzcd}$$

The first axiom of a braided category is that the morphisms $c_{A,B} : A \otimes B \to B \otimes A$ are to be natural. This means that if $\alpha : A \to A'$ and $\beta : B \to B'$ are morphisms, then

$$(\beta \otimes \alpha) \circ c_{A,B} = c_{A',B'} \circ (\alpha \otimes \beta)$$

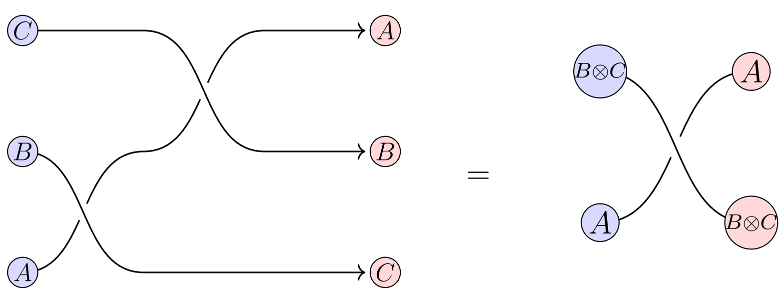


(We are representing the morphisms α, β by dots.)

The braid morphism $c_{A,B} : A \otimes B \to B \otimes A$ is sometimes called an *R-matrix*. It is subject to a couple of axioms. First, it is assumed to satisfy:

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A,B\otimes C}"] \arrow[dr, "c_{A,B}\otimes 1_C"'] & & B \otimes C \otimes A \\
& B \otimes A \otimes C \arrow[ur, "1_B \otimes c_{A,C}"'] &
\end{tikzcd}$$

We can diagram this as follows.

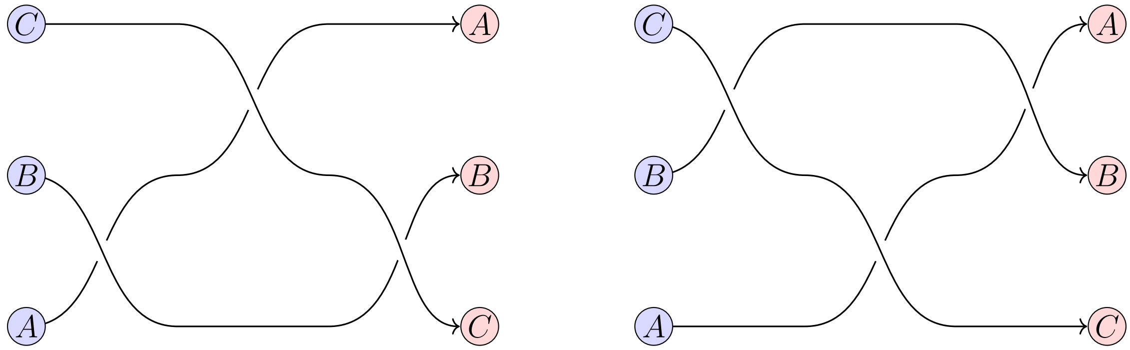


The dual axiom is also needed:

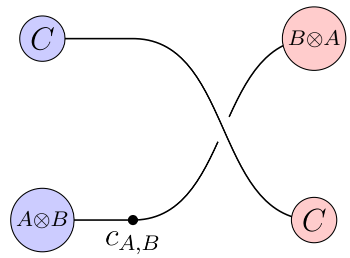
$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A\times B,C}"] \arrow[dr, "1_A\otimes c_{B,C}"'] & & C \otimes A \otimes B \\
& A \otimes C \otimes B \arrow[ur, "c_{A,C} \otimes 1_B"'] &
\end{tikzcd}$$

This completes the definition of a braided monoidal category.

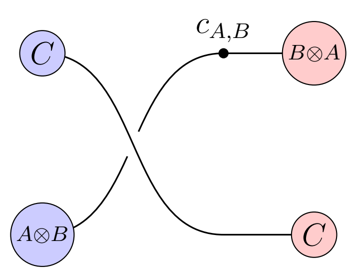
Theorem 2.1. *The Yang-Baxter equation is true in a braided monoidal category. This means we have to show the equivalence of the two following morphisms $A \otimes B \otimes C \to C \otimes B \otimes A$:*



Proof. Using one of the axioms for the braided category, the first diagram agrees with:



Using naturality, this agrees with



Now using the other axiom, this is equivalent to the morphism in the second diagram. □

3 Quantum Groups

We see that objects in a braided category, particularly if they can be realized as vector spaces, are a potential source of instances of the Yang-Baxter equation. These have applications (as we know) to solvable lattice models, but also to other areas, such as knot invariants (e.g. the Jones polynomial).

Around the same time that Joyal and Street formulated the notion of a braided category, Drinfeld [1] invented the notion of a *quasitriangular Hopf algebra*.

If H is an associative algebra, one might hope that the modules form a monoidal category. However if A, B are modules then $A \otimes B$ is not naturally a module for H, but for the tensor product algebra $H \otimes H$. A *Hopf algebra* is an associative algebra H together with an algebra homomorphism $\Delta : H \to H \otimes H$ called the *comultiplication* and some other structure (antipode, counit, various axioms). Using Δ, $A \otimes B$ becomes a module for H, and so the modules become a monoidal category.

A *quasitriangular Hopf algebra* has some further extra structure, a *universal R-matrix* $R \in H \otimes H$ satisfying certain axioms that we will not state here. (See [1, 4, 3].) What is important is that using R we may define a braiding $c_{A,B} : A \otimes B \to B \otimes A$, and Drinfeld's axioms for a quasitriangular Hopf algebra are exactly what is needed for the module category to be braided.

Drinfeld then constructed quasitriangular Hopf algebras called *quantum groups* as deformations of more familiar Hopf algebras. If $\mathfrak{g}$ is a Lie algebra, the *universal enveloping algebra* of $\mathfrak{g}$ is an associative algebra $U(\mathfrak{g})$ whose modules are the same as the modules of $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, or more generally a Kac-Moody Lie algebra or superalgebra, then it is possible to deform $U(\mathfrak{g})$ and obtain a family of Hopf algebras $U_q(\mathfrak{g})$ called *quantized enveloping algebras*. If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, it has an *affinization* $\widehat{\mathfrak{g}}$ which is infinite-dimensional. If V is a module for $\mathfrak{g}$, then $\widehat{\mathfrak{g}}$ has a family V_z of modules indexed by $z \in \mathbb{C}^\times$.

There are two choices of $\mathfrak{g}$ that have two-dimensional modules: $\mathfrak{g} = \mathfrak{sl}_2$ (or almost the same thing for this purpose, $\mathfrak{gl}_2$) or the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1|1)$. Both are related to the six-vertex model: $\widehat{\mathfrak{gl}}_2$ is related to the field-free models we have started with, while $\mathfrak{gl}(1|1)$ is related to the free-fermionic models that we will discuss later.

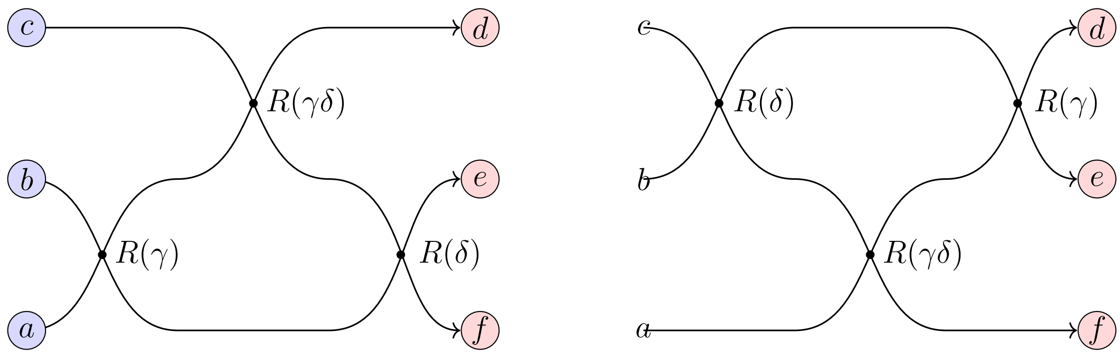
4 Parametrized Yang-Baxter equations

Let Γ be a group, and let V be a vector space. Let $R : \Gamma \longrightarrow \mathrm{GL}(V \otimes V)$ be a map such that for every $\gamma, \delta \in \Gamma$, we have a vector Yang-Baxter equation:

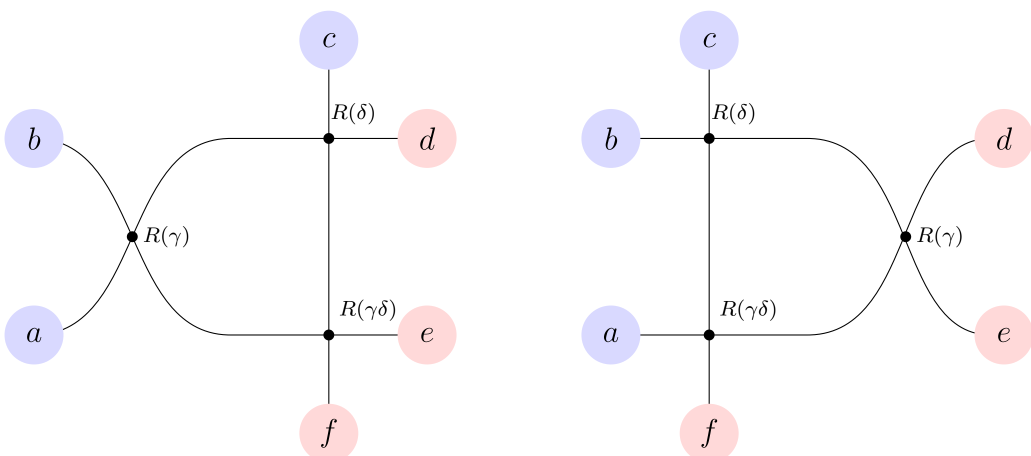
$$\begin{tikzcd}
& V \otimes V \otimes V \arrow[dl, "R_{12}(\gamma)"'] \arrow[dr, "R_{23}(\delta)"] & \\
V \otimes V \otimes V \arrow[d, "R_{23}(\gamma\delta)"'] & & V \otimes V \otimes V \arrow[d, "R_{12}(\gamma\delta)"] \\
V \otimes V \otimes V \arrow[dr, "R_{12}(\delta)"'] & & V \otimes V \otimes V \arrow[dl, "R_{23}(\gamma)"] \\
& V \otimes V \otimes V &
\end{tikzcd}$$

Then we say that we have a *parametrized Yang-Baxter equation* with parameter group Γ.

Alternatively, we could let Σ be a set and for every $\gamma \in \Gamma$ let there be a vertex type, where all edges have the spinset Γ. We will use the notation $R(\gamma)$ for this vertex type. Then we ask that for all a, b, c, d, e, f the two following partition functions are equal:



We can obtain a parametrized Yang-Baxter equation taking V to be the free vector-space on Σ and following the construction of the last section. We could alternatively orient the edges as follows:



In either case, the procedure in Lecture 3 produces a vector Yang-Baxter equation, with V being the free vector space on the spinset Σ.

We will show in the next section that the field-free Yang-Baxter equation of Section 1 gives an example of a parametrized Yang-Baxter equation.

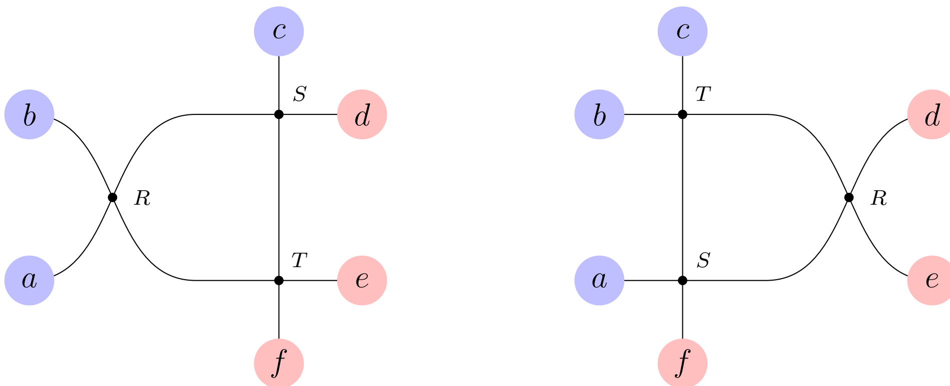
5 Parametrized Field-Free Yang-Baxter equation

Let $\Delta \in \mathbb{C}$ be fixed. We assume $\Delta \neq 0, 1, -1$. Let q be found such that $\frac{1}{2}(q + q^{-1}) = \Delta$. We will use the notation $R(a, b, c)$ for the vertex with Boltzmann weights a, b, c, as before. Let G_Δ be the set of (a, b, c) with $a, b \neq 0$ such that

$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta,$$

together with two additional elements $(\pm\Delta, 0, \Delta)$. Eventually we will give G_Δ the structure of a group.

In Lecture 1 we showed that if (a_1, b_1, c_1) and (a_2, b_2, c_2) are in G_Δ then there exists a third $(a_0, b_0, c_0) \in G_\Delta$ such that if (in the notation of Lecture 1) $R = v(a_0, b_0, c_0)$, $S = v(a_1, b_1, c_1)$ and $T = v(a_2, b_2, c_2)$, then we have a Yang-Baxter equation:



(1)

We note that the Yang-Baxter equation is homogeneous in the sense that if any one of (a_i, b_i, c_i) is multiplied by a nonzero constant then the validity of the equation is unchanged. So while R is usually determined by S and T, it is only determined up to constant multiple.

Now we want to start with R and T and compute S. This will give us our first example of a parametrized Yang-Baxter equation. We begin by noting that G_Δ can be parametrized as follows.

Lemma 5.1. *Let $x \in \mathbb{C}^\times$ and let*

$$(a, b, c) = \left(\tfrac{1}{2}(xq - (xq)^{-1}), \tfrac{1}{2}(x - x^{-1}), \tfrac{1}{2}(q - q^{-1})\right).$$

Then $(a, b, c) \in G_\Delta$.

Proof. This is a straightforward calculation. □

Theorem 5.2. *The mapping*

$$R_\Delta : \mathbb{C}^\times \longrightarrow \{\textit{field-free Boltzmann weights } (a, b, c)\}$$

is a parametrized Yang-Baxter equation with parameter group $\mathbb{C}^\times$.

Proof. The Boltzmann weights are

$$\beta_\Delta(R) = \left(\frac{1}{2}((xq) - (xq)^{-1}), \frac{1}{2}(x - x^{-1}), \frac{1}{2}(q - q^{-1})\right),$$

$$\beta_\Delta(S) = \left(\frac{1}{2}(xyq - (xyq)^{-1}), \frac{1}{2}(xy - (xy)^{-1}), \frac{1}{2}(q - q^{-1})\right),$$

$$\beta_\Delta(T) = \left(\frac{1}{2}(yq - (yq)^{-1}), \frac{1}{2}(y - y^{-1}), \frac{1}{2}(q - q^{-1})\right).$$

Checking the parametrized Yang-Baxter equation is now a matter of computation. There are 12 cases of boundary Boltzmann weights that give nontrivial identities, but acually these are reduandant and there are only 4 distinct indentities that need to be checked. I have posted a computer program called `field-free1.sage` at the class web page that checks this. □

References

[1] V. G. Drinfeld. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820. Amer. Math. Soc., Providence, RI, 1987.

[2] A. Joyal and R. Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[3] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[4] S. Majid. *A quantum groups primer*, volume 292 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.