Lecture 3

1 Gelfand-Tsetlin Patterns and States

Remark 1.1. *This section mentions some facts about Lie group representations and Schur polynomials. These are included since they may be helpful to some readers, but may be skipped. Schur polynomials will be properly introduced later and we will prove their principal properties using lattice models.*

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a sequence of nonnegative integers. We say λ is a *partition* of length $\leqslant n$ if it is weakly decreasing:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

(Note that we say the length is $\leqslant n$. The actual length is the number of nonzero entries.) We say λ is a partition of k, and write $\lambda \vdash k$ if $\sum \lambda_i = k$. The partition is *strict* if

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n \geqslant 0.$$

A strict partition is the same as a partition into unequal parts.

A closely related notion is that of a *dominant weight* for $\mathrm{GL}(n)$. Assuming some Lie theory, we may identify $\mathbb{Z}^n$ with the $\mathrm{GL}(n)$ weight lattice Λ. Then the weight $\lambda = (\lambda_1, \cdots, \lambda_n)$ is *dominant* if

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

If λ is a dominant weight, then by results of Schur and Weyl, there is a unique irreducible representation $\pi_\lambda^{\mathrm{GL}(n)}$ of $\mathrm{GL}(n, \mathbb{C})$ with highest weight λ. Its character χ_λ is essentially the *Schur polynomial* s_λ. This is a symmetric polyomial such that

$$\chi_\lambda(g) = s_\lambda(z_1, \cdots, z_n)$$

where z_i are the eigenvalues of $g \in \mathrm{GL}(n, \mathbb{C})$.

Thus a partition of length $\leqslant n$ is a dominant weight for $\mathrm{GL}(n)$. A dominant weight λ is a partition only if $\lambda_n \geqslant 0$.

Gelfand-Tsetlin patterns are triangular arrays of integers satisfying certain inequalities. We may express this by saying that the rows are weakly decreasing, and adjacent rows *interleave*. This means the following. Suppose that $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\mu = (\mu_1, \cdots, \mu_{n-1})$ are partitions or more generally dominant weights. The condition for λ and μ to interleave is that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \lambda_n.$$

Note: λ and μ interleave if and only if $\pi_\mu^{\mathrm{GL}(n-1)}$ appears in the restriction of $\pi_\lambda^{\mathrm{GL}(n)}$ to $\mathrm{GL}(n-1,\mathbb{C})$. Indeed we will later prove (using lattice models) that

$$s_\lambda(z_1,\cdots,z_{n-1},1)=\sum_{\text{dominant }\mu\text{ interleaving }\lambda} s_\mu(z_1,\cdots,z_{n-1}),$$

which is called the $\mathrm{GL}(n)\Rightarrow\mathrm{GL}(n-1)$ *branching rule.*

We may now define a Gelfand-Tsetlin pattern of size n. This is a triangular array

$$A=\left\{\begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\ & a_{21} & & a_{22} & & \cdots & & a_{2,n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & & & & & & \\ & & & & a_{n0} & & & & \end{array}\right\}$$

with $n+1$ rows such that each row is a partition, and the rows interleave.

For example, there are 8 Gelfand-Tsetlin patterns with top row $(2,1,0)$. These are:

$$\left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\},\quad \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\}\quad \color{red}{\left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\}},\quad \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{array}\right\}$$

$$\left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array}\right\},\quad \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{array}\right\}\quad \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array}\right\},\quad \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array}\right\}.$$

These patterns are all strict, except the third one, which we have marked in red.

Strict Gelfand-Tsetlin patterns with top row $(n-1,n-2,\cdots,0)$ are in bijection with another type of mathematical entity called *alternating sign matrices.* We will not discuss alternating sign matrices much, preferring to work with Gelfand-Tsetlin patterns. But due to the historical importance of alternating sign matrices, here is the rundown.

An *alternating sign matrix* of size n is a square matrix whose entries are all 0, 1 or -1. It is assumed that in every row and column, the nonzero entries alternate between 1 and -1, strating and ending with 1, so there are an odd number of nonzero entries in each row and column. For example, a permutation matrix is an alternating sign matrix.

Lemma 1.2. *There is a bijection between strict Gelfand-Tsetlin patterns with top row $(n-1,n-2,\cdots,0)$ and alternating sign matrices of size n.*

Proof. We will explain the bijection with an example. Consider the alternating sign matrix

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We "accumulate" the rows from the bottom up into annother matrix:

$$B = \begin{bmatrix} 1^3 & 1^2 & 1^1 & 1^0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus the bottom row of B is the bottom row of A, the second-from-bottom row of B is the sum of the last two rows of A, and so forth. The little numbers indicate that we have labeled the columns of this $n-1$, $n-2$, $\cdots$, 0 from right to left, with $n = 4$ in the example.

Now we read off the columns of B that have nonzero entries (all equal to 1) and these form a strict Gelfand-Tsetlin pattern:

$$\left\{ \begin{array}{ccccccc} 3 & & 2 & & 1 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{array} \right\}.$$

We leave it to the reader to convince themselves that this is a bijection. □

Alternating sign matrices originated in a method of computing determinants due to Charles Dodgson (Lewis Carroll). In the 1980's, Mills, Robbins and Rumsey investigated the number of alternating sign matrices of size n and conjectured that the number of ASM of size n

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\,. \tag{1}$$

Here are some values:

n	1	2	3	4	5	6
$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$	1	2	7	42	429	7436

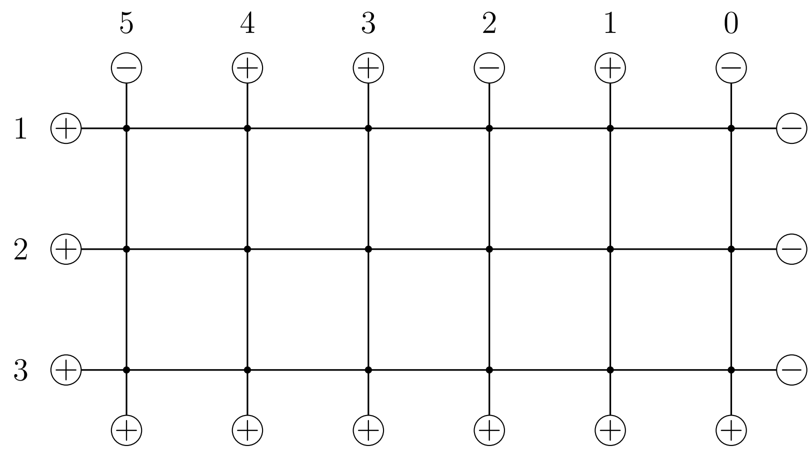
The number 7 when $n = 3$ we have already seen is the number of strict Gelfand-Tsetlin patterns with top row $(2, 1, 0)$.

Robbins and Rumsey consulted with Richard Stanley, who did not know how to prove the conjecture, but told them that the same numbers had appeared in another context in work of Andrews. They equal the number of "totally symmetric self-complementary plane partitions," a seemingly unrelated combinatorial number. The precise number remained only conjectural.

The number (1), conjectured by Mills, Robbins and Rumsey for the number of alternating sign matrices of size n, was proved correct by Zeilberger in difficult work that did not really give insight. Kuperberg then gave another proof using solvable lattice models that gave deep insight. We will eventually cover Kuperberg's proof, and the related Korepin-Izergin determinant formula.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a partition. We consider a model with domain wall boundary conditions as follows. We need a grid with $N+1$ columns, which we label from 0 to N from right to left, and n rows, labeled $1, 2, \cdots, n$ from top to bottom. We need $N \geqslant \lambda_1$. We have boundary edges on the left, right, top and bottom. On the left and bottom we put + spins, and on the right we put $-$ spins. On the top, the Lemma from Lecture 2 shows that we need n edges with $-$ spins, and the rest must be $+$. We put the $-$ spins at columns labeled $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Thus for example of $\lambda = (5, 2, 0)$ we need six columns and 3 rows, and we arrive at the following boundary conditions:



Lemma 1.3. *There is a bijection between the states of this system and the strict Gelfand-Tsetlin patterns with top row λ.*

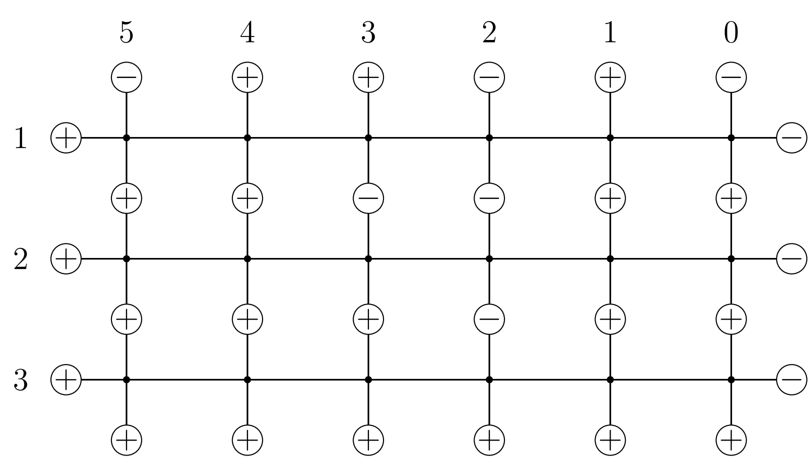
Proof. We will describe the bijection algorithmically (with an example) and leave it to the reader to convince themselves that the states described are admissible.

First, for each row $a_{i1}, \cdots, a_{i,n+1-i}$ in the given strict Gelfand-Tsetlin pattern, assign a $-$ spin to the vertical edges above the i-th row, and $+$ to the remaining vertical edges. Note that since $\lambda = (a_{11}, \cdots, a_{1n})$, this is consistent with the way we assigned the boundary spins at the top.

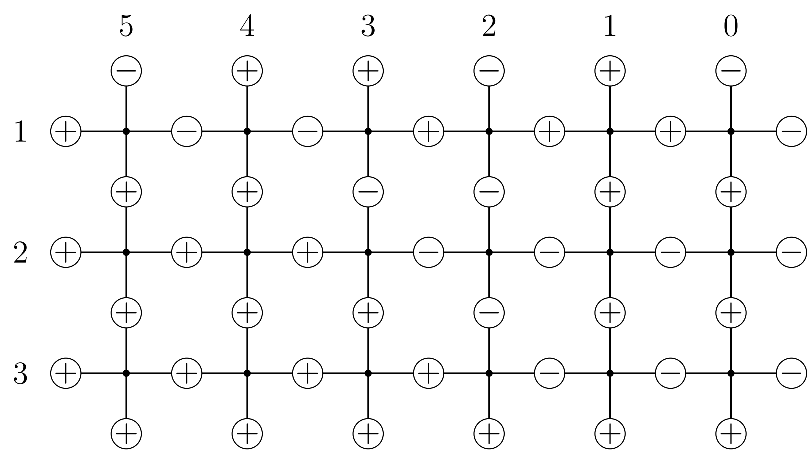
For example suppose that the Gelfand-Tsetlin pattern is:

$$\left\{ \begin{array}{ccccc} 5 & & 2 & & 0 \\ & 3 & & 2 & \\ & & 2 & & \end{array} \right\}$$

This assigns spins as follows:



It remains to be assigned spins to the horizontal edges. We may do this in each row by making use of the fact that the number of $-$ spins adjacent to every vertex is even, to figure out the rows.



This gives the configuration, and it must be checked that it is admissible for the six-vertex model. $\square$

Now let us consider the free-fermionic six-vertex model taking $a = b = c = 1$. Clearly the Boltzmann weight of each state is 1, so the partition function is equal to the number of states. Thus if we can evaluate the partition function we will have counted the number of states. In the special case where $\lambda = (n-1, n-2, \cdots, 0)$, this is equal to the number of alternating sign matrices.

This is the case where the lattice is square, with domain-wall boundary conditions, putting $+$ on the left and bottom boundary edges, and $-$ in the top and right boundary edges. In this case, there is a formula for the partition function as a determinant, due to Korepin and Izergin. This was Kuperberg's approach to the alternating sign matrix conjecture.

The proof of the Korepin-Izergin determinant formula depends again on the Yang-Baxter equation, which we will need to formulate more precisely than before. For the proof of the commutativity of the row-transfer matrices, all we needed to know about the R-matrix was its existence, but for other applications we will need to know its values, a topic that we will come to soon.

2 Vector Yang-Baxter Equation

We will give another notion of the Yang-Baxter equation. Soon we will connect it with the familiar one that we used in the last two lectures.

Let U, V and W be vector spaces. Suppose that we are given three linear transformations:

$$R : U \otimes V \longrightarrow V \otimes U,$$

$$S : U \otimes W \longrightarrow W \otimes U,$$

$$T : V \otimes W \longrightarrow W \otimes V.$$

We will consider two homomorphisms $U\otimes V\otimes W \longrightarrow W\otimes V\otimes U$. The first is the composition

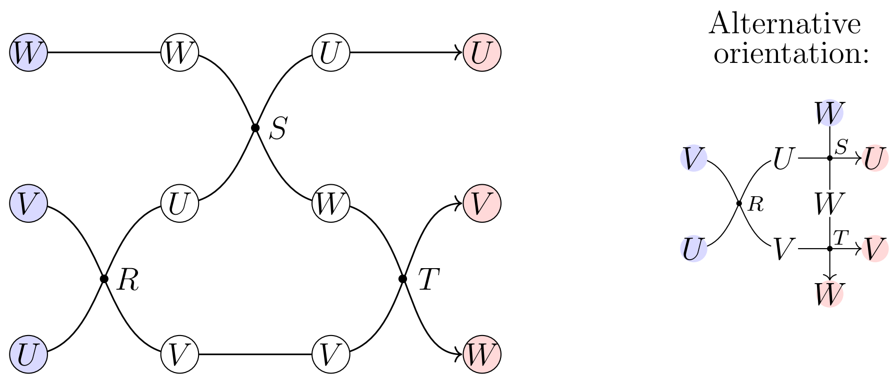
$$U\otimes V\otimes W \xrightarrow{R_{12}} V\otimes U\otimes W \xrightarrow{S_{23}} V\otimes W\otimes U \xrightarrow{T_{12}} W\otimes V\otimes U$$

where the notation is that R_{ij} means R applied to the i and j components of a tensor. Thus $R_{12}=R\otimes I_W$, $S_{23}=I_V\otimes S$ and $T_{12}=T\otimes I_U$. (The subscript notation is popular in Hopf algebra and quantum group literature.) The other homomoprhism is

$$U\otimes V\otimes W \xrightarrow{T_{23}} U\otimes W\otimes V \xrightarrow{S_{12}} W\otimes U\otimes V \xrightarrow{R_{23}} W\otimes V\otimes U.$$

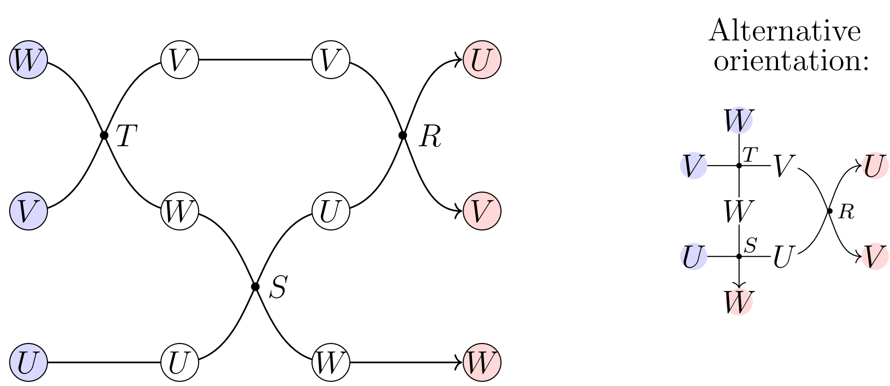
We can diagram the homomorphisms graphically as follows.

$$U\otimes V\otimes W \xrightarrow{R_{12}} V\otimes U\otimes W \xrightarrow{S_{23}} V\otimes W\otimes U \xrightarrow{T_{12}} W\otimes V\otimes U$$



and

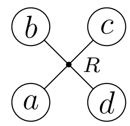
$$U\otimes V\otimes W \xrightarrow{T_{23}} U\otimes W\otimes V \xrightarrow{S_{12}} W\otimes U\otimes V \xrightarrow{R_{23}} W\otimes V\otimes U$$



If these two homomorphisms $U\otimes V\otimes W \longrightarrow W\otimes V\otimes U$ are equal, we will say that R,S,T give an instance of the *Yang-Baxter equation*. We will of course have to explain how this is related to the Yang-Baxter equations we have previously described in terms of Boltzmann weights.

For the six- or eight-vertex models, the vector spaces U,V and W can be taken to be two-dimensional, with bases indexed by the possible spins. Thus U is spanned by u_+,u_- and similarly we have bases v_+, v_- and w_+, w_- for V and W. Let us start with a vertex

R with chosen Boltzmann weights $a_1(R), a_2(R)$, etc. and encode these weights in a linear transformation $R : U \otimes V \longrightarrow V \otimes U$ by the following rule. If $a, b, c, d \in \{\pm\}$ then the Boltzmann weight of the state



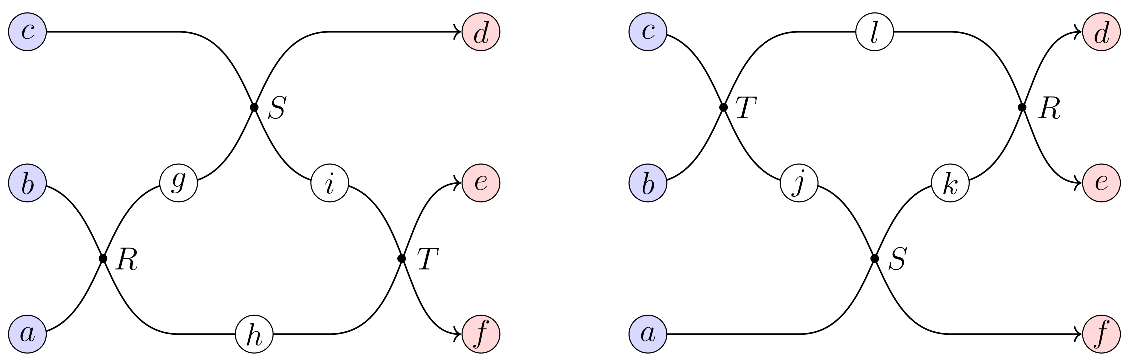
is to be the coefficient of $v_d \otimes u_c$ in $R(u_a \otimes v_b)$. We will write this coefficient in Dirac notation as $\langle v_d \otimes u_c | R | u_a \otimes v_b \rangle$, or if we are thinking of it as a Boltzmann weight as

$$\beta_R \begin{pmatrix} b & c \\ a & d \end{pmatrix}.$$

So

$$R|u_a \otimes v_b\rangle := R(u_a \otimes v_b) = \sum_{c,d} \beta_R \begin{pmatrix} b & c \\ a & d \end{pmatrix} |v_d \otimes u_c\rangle.$$

Lemma 2.1. *The partition function of the systems*



equal $\langle w_f \otimes v_e \otimes u_d | T_{12} S_{23} R_{12} | u_a \otimes v_b \otimes w_c \rangle$ *and* $\langle w_f \otimes v_e \otimes u_d | R_{13} S_{23} R_{12} | u_a \otimes v_b \otimes w_c \rangle$.

Proof. We compute

$$T_{12}S_{23}R_{12}|u_a \otimes v_b \otimes w_c\rangle = \sum_{g,h} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} T_{12}S_{23}|v_h \otimes u_g \otimes w_c\rangle$$

$$= \sum_{g,h} \sum_{i,d} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} T_{12}|v_h \otimes w_i \otimes u_d\rangle$$

$$= \sum_{g,h} \sum_{d,i} \sum_{e,f} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} \beta_T \begin{pmatrix} i & e \\ h & f \end{pmatrix} |w_f \otimes v_e \otimes u_d\rangle.$$

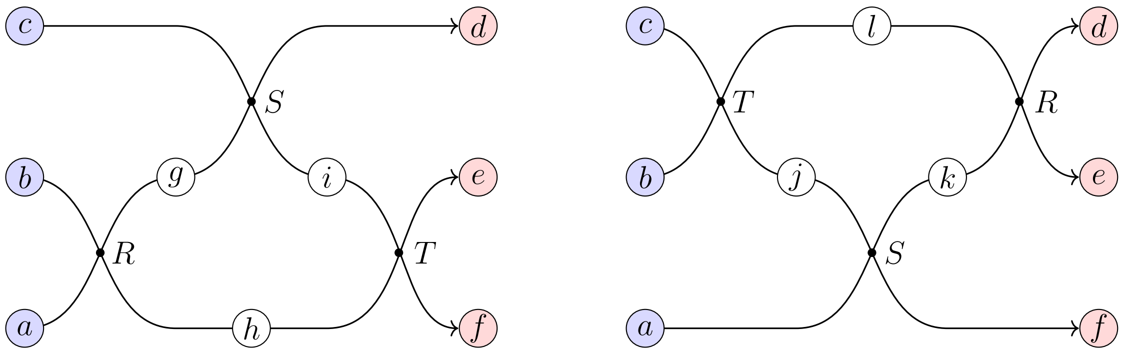
Therefore

$$\langle w_f \otimes v_e \otimes u_d | T_{12}S_{23}R_{12} | u_a \otimes v_b \otimes w_c \rangle = \sum_{g,h} \sum_{d,i} \sum_{e,f} \beta_R \begin{pmatrix} b & g \\ a & h \end{pmatrix} \beta_S \begin{pmatrix} c & d \\ g & i \end{pmatrix} \beta_T \begin{pmatrix} i & e \\ h & f \end{pmatrix}.$$

The right hand side is the partition function of the left-side of the Yang-Baxter equation system. As usual, the boundary spins a, b, c, d, e, f are fixed, and the spins of the interior edges g, h, i or j, k, l are summed over in the partition function. We leave the reader to check the other side. □

Therefore:

Theorem 2.2. *Let R, S, T be vertex types, and let U, V, W be as above, and define homomorphisms $R : U \otimes V \longrightarrow V \otimes U$ as above. If for all choices of boindary spins the partition functions of the systems*



agree, then the vector Yang-Baxter equation is satisfied.

One may also reorient the edges and work instead with the systems: