Lecture 20

This lecture contains the proof of Theorem 6.1 of Lecture 19, expressing the row transfer matrix $T_\Delta(z;q)$ as the exponential of the Hamiltonian

$$H_+(z;q) = \sum_{k=1} \frac{1}{k}(1-q^k)z^k J_k.$$

There is a corresponding result for T_Γ and H_+ but we will omit that. (It can be deduced from the T_Δ case by taking adjoints, as at the end of Section 4 in [1].)

1 Fermionic operators

We introduce *fermionic creation operators* ψ_n^* $(n \in \mathbb{Z})$ on $\mathfrak{F}$ that create particles by

$$\psi_n^*(\eta) = u_n \wedge \eta.$$

If η is a basis vector of $\mathfrak{F}_m$, say

$$\eta = |\mathbf{j}\rangle = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots,$$

then $\psi^*(\eta) = 0$ if n is among the indices $j_m, j_{m-1}, \cdots$. Otherwise, $\psi_n^*(\eta)$ can be calculated by moving u_n to its proper place among the indices. This can involve interchanging some u_j, which can introduce sign changes and so $\psi_n^*(\eta)$ is either zero or $\pm|\mathbf{j}'\rangle$, where $\mathbf{j}'$ is obtained by sorting $\{n, j_m, j_{m-1}, \cdots\}$ into descending order. We see that $\psi_n^* : \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+1}$.

Dual to the creation operators ψ_n^* are their adjoints $\psi_n : \mathfrak{F}_{m+1} \longrightarrow \mathfrak{F}_m$. The operator ψ_n deletes u_n from the semi-infinite monomial if $n \in \{j_m, j_{m-1}, \cdots\}$, which can result in a sign change. If $n \notin \{j_m, j_{m-1}, \cdots\}$ then $\psi_n|\mathbf{j}\rangle = 0$.

Lemma 1.1. *We have*

$$[J_k, \psi_j^*] = \psi_{j-k}^*.$$

Proof. From the Leibnitz rule, if $\eta \in \mathfrak{F}$, then

$$J_k\psi_j^*\eta = J_k(u_j \wedge \eta) = J_k(u_j) \wedge \eta + u_j \wedge J_k(\eta) = u_{j-k} \wedge \eta + \psi_j^*(J_k\eta).$$

Rearranging,

$$[J_k, \psi_j^*]\eta = u_{j-k} \wedge \eta = \psi_{j-k}^*(\eta).$$

□

Now let us introduce the *fermion field*

$$\psi(x) = \sum_{j\in\mathbb{Z}} \psi_j^* x^j.$$

For our purposes this is just a formal expression that we can use to do a calculation. (The "field" terminology comes from quantum field theory.)

Proposition 1.2. *We have*

$$[H_+(z;q), \psi^*(x)] = \log\left(\frac{1-qxz}{1-xz}\right)\psi^*(x). \tag{1}$$

Proof. Note that by Lemma 1.1 we have

$$[J_k, \psi^*(x)] = \sum_j x^j [J_k, \psi_j^*] = \sum_j x^j [J_k, \psi_j^*] = \sum_j x^j \psi_{j-k}^* = x^k \psi^*(x).$$

Now the left-hand side of (1) equals

$$\sum_k \frac{1}{k}(1-q^k) z^k [J_k, \psi^*(x)] = \sum \frac{1}{k}(1-q^k)(xz)^k \psi^*(x) = -\log(1-xz) + \log(1-qxz)$$

from the identity

$$-\log(1-t) = \sum_{k=1}^{\infty} \frac{t^k}{k}.$$

□

Lemma 1.3. *Suppose that $xa - ax = ca$, where $c \in \mathbb{C}^\times$. Then*

$$e^x a e^{-x} = e^c a.$$

Proof. This is a special case of the Baker-Campbell-Hausdorff formula. We treat this as a formal identity, disregarding convergence. We need the following identity, for $k \geqslant 0$:

$$\sum_j \binom{k}{j} (-1)^j x^{k-j} a x^j = c^k a \tag{2}$$

To avoid some bookkeeping we sum over all $j \in \mathbb{Z}$ but regard $\binom{k}{j}$ as zero unless $0 \leqslant j \leqslant k$, so most terms are zero. Assuming this true for $k-1$, we may establish (2) by induction, writing $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$. The left-hand side equals

$$x \cdot \left[\sum_j \binom{k-1}{j-1} (-1)^j x^{k-1-j} a x^j\right] - \left[\sum_j \binom{k-1}{j-1} (-1)^{j-1} x^{k-j} a x^{j-1}\right] \cdot x.$$

Both terms in brackets equal $c^{k-1}a$ by induction, so we obtain $c^{k-1}[x,a] = c^k a$. This proves (2).

Now expand the exponentials and collect terms of degree k to write

$$e^x a e^{-x} = \sum_k \frac{1}{k!} \sum_j \binom{k}{j} (-1)^j x^{k-j} a x^j = \sum_k \frac{1}{k!} c^k a = e^c a,$$

as required. □

Proposition 1.4. *Let $H = H_+(z;q)$. We have*

$$e^H \psi^*(x) e^{-H} = \frac{1-qxz}{1-xz}\psi^*(x). \tag{3}$$

Proof. This follows from our Proposition 1.2 by exponentiating (using Lemma 1.3). $\square$

Now the key point is to show that the row transfer matrices $T_\Delta(z;q)$ satisfy the same identity in Proposition 4. Let us introduce the operator $\rho_k(z) : \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+1}$ defined by:

$$\rho_k(z) = \psi_k^* - z\psi_{k-1}^*$$

Lemma 1.5. *Granted the invertibility of e^H, the identity (3) is equivalent to*

$$e^H \rho_k(z) = \rho_k(qz) e^H. \tag{4}$$

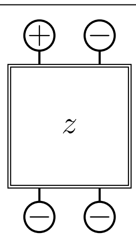
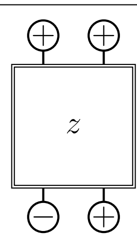
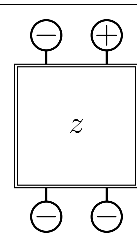
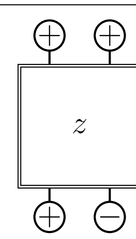
for $k \in \mathbb{Z}$.

Proof. We rewrite (3) in the form

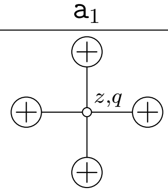
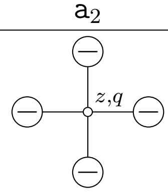
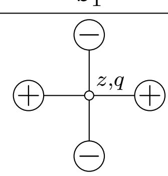
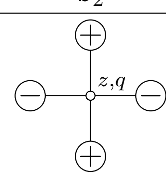
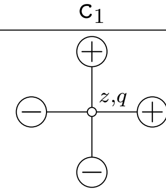
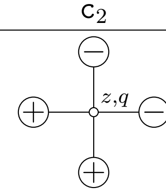
$$(1-xz)e^H\psi^*(x) = (1-qxz)\psi^*(x)e^H.$$

This is a formal identity that can be expanded in powers of x. Comparing the coefficient of x^k gives exactly the identity (4). $\square$

Our goal is to show that the row $T = T_\Delta(z;q)$ satisfies the same identity $T\rho_k(z) = \rho_k(qz)T$ as e^H. Let us represent ρ_k graphically as a "gate" that can be attached to the lattice model. Remembering that ψ_k creates a particle in the k-th column, and that $+$ denotes the absence of a particle, $-$ its presence, we see that we have the following Boltzmann weights:

			
1	1	$-z$	$-z$

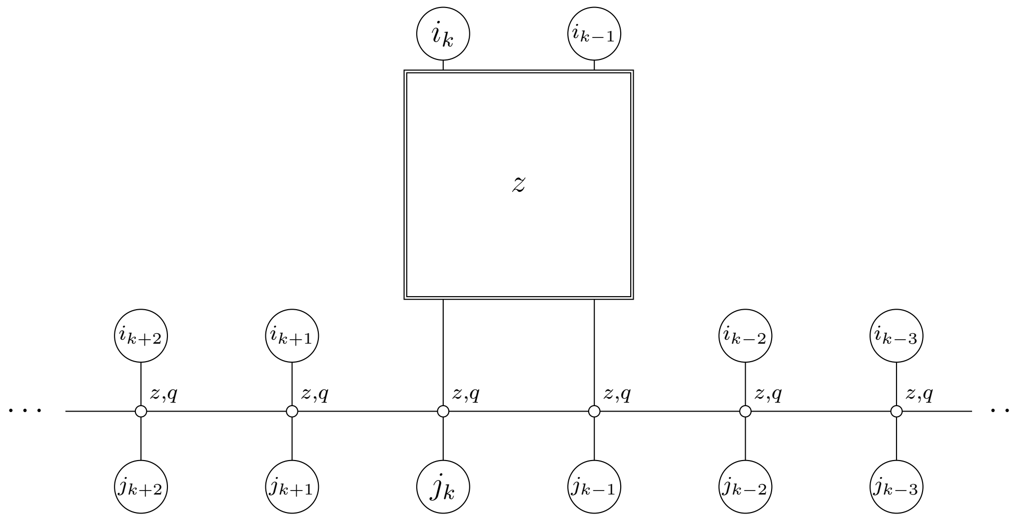
For reference, here are the Delta Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Δ-ice						
	1	$-qz$	1	z	$(1-q)z$	1

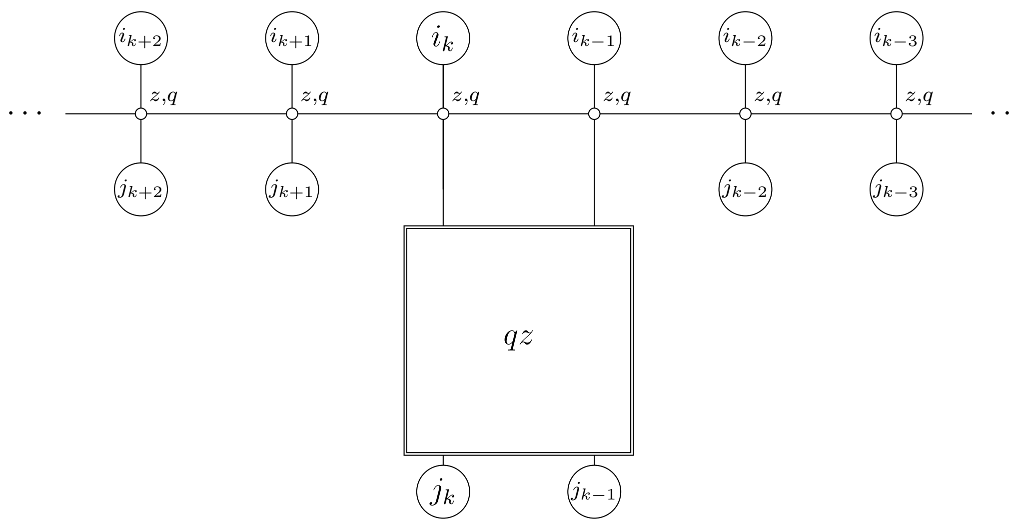
Proposition 1.6. *The row transfer matrix*

$$T\rho_k(z) = \rho_k(qz)T.$$

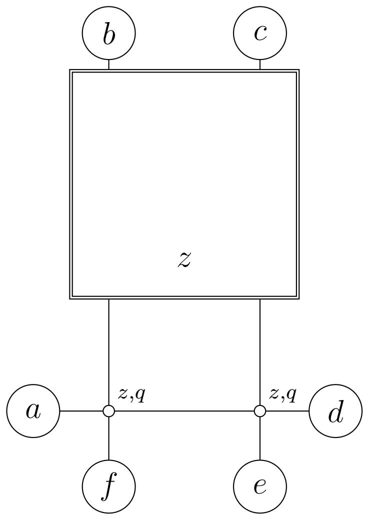
Proof. Graphically this means that we must show the equivalence of the two following partition functions:



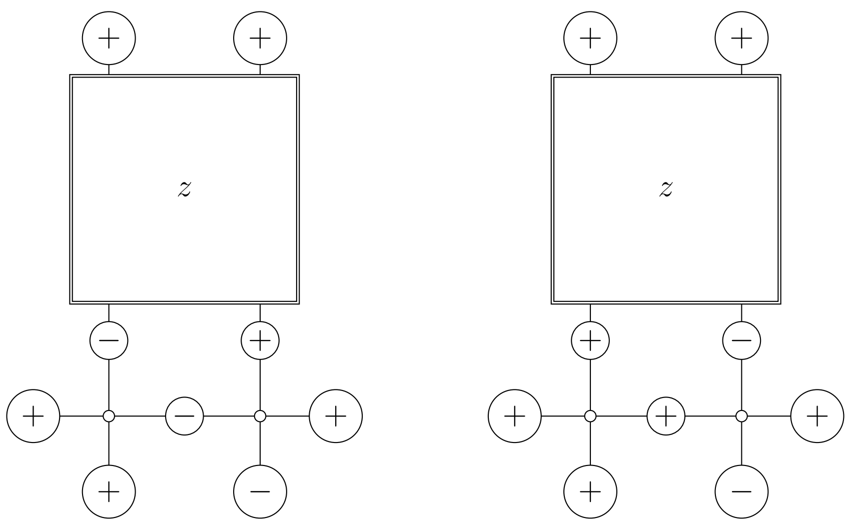
and



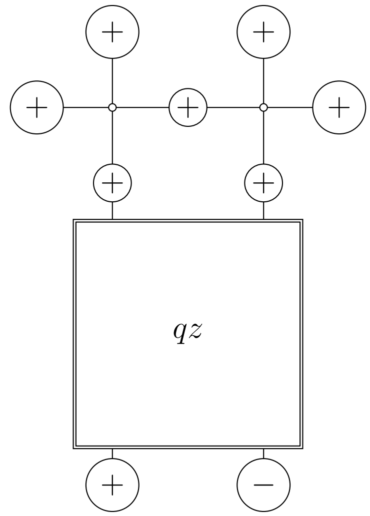
We can clip out the middle part and just prove the equivalence of these systems:



This can be thought of as a kind of a Yang-Baxter equation, but of the sort mentioned in Lecture 16 Section 1, where the R-matrix changes as it moves past the vertices. This verification is now subject to case by case verification. Let us check just one case. Suppose that the boundary values are $(a, b, c, d, e, f) = (+, +, +, +, -, +)$. On the left-hand side there are two admissible states:



Their Boltzmann weights are, respectively $(1 - q)z$ and $-z$, for a total of $-qz$. On the right-hand side there is only one admissible state:



The Boltzmann weight is $-qz$. Since $(1-q)z+(-z)=-qz$, the required identity is satisfied in this case, and the remaining cases are similar. $\square$

2 Proof of Theorem 6.1 of Lecture 19

We will only prove that $e^{H_+(z;q)}=T_\Delta(z;q)$. The identity $e^{H_-(z;q)}=T_\Gamma(z;q)$ can be deduced using adjointness considerations, as in [1].

As in the last section, we abbreviate $H=H_+(z;q)$ and $T=T_\Delta(z;q)$. We have proved that both operators e^H and T both satisfy the same identities

$$e^H\rho_k(z)=\rho_k(qz)e^H, \qquad T\rho_k(z)=\rho_k(qz)T.$$

We need to show that there is enough information in this fact to deduce that $T|\mathbf{j}\rangle=e^H|\mathbf{j}\rangle$ for every semi-infinite monomial $|\mathbf{j}\rangle\in\mathfrak{F}$.

Recall that the *energy* of $|\mathbf{j}\rangle$, with $\mathbf{j}=(j_m,j_{m-1},\cdots)\in\mathfrak{F}_m$ is $\sum_k(j_k-k)$. This is actually a finite sum. The unique basis vector in $\mathfrak{F}_m$ of energy 0 is the vacuum

$$|\varnothing\rangle_m=u_m\wedge u_{m-1}\wedge\cdots.$$

The identity

$$e^{H_-(z;q)}|\varnothing\rangle_m=T_\Delta(z;q)|\varnothing\rangle_m$$

is clear since both sides are $|\varnothing\rangle_m$.

So assume that $|\mathbf{j}\rangle_m$ is not the vacuum. Then it has positive energy. This means $j_m>m$. We will show

$$e^{H_-(z;q)}|\mathbf{j}\rangle_m=T_\Delta(z;q)|\mathbf{j}\rangle_m. \tag{5}$$

We are assuming inductively that the identity is known for states of lower energy.

Let $|\mathbf{j}'\rangle=u_{j_{m-1}}\wedge u_{j_{m-2}}\wedge\cdots\in\mathfrak{F}_m$, so $|\mathbf{j}\rangle_m=\psi^*_{j_m}|\mathbf{j}'\rangle_{m-1}$. We have

$$|\mathbf{j}\rangle_m=\rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1}+z\xi, \tag{6}$$

where

$$\xi = u_{j_m - 1} \wedge |\mathbf{j}'\rangle.$$

Now *both* terms on the right-hand side of (6) have lower energy than $|\mathbf{j}\rangle_m$. It is possible that $\xi = 0$ (if $j_{m-1} = j_m - 1$) but if $\xi \neq 0$ it has lower energy than $|\mathbf{j}\rangle_m$. So by our induction hypothesis

$$e^H |\mathbf{j}'\rangle_{m-1} = T|\mathbf{j}'\rangle_{m-1}, \qquad e^H \xi = \xi.$$

Now we have

$$e^H |\mathbf{j}\rangle_m = e^H \rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1} + ze^H \xi = \rho_{j_m}(qz)e^H|\mathbf{j}'\rangle_{m-1} + ze^H\xi,$$

$$T|\mathbf{j}\rangle_m = T\rho_{j_m}(z)|\mathbf{j}'\rangle_{m-1} + zT\xi = \rho_{j_m}(qz)T|\mathbf{j}'\rangle_{m-1} + zT\xi,$$

and using (2) we obtain (5). So the theorem is proved.

3 Delta Ice and U-Turn models

Delta ice, which we introduced in Lecture 19, plays well with Gamma ice, and they often appear together. The distinction between them is in the horizontal edges, not the vertical. This situation persists in the colored case.

The Yang-Baxter equation can be used to prove:

- The row transfer matrices $T_\Gamma(z)$ commute with each other for varying z.
- The row transfer matrices $T_\Delta(z)$ commute with each other for varying z.

(There are versions of these statements for both infinite and finite grids.)

But the row transfer matrices $T_\Gamma(z)$ and $T_\Delta(w)$ do not commute, though

$$T_\Gamma(z)T_\Delta(w) = \text{const} \times T_\Delta(w)T_\Gamma(z),$$

for a computable constant. This can be proved using the Yang-Baxer equation. For the infinite grids, it can also be deduced from the $T_\Gamma(z) = e^{H_-(z;q)}$ and $T_\Delta(w) = e^{H_+(z;q)}$ using the technique of the first section of this lecture.

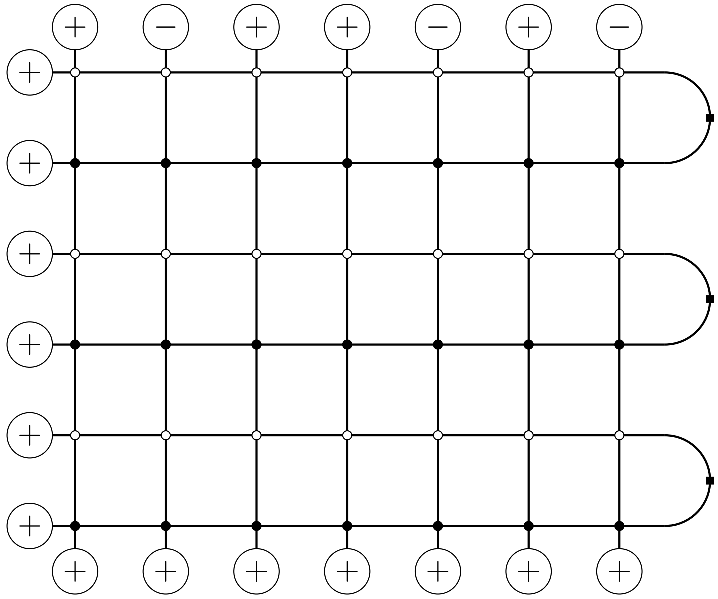
Gamma ice and Delta ice appeared in [3]. (Use the arxiv version of this paper.) We considered Gamma ice in Lectures 5 and 6, and computed the partition function as

$$s_\lambda(\mathbf{z}) \prod_{i<j} x_i - qx_j.$$

There is a similar Tokuyama result for Delta ice.

Ivanov [9] gave a Tokuyama result for characters of symplectic groups. The lattice models had been considered previously by Hamel and King [8], but we prefer Ivanov's treatment since Hamel and King do not use the Yang-Baxter equation, but instead combinatorial arguments based on *jeu de taquin*. (They also preceded [3] in reinterpreting the formula of Tokuyama [13] in terms of lattice models.)

The models look like this, with alternating rows of Gamma and Delta ice:



Boltzmann weights at the "cap" vertices on the right edge must be specified, resulting in what is sometimes called "U-turn models." We are changing the Boltzmann weights from Ivanov by switching + and − on the Delta ice. With this convention, the "paths" switch from + and − when they cross a cap. Paths move right on the Delta rows (marked ∘) along the − horizontal edges, and left along the Gamma rows (marked •) eventually exiting on the left.

Changing just the cap weights results in another interesting model [2] related to metaplectic Whittaker functions. Both the models of [9, 2] were vastly generalized in [7], which can now be understood as colored variants of the original model. U-turn models are also employed in [11, 5, 14]

U-turn models and other exotic variations of the standard grid go back to [10]. In [12, 4], many of these exotic variations of the grid are used in interesting models that are deformations of the Weyl character formula. However they are different from the results of [8], [3] and [9]. In those papers, models are exhibited whose partition functions are of the form

$$\prod_{\alpha\in\Phi^+}(1-q\mathbf{z}^{-\lambda})\chi_\lambda(\mathbf{z}),$$

where χ_λ is either a character of $\mathrm{GL}(n,\mathbb{C})$ (that is, a Schur function) or in Ivanov's case of $\mathrm{Sp}(2n,\mathbb{C})$. The character itself is undeformed. We will call such a result a *Tokuyama formula*. These are significant because of the similarity to the Casselman-Shalika formula [6]. The models of Tabony, Brubaker and Schultz are not Tokuyama results since the character itself is also deformed. Finding a Tokuyama formula for orthogonal groups is an open problem.

References

[1] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Vertex operators, solvable lattice models and metaplectic Whittaker functions. *Comm. Math. Phys.*, 380(2):535–579, 2020.

[2] B. Brubaker, D. Bump, G. Chinta, and P. E. Gunnells. Metaplectic Whittaker functions and crystals of type B. In *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 93–118. Birkhäuser/Springer, New York, 2012.

[3] B. Brubaker, D. Bump, and S. Friedberg. Schur polynomials and the Yang-Baxter equation. *Comm. Math. Phys.*, 308(2):281–301, 2011, https://arxiv.org/abs/0912.0911.

[4] B. Brubaker and A. Schultz. The six-vertex model and deformations of the Weyl character formula. *J. Algebraic Combin.*, 42(4):917–958, 2015.

[5] V. Buciumas and T. Scrimshaw. Quasi-solvable lattice models for Sp_{2n} and SO_{2n+1} Demazure atoms and characters. *Forum Math. Sigma*, 10:Paper No. e53, 34, 2022.

[6] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[7] N. Gray. Metaplectic ice for Cartan type C, 2017, arXiv:1709.04971.

[8] A. M. Hamel and R. C. King. Bijective proofs of shifted tableau and alternating sign matrix identities. *J. Algebraic Combin.*, 25(4):417–458, 2007.

[9] D. Ivanov. Symplectic ice. In *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 205–222. Birkhäuser/Springer, New York, 2012.

[10] G. Kuperberg. Symmetry classes of alternating-sign matrices under one roof. *Ann. of Math. (2)*, 156(3):835–866, 2002.

[11] K. Motegi. Dual wavefunction of the symplectic ice. *Rep. Math. Phys.*, 80(3):391–414, 2017.

[12] S. J. Tabony. *Deformations of characters, metaplectic Whittaker functions, and the Yang-Baxter equation*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[13] T. Tokuyama. A generating function of strict Gelfand patterns and some formulas on characters of general linear groups. *J. Math. Soc. Japan*, 40(4):671–685, 1988.

[14] C. Zhong. Stochastic symplectic ice. 2021, arXiv:2102.00660.