Lecture 2

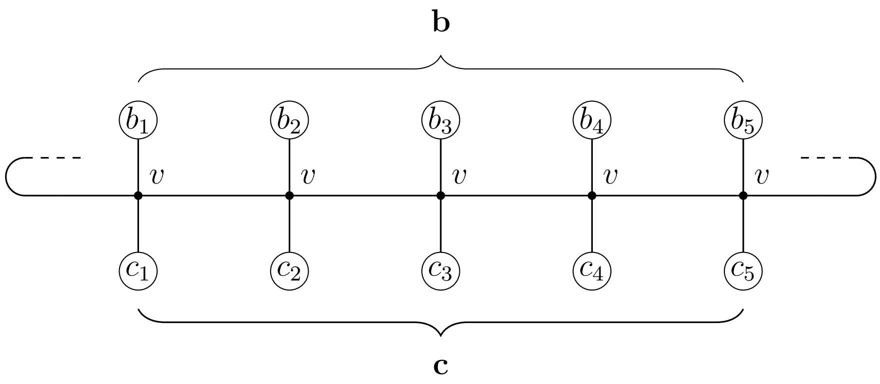
1 Row Transfer Matrices

We will consider systems $\mathfrak{S}$ built up from graphs Γ as in Lecture 1. Recall that a *graph* for us consists of vertices and edges, with an incidence relation between them. Every edge is adjacent to one or two vertices. An edge that is adjacent to two vertices is called *interior*, and an edge that is adjacent to only one vertex is called a *boundary* edge. Every edge $\mathcal{E}$ is assigned a *spinset* $\Sigma_{\mathcal{E}}$ of possible states called *spins*. The spins of boundary edges are fixed, and are part of the data defining the system. A state of the system consists of an assignment of spins to the interior edges.

Also required for the specification of the system $\mathfrak{S}$ is, for every vertex $v \in \Gamma$ a rule β that assigns to a state $\mathfrak{s}$ and a vertex v a weight $\beta(v, \mathfrak{s})$. This should only depend on the spins of the edges adjacent to v. The Boltzmann weight $\beta(\mathfrak{s})$ is the product of the $\beta(v, \mathfrak{s})$ over all vertices, and the partition function is

$$Z(\mathfrak{S}) = \sum_{\text{states } \mathfrak{s}} \beta(\mathfrak{s}).$$

We wish to discuss the concatenation of two systems. To have an example in mind, consider a system consisting of a single row of vertices:



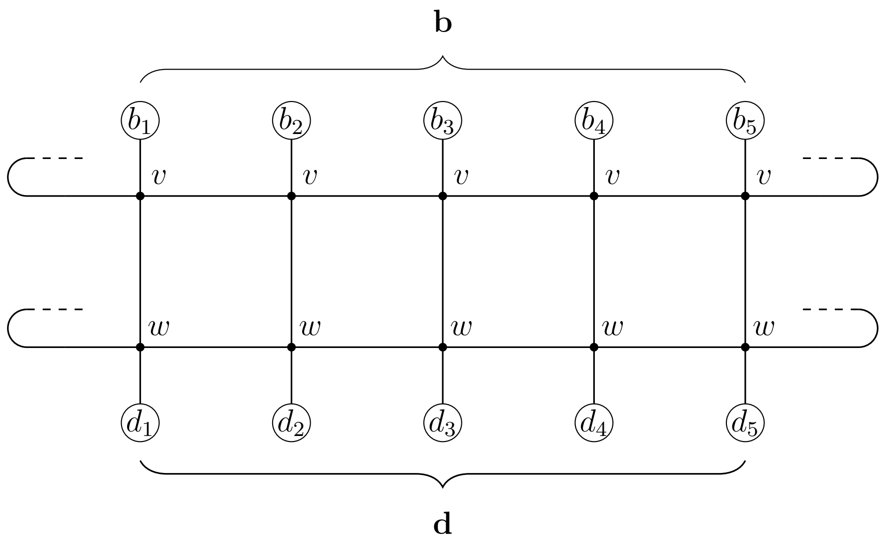
We are imagining that every vertex has the same Boltzmann weight v, so we are giving every vertex the same label. We are imagining that one edge wraps around the back, so the system is periodic. We will refer to this as *cyclindric boundary conditions*.

We have partitioned the boundary edges into two sets, $\mathbf{b} = (b_1, \cdots, b_n)$ where in the picture $n = 5$, and $\mathbf{c} = (c_1, \cdots, c_n)$. The partition function thus depends on $\mathbf{b}$ and $\mathbf{c}$, and we think of $\mathbf{b}$ (somewhat arbitrarily) as *inputs* and $\mathbf{c}$ as outputs, and write

$$Z(\mathfrak{S}) = \langle \mathbf{c} | T_v | \mathbf{b} \rangle$$

where we are using Dirac notation to indicate T_v as a matrix with row entries $\mathbf{b}$ and column entries $\mathbf{c}$. We can think of it as an operator on the free vector space on the set of possible input spins $b_1, \cdots, b_n$, assuming that the spinsets match, so $\Sigma_{b_i} = \Sigma_{c_i}$. We call T_v the *row transfer matrix*.

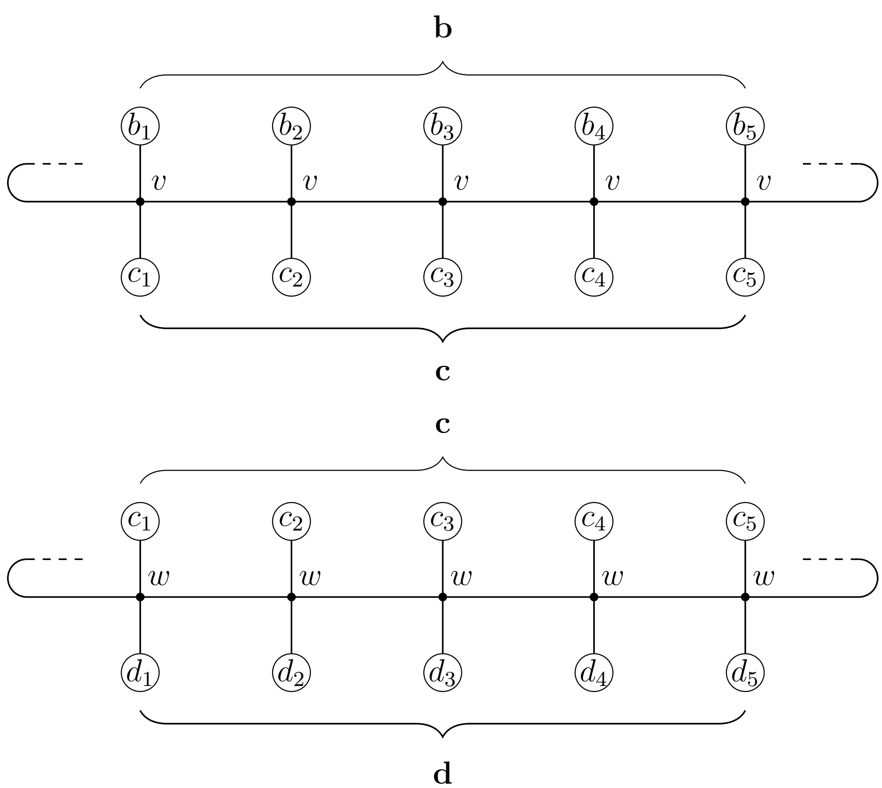
Now let w be another vertex type, and let us consider a system with two layers:



We can express this in terms of the product of two row transfer matrices:

$$\langle \mathbf{d} | T_w T_v | \mathbf{b} \rangle$$

Indeed, we may concatenate the two smaller systems:



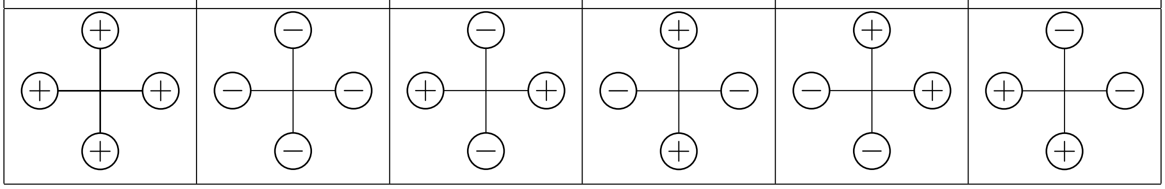
Now the common edges, labeled $\mathbf{c}$ in both cases have become interior edges, so by our rules, we have to sum over the possible states, to obtain:

$$\sum_{\mathbf{c}} \langle \mathbf{d}|T_w|\mathbf{c}\rangle\langle \mathbf{c}|T_v|\mathbf{b}\rangle = \langle \mathbf{d}|T_wT_v|\mathbf{b}\rangle$$

by the usual rule for matrix multiplication.

In preparation for applying the Yang-Baxter equation, we write $v = v(a, b, c)$, where a, b, c are real or complex parameters and the Boltzmann weights are as in the previous lecture.

a	a	b	b	c	c



Baxter's great insight was the use of the Yang-Baxter equation to prove that under certain conditions, row transfer matrices commute.

Theorem 1.1 (Baxter)**.** *Let $\Delta \in \mathbb{C}^\times$, and let a, b, c, a', b', c' be such that*

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{(a')^2 + (b')^2 - (c')^2}{2a'b'} = \Delta.$$

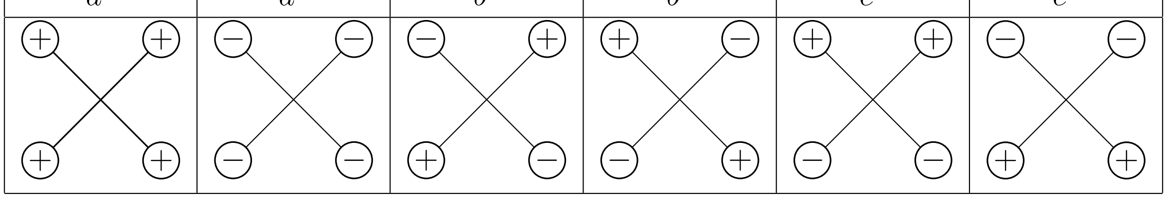
Let $v = v(a, b, c)$ and $w = v(a', b', c')$ be the two corresponding vertex types. Then the corresponding row transfer matrices commute:

$$T_wT_v = T_vT_w.$$

We should think of this in the context of "diagonalizing" the matrix T_v, for it is often easier to diagonalize a large family of commuting operators than a single operator.

Proof. To prove this, we will make use of the Yang-Baxter equation, with the R-matrix r from the last section. We recall from the last lecture that this is the vertex $v(a'', b'', c'')$ which we draw in a rotated orientation, thus:

a''	a''	b''	b''	c''	c''



where

$$a'' = \frac{ba'b' - a(b')^2 + a(c')^2}{a'} = \frac{aba' - a^2b' + c^2b'}{b},$$
$$b'' = ba' - ab', \qquad c'' = cc'.$$

We recall that also

$$\frac{(a'')^2 + (b'')^2 - (c'')^2}{2a''b''} = \Delta.$$

The matrix r is invertible in the following sense. We think of the two vertices to the right of the matrix as "inputs" and the vertices to the left as "outputs" so that r is represented as a matrix

$$r = \begin{pmatrix} a'' & & & \\ & c'' & b'' & \\ & b'' & c'' & \\ & & & a'' \end{pmatrix}$$

with inverse (as usual matrices):

$$\begin{pmatrix} a''' & & & \\ & c''' & b''' & \\ & b''' & c''' & \\ & & & a''' \end{pmatrix} \begin{pmatrix} a''' & & & \\ & c''' & b''' & \\ & b''' & c''' & \\ & & & a''' \end{pmatrix} =$$

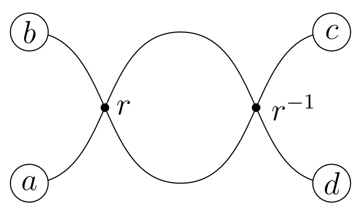
It may be computed that

$$a''' = \frac{1}{a''}, \qquad b''' = \frac{-b''}{(c'')^2 - (b'')^2}, \qquad c''' = \frac{c''}{(c'')^2 - (b'')^2}.$$

Then we compute that also

$$\frac{(a''')^2 + (b''')^2 - (c''')^2}{2a'''b'''} = \Delta.$$

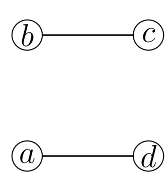
Now we may concatenate the matrices r and r^{-1}, and this is done by ordinary matrix multiplication. In other words, if we compute the partition function of the following system:



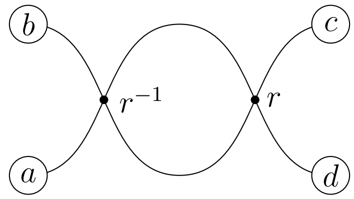
we get 1 if $a = d$ and $b = c$ but 0 otherwise. This is because summing over the middle column (four possibilities) really amounts to just multiplying matrices:

$$\begin{pmatrix} a'' & & & \\ & c'' & b'' & \\ & b'' & c'' & \\ & & & a'' \end{pmatrix}$$

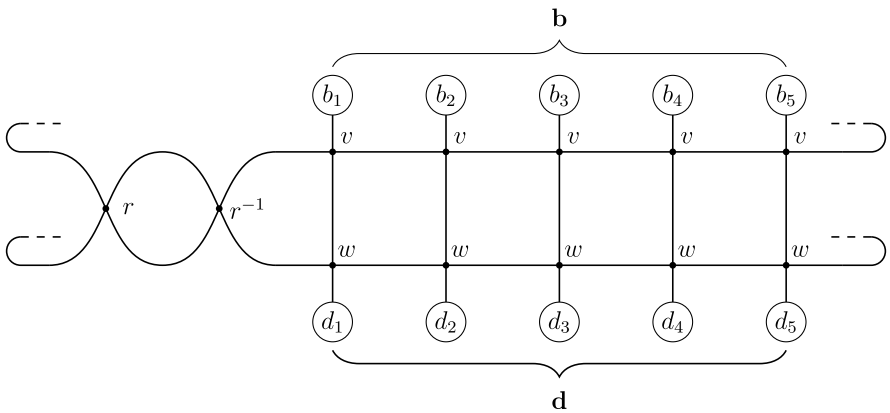
So this concatenation of r and r^{-1} is equivalent to:



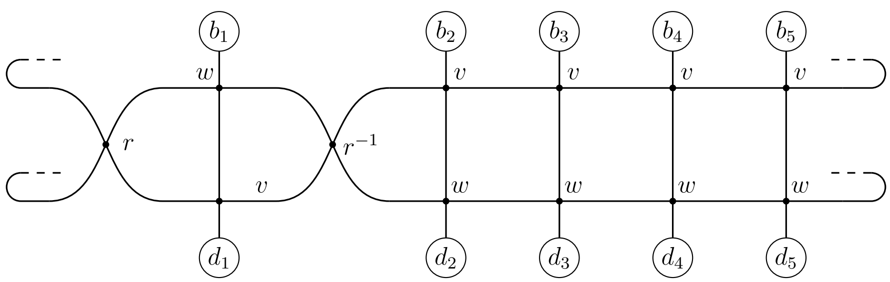
This is also equivalent to



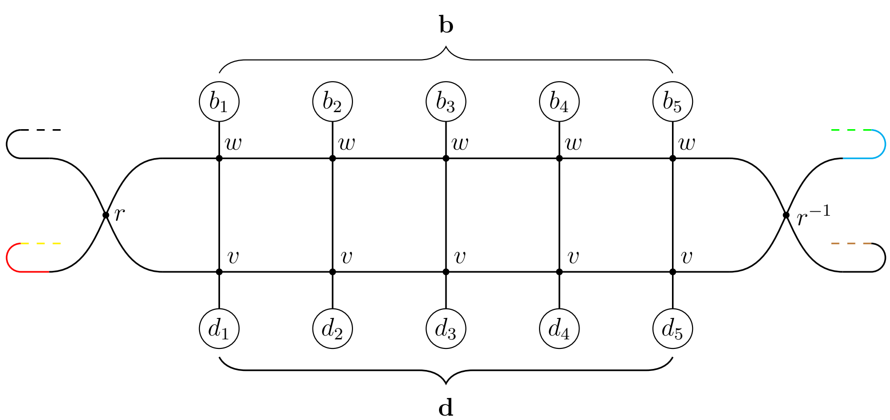
We may insert r and r^{-1} into our system representing $\langle \mathbf{d} T_w T_v | \mathbf{b} \rangle$ to obtain:



Now we use the Yang-Baxter equation to see that this system is equivalent to:



We may repeat this process several times to obtain this system:

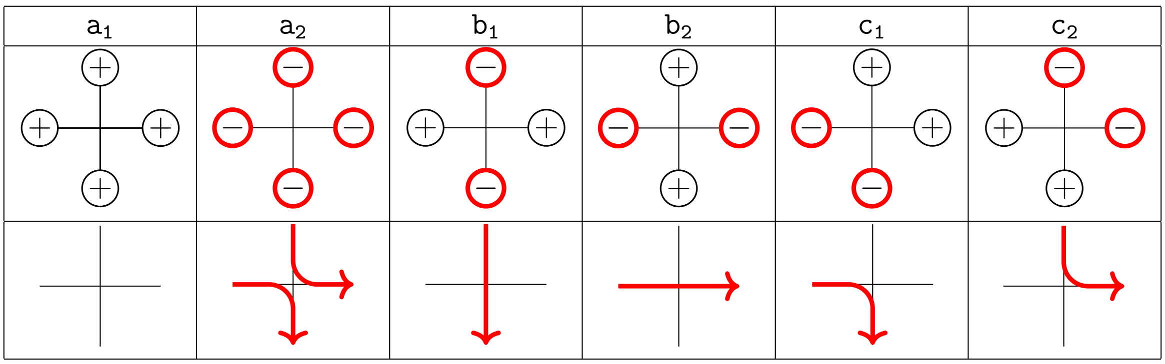


Now due to the cylindric boundary conditions, the r and r^{-1} are again adjacent and may be discarded. But now the system represents $\langle \mathbf{d} T_v T_w | \rangle$. We have proven that the two row transfer matrices commute. $\square$

2 Paths

In many models we may visualize states in terms of paths (or lines) through the lattice. Let us see how this works with the six-vertex model.

We will interpret a $-$ state as the presence of a particle, and $+$ as the absence of a particle. We will visualize the particles as moving from top to bottom, and from left to right.



We have drawn the particles in red, then visualized the paths they must take. In the case of $\mathtt{a}_2$ we have elected not to allow the paths to cross, though in other schemes they *might* cross.

In the last section we considered cylindric boundary conditions, wrapping the grid around into a cylinder. We might also consider *toroidal* boundary conditions, additionally wrapping the top to bottom so that there are no boundary edges. Now, however, we want to do no wrapping, envisioning a rectangular grid with boundary edges on the left, right, top and bottom. To specify the system, we must specify which of these will be $-$ and which will be $+$. We refer to this specification as *domain wall boundary conditions*.

Lemma 2.1. *Let us consider a grid with domain wall boundary conditions. The number of $-$ on the top and left must equal the number of $-$ on the right and bottom, or else the system will have no admissible states.*

Proof. Every line must start at the top or left and finish on the right or bottom. This gives a bijection between the $-$ spins on the top or left and those on the right or bottom. $\square$