Lecture 19

1 Introduction

In Lecture 17 we saw that in the field-free six-vertex model there is a Hamiltonian H and also a commuting family of six-vertex model row transfer matrices T_θ acting on a Hilbert space, which in that case was $\mathcal{H} = \otimes^N \mathbb{C}^2$. The main theorem is that H commutes with T_θ, which was proved by showing that $H = (T_\theta^{-1} T'_\theta)|_{\theta=\chi} + cI_{\mathcal{H}}$ for a suitable constant c. This result was proved by Baxter, in the greater generality of the 8 vertex model.

For the free-fermionic six-vertex model, there is a similar result, due to Brubaker and Schultz [2]. In the proof (Lecture 20) we will follow [1], where a more general result is proved. (The models in [1] may be regarded as generalizations of the result in [2] to a colored model.) In this free-fermionic case there is a Hamiltonian operator H and a row transfer matrix T, and the result is now in the form $e^H = T$. But the conclusion is the same: the Hamiltonian H commutes with the row transfer matrix T.

The identity $e^H = T$ can be thought of as an expansion of T in terms of operators J_k which move particles right or left to lower or higher energy levels. If $k > 0$, then J_k moves the particle right to a lower energy level, and if $k < 0$ it moves the particle to the left. There are correspondingly two versions of both the Hamiltonian, and two versions of the row transfer matrix.

2 The Fermionic Fock space

The *fermionic Fock space* was invented by Dirac in the theory of the electron. The electron is described by the *Dirac equation*, which we will not discuss, except to mention that the energy levels are quantized, and there are solutions of arbitrary negative energy. This seems unphysical, since a particle could radiate an arbitrarily large amount of energy by falling to lower and lower energy levels.

But Dirac proposed a solution to this. Since the Dirac equation is linear, solutions can exist in superposition. The electron is a fermion, subject to the Pauli exclusion principle, meaning that no two electrons can occupy the same state. Dirac's proposal was that all sufficiently large negative energy level states are occupied, and all sufficiently large positive energy levels are unoccupied.

Mathematically, the states are vectors in a Hilbert space that is now called the *fermionic Fock space* $\mathfrak{F}$, which we will now describe. This is based on another Hilbert space that we will call V, with basis vectors u_i ($i \in \mathbb{Z}$). Each u_i represents a particle with a definite energy level equal to i. Let us fix $m \in \mathbb{Z}$ and consider a sequence $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ where

$j_m > j_{m-1} > \cdots$ and $j_k = k$ for k sufficiently negative. Define the *charge m fermionic Fock space*, denoted $\mathfrak{F}_m$ to be the free vector space on formal symbols

$$|\mathbf{j}\rangle := |\mathbf{j}\rangle_m = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots, \qquad \mathbf{j} = (j_m, j_{m-1}, j_{m-2}, \cdots). \tag{1}$$

The Fock space $\mathfrak{F}$ resembles the exterior algebra $\bigwedge V$, except that the basis vectors are *infinite* wedges (called *semi-infinite monomials*).

We extend the notation $\xi_{\mathbf{j}}$ to sequences $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ where $j_k = k$ for k sufficiently negative, dropping the assumption that the sequence is strictly decreasing, by the usual rules for $\wedge$ in the exterior algebra. Thus $|\mathbf{j}\rangle = 0$ if $j_k = j_l$ for any distinct $k, l < m$. And interchanging two adjacent indices changes the sign of $|\mathbf{j}\rangle$.

We can visualize the vector $|\mathbf{j}\rangle$ by a *Maya diagram* in which sites numbered by integers are filled with stones. If the site n equals j_k for some k, the site is *occupied*, otherwise it is *unoccupied*. We put a black stone at the occupied sites, and a white stone at the unoccupied sites.

For example, if $\mathbf{j} = (4, 2, -1, -2, -3, -4, \cdots)$, so

$$|\mathbf{j}\rangle = u_4 \wedge u_2 \wedge u_{-1} \wedge u_{-2} \wedge u_{-3} \wedge u_{-4} \wedge \cdots$$

then the Maya diagram looks like this:

$\cdots$	6	5	4	3	2	1	0	−1	−2	−3	−4	$\cdots$
$\cdots$	○	○	●	○	●	○	○	●	●	●	●	$\cdots$

The main point is that every sufficiently negative site is occupied, and every sufficiently positive site is unoccupied. Although Maya diagrams are traditional (originating in soliton theory with M. Sato and his collaborators), because we want to relate this story to the six vertex model as we have been we prefer to use $-$ and $+$ for the occupied and unoccupied sites respectively, so the Maya diagram looks like this:

$\cdots$	6	5	4	3	2	1	0	−1	−2	−3	−4	$\cdots$
$\cdots$	⊕	⊕	⊖	⊕	⊖	⊕	⊕	⊖	⊖	⊖	⊖	$\cdots$

For this state the charge $m = 1$.

If $j_k = k$ for all $k \leqslant m$ then we obtain the charge m *vacuum vector* for which we have an alternative notation

$$|\varnothing\rangle_m = u_m \wedge u_{m-1} \wedge \cdots .$$

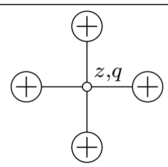
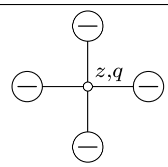
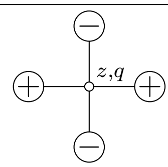
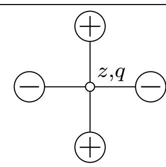
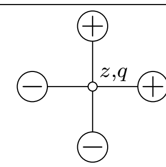
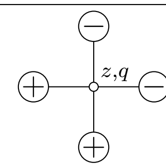
In general we may define the *energy* of $|\mathbf{j}\rangle_m$ to be $\sum_{k \leqslant m}(j_k - k)$. This is a finite sum. The vacuum is the unique semi-infinite monomial in $\mathfrak{F}_m$ of energy 0.

3 The Row Transfer matrix $T_\Delta(z;q)$

We will describe a kind of free-fermionic six-vertex model that we will call *Delta ice*. The grid will now be of infinite width, and the Boltzmann weights in each row will depend on a parameter $z \in \mathbb{C}^\times$.

Remark 1. The Δ here is different from Baxter's Δ, which is $(a_1a_2 + b_1b_2 - c_1c_2)/2a_1b_1$. Baxter's Δ is zero here, since all weights in this lecture are free-fermionic.

Now let $\mathbf{i} = (i_m, i_{m-1}, \cdots)$ and $\mathbf{j} = (j_m, j_{m-1}, \cdots)$ be two sequences such that $i_m > i_{m-1} > \cdots$ and $j_m > j_{m-1} > \cdots$ and $i_i = j_k = k$ for k sufficiently negative. We will define a simple system consisting of a single row, and either no states or a single state. We consider a grid with only one row that is infinite in both directions. As boundary conditions, the spins of the vertical edges at the top will be given by the Maya diagram for $\xi_{\mathbf{i}}$, and for the vertical edges at the bottom, by the Maya diagram for $\xi_{\mathbf{j}}$. There is also a "boundary condition" for the horizontal edges, that there are only finitely many + spins. We use these Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Δ-ice						
	1	$-qz$	1	z	$(1-q)z$	1

Since as part of the boundary conditions there are only finitely many horizontal edges with + spins, all but finitely factors in the Boltzmann weight of a state are of type $\mathtt{b}_1$ (for vertices far to the left) or of type $\mathtt{a}_1$ (for vertices far to the right). Therefore the Boltzmann weight of a state is an infinite product with only finitely many terms not equal to 1, and so has a well-defined finite value.

Lemma 3.1. *The condition for the partition function to have a state (which is therefore unique) is that*

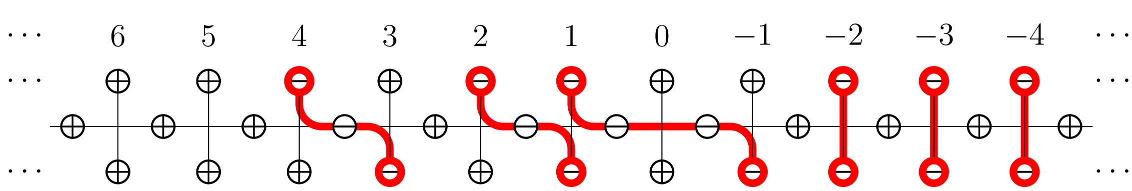
$$i_m \geqslant j_m \geqslant i_{m-1} \geqslant j_{m-2} \geqslant \cdots . \tag{2}$$

We express equation (2) by saying that the sequences $\mathbf{i}$ and $\mathbf{j}$ *interleave*.

Proof. This may be seen by consideration of the paths, which we recall from Lecture 2 Section 2 are obtained by joining edges with spin $-$. Because of our boundary condition, that there are only finitely many horizontal edges with spin $-$, each path must begin at the top and exit at the bottom for this system. For example, suppose that $m = 1$ and

$$\mathbf{i} = (4, 2, 1, -2, -3, -4, \cdots), \qquad \mathbf{j} = (3, 1, -1, -2, -3, -4, \cdots).$$

Then we have the following state.



Every path must start in the i_k column and end in the j_k column. Call this the k-th path. We must have $i_k \geqslant j_k$ since the paths move down and to the right. We also need $j_k \geqslant i_{k-1}$ since otherwise two paths will overlap between the i_{k-1} column and the j_k column. □

We quickly review Dirac notation for operators. Let $\mathcal{H}$ be a Hilbert space. A vector in $v \in \mathcal{H}$ is denoted alternatively as $|v\rangle$, and called a *ket*. On the other hand, a vector w gives rise to a linear functional $v \to (v, w)$ using the inner product on $\mathcal{H}$, and we denote this linear functional as $\langle w|$. The notation works well in quantum mechanics due to the emphasis on Hermitian (self-adjoint) operators. If T is Hermitian, then $(Tv, w) = (v, Tw)$, which we denote $\langle w|T|v\rangle$. We can either think of this as either the linear functional $\langle w|$ applied to the vector $T|v\rangle$, or as the linear functional $\langle w|T$ applied to the vector v.

As a special case, the partition function of the monostatic system above will be denoted

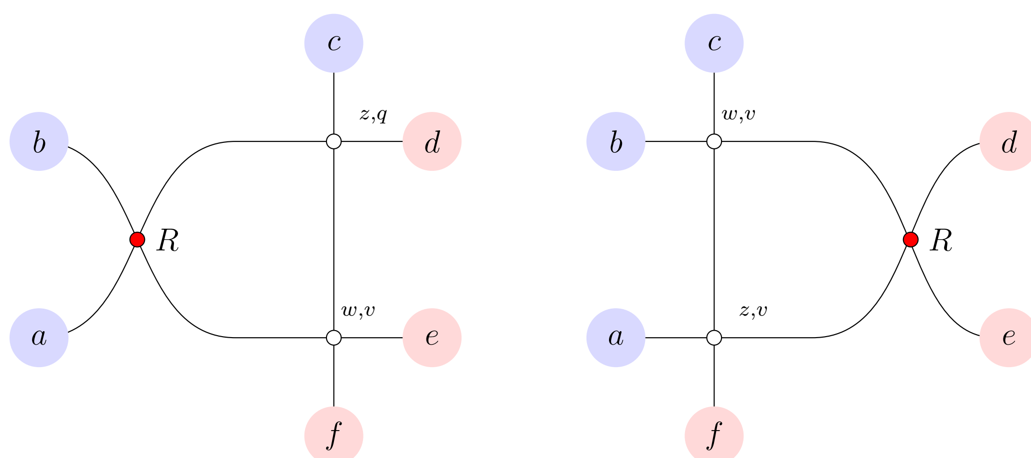
$$\langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle,$$

and we are now thinking of $T_\Delta(z;q)$ as being an operator on $\mathcal{H}$.

Theorem 3.2. *The operators $T_\Delta(z;q)$ all commute. That is, if w and v are other parameters, we have*

$$T_\Delta(z;q)T_\Delta(w,v) = T_\Delta(w;v)T_\Delta(z,q).$$

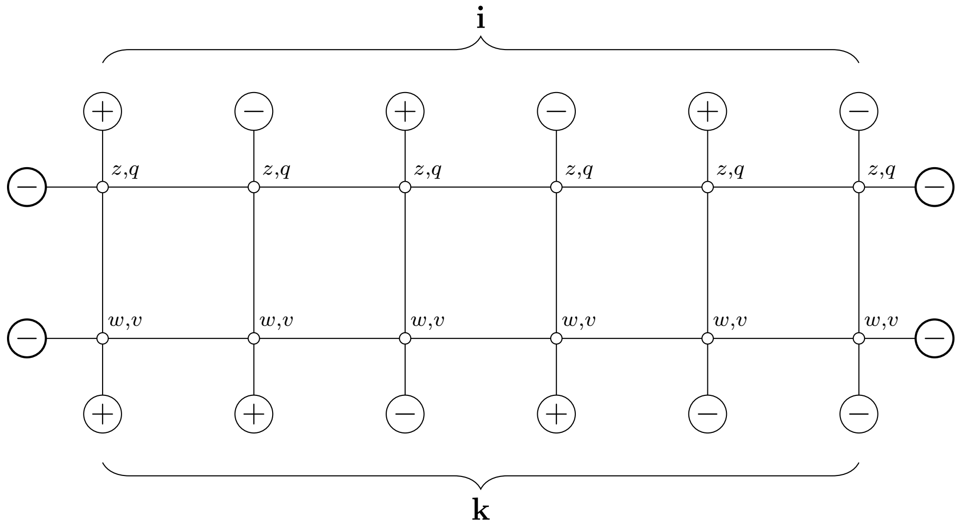
Proof. We make use of the general free-fermionic Yang-Baxter equation from Lecture 7. By Theorem 1.1 of Lecture 7, there exists an R-matrix R depending on z, q, w, v such that we have a Yang-Baxter equation in the form



It is of course not hard to compute the Boltzmann weights but we do not need them for this proof. We only need that the a_2 weight of R is nonzero. We fix $\mathbf{i}$ and $\mathbf{k}$ and will show that

$$\langle \mathbf{k}|T_\Delta(w;v)T_\Delta(z,q)|\mathbf{i}\rangle = \langle \mathbf{k}|T_\Delta(z;q)T_\Delta(w,v)|\mathbf{i}\rangle. \tag{3}$$

The left-hand side is the partition function of a 2-rowed infinite grid, but we may truncate this to a finite grid such that all sites of $|\mathbf{i}\rangle$ and $|\mathbf{k}\rangle$ to the right are occupied, and all sites to the left are unoccupied. This partition function looks like this:



All vertices outside this finite grid have type $\mathtt{a}_1$ or $\mathtt{b}_1$, and Boltzmann weight 1, so discarding them does not change the partition function. So the partition function of this system is

$$\langle \mathbf{k}|T_\Delta(w;v)T_\Delta(z;q)|\mathbf{i}\rangle.$$

Now we attach the R-matrix, which multiplies the Boltzmann weight by $a_2(R)$. We apply the train argument, and discard the R-matrix on the right, which divides the Boltzmann weight by the same constant $a_2(R)$. The resulting system has the rows switched, proving (3). Since this is true for all $\mathbf{i}$ and $\mathbf{k}$, the row transfer matrices are proved to commute. □

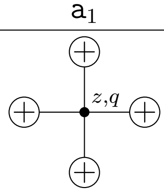
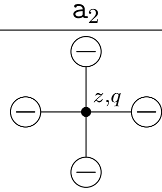
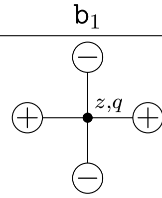
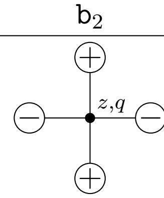
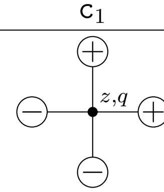
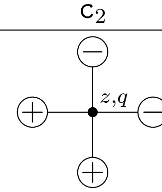
We can define $T_\Delta(z;q)$ as an operator on $\mathfrak{F}$ by

$$T_\Delta(z;q)|\mathbf{i}\rangle = \sum_{\mathbf{j}} \langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle\,|\mathbf{j}\rangle. \tag{4}$$

The sum on the right is finite, so this defines an element of $\mathfrak{F}$. However $T_\Delta(z;q)$ is not a bounded operator. That is, if we make $\mathfrak{F}$ into a Hilbert space where the semi-infinite monomials $|\mathbf{i}\rangle$ are an orthonormal basis, since the number of terms on the right side of (4) can be arbitrarily large, the map $T_\Delta(z;q)$ defined on basis elements does extend to an operator with bounded operator norm.

4 The Row Transfer Matrix $T_\Gamma(z)$

There is another type of six-vertex model that is in a sense dual to the models in Section 3. For these we use the following Boltzmann weights:

	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Γ-ice						
	z^{-1}	1	$-qz^{-1}$	1	$1-q$	z^{-1}

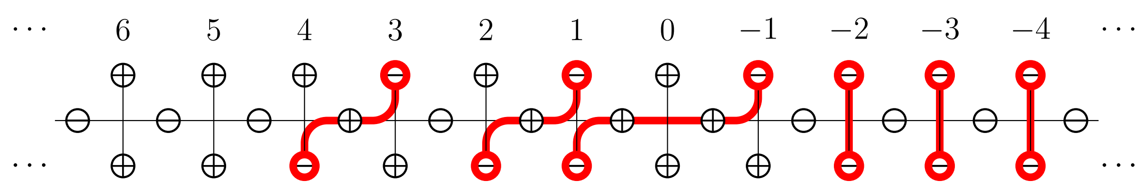
Remark 2. These are the same as the weights Tokuyama models introduced in Lecture 5, Section 2, divided by z. Since every weight is divided by the same constant, we could use these weights in the Tokuyama model, and the partition functions would be essentially unchanged, altered only be a constant monomial. However our boundary conditions will be different from the Tokuyama models.

Now we change the boundary conditions. We will requre all but finitely many horizontal spins to be $-$. This guarantees that the row transfer matrix will be an essentially finite product, since all but finitely many spins will be of type $\mathtt{a}_2$ or $\mathtt{b}_2$.

We can define $\langle \mathbf{j}|T_\Delta(z;q)|\mathbf{i}\rangle$ as before, but now the condition for this to be nonzero is changed: now we require

$$j_m \geqslant i_m \geqslant j_{m-1} \geqslant i_{m-1} \geqslant \cdots . \tag{5}$$

Here is a sample state with $\mathbf{i} = (3, 1, -1, -2, -3, \cdots)$ and $j = (4, 2, 1, -2, -3, \cdots)$. We modify the rule for describing the paths: now the paths follow the $-$ spins on vertical edges, and $+$ spins on the horizontal edges. This means that the paths move down and to the left, so the row transfer matrix is energy raising, in accordance with (5).



We can try to define $T_\Gamma(z;q)$ as an operator,

$$T_\Gamma(z;q)|\mathbf{i}\rangle = \sum_{\mathbf{j}} \langle \mathbf{j}|T_\Gamma(z;q)|\mathbf{i}\rangle\, |\mathbf{j}\rangle.$$

However (in contrast with Δ-ice) the sum on the right-hand side is no longer finite.

5 The Heisenberg Lie Algebra

We now come to a representation of the *Heisenberg Lie algebra* $\mathfrak{s}$ with generators

$$\{j_k | k \in \mathbb{Z}\} \qquad \text{and} \qquad \mathbf{1},$$

with $\mathbf{1}$ central, and

$$[j_k, j_l] = \begin{cases} k & \text{if } k = -l, \\ 0 & \text{otherwise.} \end{cases}$$

The center of $\mathfrak{s}$ is spanned by $\mathbf{1}$ and j_0. This representation is at the heart of the *boson-fermion correspondence*. This is a relationship between the fermionic Fock space and the bosonic Fock space which originated in mathematical physics, and has important applications to representation theory and algebraic combinatorics ([4, 7, 9]).

We remind the reader that we have defined

$$u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots$$

even if we do not have $j_m \geqslant j_{m-1} \geqslant \cdots$. It is only necessary that $j_k = k$ for k sufficiently negative. However this monomial might be zero (if some index is repeated) or the negative of a basis element if putting the vectors in order produces an odd number of sign changes. If $j_m \geqslant j_{m-1} \geqslant \cdots$ we will denote this vector as $|\mathbf{j}\rangle$. Otherwise we will avoid this notation.

Let $k \in \mathbb{Z}$. For the time being assume that $k \neq 0$. We define an operator J_k on V by $J_k(u_n) = u_{n-k}$. Then we transport J_k to acting on $\mathfrak{F}$ by the Leibnitz rule, so that

$$J_k|\mathbf{j}\rangle = (u_{j_m-k} \wedge u_{j_{m-1}} \wedge \cdots) + (u_{j_m} \wedge u_{j_{m-1}-k} \wedge \cdots) + \cdots \ .$$

In other words, to apply J_k, we pick one occupied location, and move the particle at that location k steps lower or higher (depending on the sign of k) to an unoccupied location. We also define J_0 to have eigenvalue m on $\mathfrak{F}_m$.

Theorem 5.1. *The operators J_k on $\mathfrak{F}_m$ satisfy*

$$[J_k, J_l] = \begin{cases} k \cdot 1_{\mathfrak{F}_m} & \text{if } k = -l, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $j_k \mapsto J_k$ defines a representation of the Heisenberg Lie algebra.

Proof. Let us first show that

$$J_k J_{-k}|\mathbf{j}\rangle - J_{-k} J_k|\mathbf{j}\rangle = k|\mathbf{j}\rangle. \tag{6}$$

We may assume $k > 0$ since the statements for k and $-k$ are trivially equivalent.

First suppose that $|\mathbf{j}\rangle = |\varnothing\rangle_m$ is the vacuum. Then $J_k|\varnothing\rangle_m = 0$. On the other hand, $J_{-k}|\varnothing\rangle$ is a sum of k terms, and applying J_k to each of these produces a copy of $|\varnothing\rangle_m$. Now we prove (6) for general $\mathbf{j}$. If $|\mathbf{j}\rangle = |\mathbf{j}\rangle_m$ is not the vacuum may write $|\mathbf{j}\rangle_m = u_j \wedge \eta$ where $j = j_m$ and

$$\eta = u_{j_{m-1}} \wedge u_{j_{m-1}} \wedge \cdots$$

has strictly smaller energy than $|\mathbf{j}\rangle_m$. By induction on enery we may assume that (6) is true for η. Now we have $J_{-k} = u_{j+k} \wedge \eta + u_j \wedge J_{-k}\eta$ and so

$$J_k J_{-k}|\mathbf{j}\rangle_m = u_j \wedge \eta + u_{j+k} \wedge J_k\eta + u_{j-k} \wedge J_{-k}\eta + u_j \wedge J_k J_{-k}\eta.$$

Similarly

$$J_{-k} J_k|\mathbf{j}\rangle_m = u_j \wedge \eta + u_{j-k} \wedge J_{-k}\eta + u_{j+k} \wedge J_k\eta + u_j \wedge J_{-k} J_k\eta.$$

Subtracting,

$$J_k J_{-k}|\mathbf{j}\rangle_m - J_{-k} J_k|\mathbf{j}\rangle_m = u_j \wedge (J_k J_{-k}\eta - J_{-k} J_k\eta) = u_j \wedge k\eta = k|\mathbf{j}\rangle_m,$$

where we have used our induction hypothesis.

We leave it to the reader to show that J_k and J_l commute unless $k = -l$. □

6 Row Transfer Matrices as Vertex Operators

We emphasize that the J_k with $k > 0$ all commute, and the J_{-k} with $-k < 0$ all commute, so we have two large commuting families of "operators" on $\mathfrak{F}$ or $\mathfrak{F}_m$. The J_{-k} are not operators in the usual sense, since each turns each basis vector into an infinite sum of basis vectors, which is not in $\mathfrak{F}$. Still, the two-point functions

$$\langle \mathbf{i}|J_k|\mathbf{j}\rangle$$

do make sense for all k, and as long as we couch our results in terms of these, there are no difficulties.

Now let us introduce two "Hamiltonians"

$$H_+(z;q) = \sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^k J_k, \qquad H_-(z;q) = \sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^{-k} J_{-k}$$

Theorem 6.1 ([2]). *We have*

$$e^{H_+(z;q)} = T_\Delta(z;q), \qquad e^{H_-(z;q)} = T_\Gamma(z;q), \tag{7}$$

The operator $H_+(z;q)$ commutes with $T_\Delta(w;v)$ for all w,v, and the operator $H_-(z;q)$ commutes with $T_\Gamma(w;v)$ for all w,v.

Proof. We will take this up in Lecture 20. For now we point out that the identities (7) imply the commutativity statements, since for example the operators $T_\Delta(w;v)$ and the operator $H_+(z;q)$ are all seen to be expressible in terms of the J_k with $k > 0$, which commute with each other. We also obtain a new proof of the commutativity statement in Theorem 3.2 from this observation. □

"Operators" such as $e^{H_+(z;q)}$ and $e^{H_-(z;q)}$, particularly in combinations such as:

$$e^{H_-(z;q)}e^{H_+(z;q)} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^{-k}J_{-k}\right)\exp\left(\sum_{k=1}^{\infty} \frac{1}{k}(1-q^k)z^k J_k\right) \tag{8}$$

are called *vertex operators.* Here "operators" is in quotation marks since there is a nontrivial problem in making sense of this. Similar expressions appear in conformal field theory and in soliton theory. A purely algebraic and rigorous axiomatization of the underlying mathematics may be found in the theory of *vertex algebras.* In this context, expressions such as (8) appear in *lattice vertex algebras* ([3] Chapter 5 or [6] Section 5.4). See also [8] and [5].

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