# Lecture 18

## 1 Comparison of two similar models

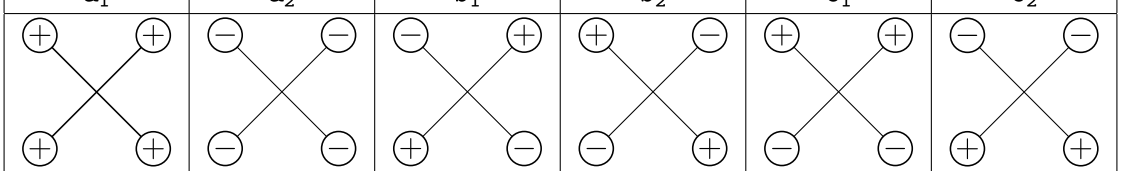
This lecture will be a very quick introduction to Lie superalgebras. Our reason for wanting to introduce this topic is that some very interesting models are associated with the quantized enveloping algebras of Lie superalgebras. For more information about Lie superalgebras, we recommend Cheng and Wang [9] and Musson [14].

Let us compare two similar models, the Tokuyama model of Lectures 5 and 6, and the bosonic models of Lectures 8 and 16. The partition functions of the Tokuyama models are, we saw in Lectures 5 and 6, Schur polynomials times deformed Weyl denominators:

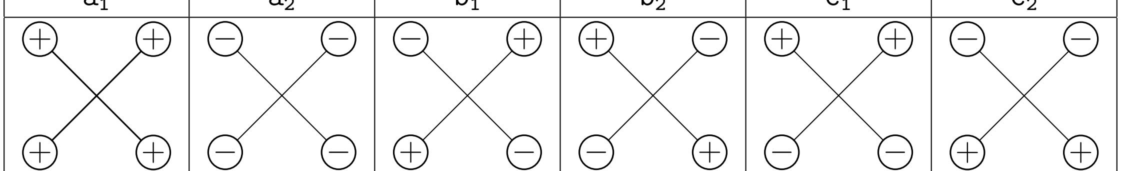
$$\mathbf{z}^{\rho} \prod_{\alpha \in \Phi^+} (1 - q\mathbf{z}^{-\alpha}) s_{\lambda}(\mathbf{z}).$$

The partition functions of the bosonic models are Hall-Littlewood polynomials [8]. Both models have colored variants that produce nonsymmetric polynomials ([5, 8]). It is possible to argue that these two symmetric polynomials are closely related "twins," a parallel that would extend to other Cartan types. We will not explain this point but see [7, 4].

Now let us compare the R-matrices. Here is the R-matrix for the bosonic models. The relevant quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$.

| $\mathtt{a}_1$ | $\mathtt{a}_2$ | $\mathtt{b}_1$ | $\mathtt{b}_2$ | $\mathtt{c}_1$ | $\mathtt{c}_2$ |
|---|---|---|---|---|---|
|  | | | | | |
| $z - qw$ | $z - qw$ | $q(z - w)$ | $z - w$ | $(1 - q)z$ | $(1 - q)w$ |

The R-matrix for the Tokuyama models is extremely similar, the only different entry being the $\mathtt{a}_1$ entry:

| $\mathtt{a}_1$ | $\mathtt{a}_2$ | $\mathtt{b}_1$ | $\mathtt{b}_2$ | $\mathtt{c}_1$ | $\mathtt{c}_2$ |
|---|---|---|---|---|---|
|  | | | | | |
| $w - qz$ | $z - qw$ | $q(z - w)$ | $z - w$ | $(1 - q)z$ | $(1 - q)w$ |

It can be shown that the R-matrix is related to the standard representations of $U_q(\widehat{\mathfrak{gl}}(1|1))$, a superalgebra quantum group. See [17, 12, 18] for results on R-matrices of $U_q(\widehat{\mathfrak{gl}}(m|n))$.

The similarity between the R-matrices for $U_q(\widehat{\mathfrak{sl}}_{m+n})$ and $U_q(\widehat{\mathfrak{gl}}(m|n))$ extends to general $m$ and $n$. The R-matrices correspond to colored models ([6, 1]).

Now the interesting thing is that this similarity between the Tokuyama and bosonic models also applies to the vertical edges. For the models in [5], which are related to $U_q(\widehat{\mathfrak{gl}}(n|1))$, the vertical edges have spinset of cardinality $2^n$, in bijection with the set of subsets of $n$ colors. The vertical edges can be understood in terms of fusion as in the bosonic models of [8]. The $U_q(\widehat{\mathfrak{sl}}_{n+1})$-modules in [8], we saw in Lecture 16, correspond to Verma modules, which are isomorphic to $\mathrm{Sym}(\mathbb{C}^n)$. Now $U(\mathfrak{gl}(n|1))$ and its quantized affinization $U_q(\widehat{\mathfrak{gl}}(n|1))$ have a kind of Verma module called a *Kac module* that is isomorphic to the exterior algebra $\bigwedge \mathbb{C}^n$ of cardinality $2^n$, which is expected to be related to the vertical edges in these models. The purpose of this lecture will be to introduce Lie superalgebras and to define the Kac modules that we claim are to explain the vertical edges in the colored fermionic models.

# 2 Lie Superalgebras

Lie superalgebras are a generalization of a Lie algebra. They emerged in the 1970's from physical theories having symmetries that connect fermions and bosons [10]. In 1981 Perk and Schultz [15] found some new solvable lattice models which were explained by Yamane [17] as being related to the superalgebra quantum group $U_q(\mathfrak{gl}(m|n))$, whose R-matrices he computed. Later Brubaker, Bump and Buciumas [3] found supersymmetric lattice models whose partition functions are Whittaker functions on $p$-adic metaplectic groups. Such models were substantially generalized by Brubaker, Buciumas, Bump and Gustafsson [6]. Other supersymmetric models were found by Aggarwal, Borodin and Wheeler [2].

In retrospect, the Tokuyama models, which we have already looked at are supersymmetric, being associated with $U_q(\widehat{\mathfrak{gl}}(1|1))$. Colored variants associated with $U_q(\widehat{\mathfrak{gl}}(n|1))$. For these models, the quantum group is identified from the R-matrix. But the vertical edges are associated with *Kac modules*, which are Verma modules for superalgebras that are finite-dimensional.

A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, where $V_0$ is the *even part* and $V_1$ is the *odd part*. Elements of $V_0$ and $V_1$ are called *homogeneous*. It will be convenient to denote $|a| = i$ if $a \in V_i$, so $|a| = 0$ if $a$ is even and $|a| = 1$ if $a$ is odd. We define the super dimension of $V$ to be $n|m$ where $n = \dim(V_0)$ and $m = \dim(V_1)$.

Other things such as associative algebras and Lie algebras have super variants. The common feature is that interchanges involve sign changes. The rule is that when an odd element "moves past" an even element, a minus sign is introduced.

For example, an associative superalgebra is just a $\mathbb{Z}_2$-graded associative algebra. No modification of the associative law is needed, because in the identity $a(bc) = (ab)c$, the elements occur in the same order. But what does it mean for an associative superalgebra to be *commutative?* The rule is that if $a$ and $b$ are homogeneous, then $ab = ba$ unless both are odd, in which case $ab = -ba$. We can write this more succinctly as $ab = (-1)^{|a|\cdot|b|}ba$.

For example, the exterior algebra of a vector space, or the cohomology ring of a topological space are commutative superalgebras.

If $V$ is an ordinary vector space, we will denote by $S(V)$ and $\bigwedge V$ the symmetric and exterior algebras. If $V$ is a super vector space, our convention is that the symmetric and exterior algebras are

$$S(V) = S(V_0) \otimes \bigwedge V_1, \qquad \bigwedge V = \bigwedge V_0 \otimes S(V_1).$$

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space with a bilinear operation $[\ ,]$ such that (for $x, y$ and $z$ homogeneous)

$$[x, y] = -(-1)^{|x|\cdot|y|}[y, x] \tag{1}$$

and the Jacobi identity holds, in the form

$$(-1)^{|x|\cdot|z|}[[x, y], z] + (-1)^{|y|\cdot|x|}[[y, z], x] + (-1)^{|z|\cdot|y|}[[z, x], y] = 0.$$

The even part $\mathfrak{g}_0$ of the Lie superalgebra $\mathfrak{g}$ is an ordinary Lie algebra.

**Example 2.1.** Let $V$ be a super vector space. Then $\mathfrak{gl}(V) = \mathrm{End}(V)$, with the following grading If $\phi \in \mathrm{End}(V)$. We can write

$$\phi = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11} \end{pmatrix}$$

where $\phi_{ij} \in \mathrm{Hom}(V_j, V_i)$. We make $\mathfrak{gl}(V)$ into a super vector space in which $\phi_{00}, \phi_{01}, \phi_{10}$ are even, and $\phi_{11}$ is odd. As a particular case, let $\mathbb{C}^{m|n}$ denote the super vector space $V_0 \oplus V_1$ where the even part $V_0 = \mathbb{C}^m$ and the odd part $V_1 = \mathbb{C}^n$. With $V = \mathbb{C}^{m|n}$ we will denote $\mathfrak{gl}(V) = \mathfrak{gl}(m|n)$.

We are assuming that the ground field has characteristic not equal to 2 or 3. The enveloping algebra $U(\mathfrak{g})$ is the associative superalgebra generated by $\mathfrak{g}$ modulo the relations $[x, y] = xy - (-1)^{|x|\cdot|y|}yx$.

**Lemma 2.2.** *If $x \in \mathfrak{g}$ is odd, then $[x, x] = 0$, and $x^2 = 0$ in $U(\mathfrak{g})$.*

*Proof.* By (1) we have $0 = [x, x] = -[x, x]$ so $[x, x] = 0$ in $\mathfrak{g}$. Since $x$ is odd, the relation $[x, x] = x^2 - (-1)^{|x|\cdot|x|}x^2 = 2x^2$, so $x^2 = 0$ in $\mathfrak{g}$. □

There is a PBW Theorem.

**Theorem 2.3.** *For simplicity let us assume that $\mathfrak{g}$ is finite-dimensional. Let us choose a basis $x_1, \cdots, x_d$ consisting of homogeneous elements. Then $U(\mathfrak{g})$ has a basis consisting of elements*

$$x_1^{k_1} \cdots x_d^{k_d}$$

*where $k_i \in \mathbb{N}$ if $x_i$ is even, and $k_i \in \{0, 1\}$ if $x_i$ is odd.*

*Proof.* See [16], Theorem 2.1, or [14] Chapter 6. □

**Proposition 2.4.** *Let $\mathfrak{g}$ be an abelian Lie superalgebra, so $[\mathfrak{g},\mathfrak{g}]=0$. Then*

$$U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong S(\mathfrak{g}_0) \otimes \bigwedge \mathfrak{g}_1 \,.$$

*Proof.* We leave this to the reader. □

Now let $\mathfrak{g}$ be a Lie superalgebra. The even part $\mathfrak{g}_0$ is a Lie algebra. We may choose a maximal abelian Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$, and decompose $\mathfrak{g}$ into root spaces as in Lecture 15:

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H,X] = \alpha(H)X \text{ for } H \in \mathfrak{h}\} \,.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, we call $\alpha \in \mathfrak{h}$ a root. The roots may be divided into even and odd roots, where a root is *even* if the *root space* $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$, and *odd* if $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$. The set $\Phi$ of roots can also be divided into positive and negative roots. Then we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}_-^{\text{odd}} \oplus \mathfrak{g}_0 \oplus \mathfrak{u}_+^{\text{odd}},$$

where $\mathfrak{u}_-^{\text{odd}}$ is the sum of the root spaces for the odd negative roots, and $\mathfrak{u}_+^{\text{odd}}$ is the sum of the root spaces for the odd positive roots.

For example, if $\mathfrak{g} = \mathfrak{gl}(2|2)$, the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ is $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$, and we may take $\mathfrak{h}$ to be the diagonal subalgebra. Then

$$\mathfrak{g}_0 = \left(\begin{array}{cc|cc} * & * & 0 & 0 \\ * & * & 0 & 0 \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right), \qquad \mathfrak{u}_+^{\text{odd}} = \left(\begin{array}{cc|cc} 0 & 0 & * & * \\ 0 & 0 & * & * \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \mathfrak{u}_-^{\text{odd}} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline * & * & 0 & 0 \\ * & * & 0 & 0 \end{array}\right).$$

Now by the PBW theorem we have

$$U(\mathfrak{g}) \cong U(\mathfrak{u}_-^{\text{odd}}) \otimes_{\mathbb{C}} U(\mathfrak{g}_0) \otimes_{\mathbb{C}} U(\mathfrak{u}_+^{\text{odd}})$$

where $\otimes = \otimes_{\mathbb{C}}$ is the tensor product of associative superalgebras, a modification of the usual tensor product of associative algebras in which (for homogeneous elements $a,b,c,d$)

$$(a \otimes b)(c \otimes d) = (-1)^{|b|\cdot|d|}(ac \otimes bd).$$

Assuming that $\mathfrak{u}_-^{\text{odd}}$ is abelian, which it is in the example of $\mathfrak{gl}(m|n)$, we have

$$U(\mathfrak{u}_-^{\text{odd}}) \cong \bigwedge \mathfrak{u}_-^{\text{odd}}, \tag{2}$$

the exterior algebra, of dimension $2^{\dim(\mathfrak{u}_-^{\text{odd}})}$.

Now we can explain the Kac modules that are intended as an explanation for the vertical edges in the colored fermionic models mentioned at the beginning of the lecture, including the (uncolored) Tokuyama model. Let $V$ be a finite-dimensional $\mathfrak{g}_0$-module. We extend it to a $\mathfrak{p}$-module where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{u}_+^{\text{odd}}$ by letting $\mathfrak{u}_+^{\text{odd}}$ act trivially. Then the induced module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V \cong U(\mathfrak{u}_-^{\text{odd}}) \otimes_{\mathbb{C}} U(\mathfrak{p}) \otimes_{U(\mathfrak{p})} V \cong \bigwedge \mathfrak{u}_-^{\text{odd}} \otimes V$$

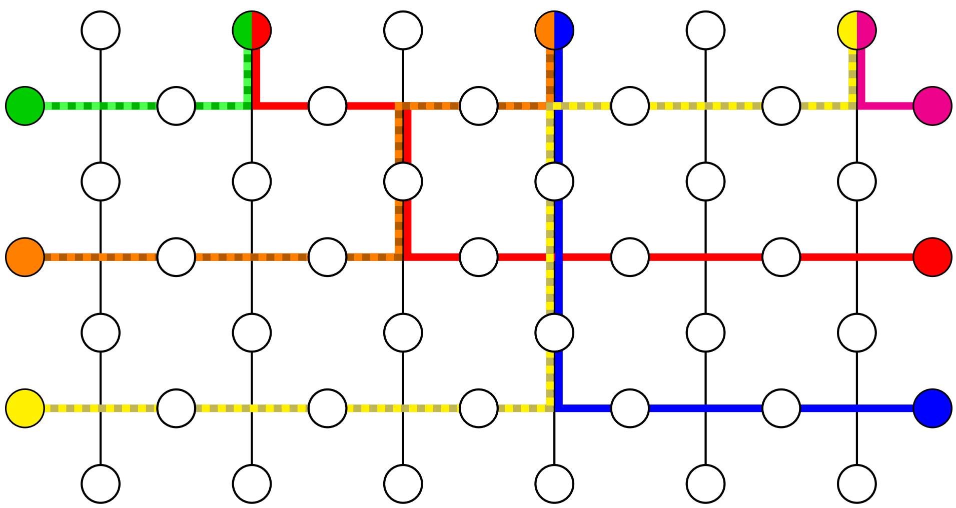
is called the *Kac module*. As we can see it is a kind of Verma module that happens to be finite-dimensional. Depending on $V$, it is usually irreducible, but not always [11]. Its character is easy to describe. See [13] for Kac modules of $U_q(\mathfrak{gl}(m|n))$.

**Conjecture 2.5.** *For $\mathfrak{g} = \mathfrak{gl}(m|n)$. There exists a one-dimensional representation $V$ of $\mathfrak{g}_0$ whose Kac module explains the vertical edges in the colored fermionic models of [6].*

To illustrate these ideas, we describe some $U_q(\widehat{\mathfrak{gl}}(m|n))$ models from [6]. Other $U_q(\mathfrak{gl}(m|n))$ models may be found in [1]. We will omit full description of the Boltzmann weights but will discuss how these models make use of Kac modules for $U_q(\mathfrak{gl}(m|n))$. These models are generalizations of the Tokuyama model, and it is helpful to consider them even if one is only interested in simpler cases.

We require $m$ colors $\mathcal{C} = \{c_1, \cdots, c_m\}$ and another palette of $n$ "supercolors," which we will denote $\mathcal{D} = \{d_1, \cdots, d_n\}$. Colors move down and to the right, while supercolors move down and to the left. The spinset of the horizontal edges is $\mathcal{C} \cup \mathcal{D}$. Thus a horizontal edge can carry a color or supercolor (but not both).

The vertical edges carry color-supercolor pairs, such as $(c_i, d_j)$. There are $mn$ such pairs. A vertical edge may carry several of those, but the models are fermionic, so it may not carry multiple pairs. So the spinset of the vertical edge is the power set of $\mathcal{C} \cup \mathcal{D}$, and its cardinality is $2^{mn}$. Here we illustrate a state of such a system with 3 colors and 3 supercolors. The paths of colors are represented by solid lines, and supercolor paths are represented by broken lines.



We omit the Boltzmann weights, which can be described by fusion (Lecture 16) of $mn$ monochrome edges, each of which can carry only a single color-supercolor pair.

We note that the color-supercolor pairs are in bijection with the $mn$ odd negative roots of $\mathfrak{g} = \mathfrak{gl}(m|n)$, so by (2), the Kac module of a one-dimensional representation of $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ can be identified with the exterior algebra of the free-vector space on the set $\mathcal{C} \times \mathcal{D}$ of such pairs. Its dimension is $2^{mn}$. Hence it is natural to believe that the module associated with these vertical edges is such a Kac module.

# References


[1] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. *Comm. Amer. Math. Soc.*, 3:400–630, 2023.

[2] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. *Comm. Amer. Math. Soc.*, 3:400–630, 2023.

[3] B. Brubaker, V. Buciumas, and D. Bump. A Yang-Baxter equation for metaplectic ice. Appendix joint with Nathan Gray. *Commun. Number Theory Phys.*, 13(1):101–148, 2019.

[4] B. Brubaker, V. Buciumas, D. Bump, and S. Friedberg. Hecke modules from metaplectic ice. *Selecta Math. (N.S.)*, 2017, arXiv:1704.00701.

[5] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Colored vertex models and Iwahori Whittaker functions, 2019, arXiv:1906.04140.

[6] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Metaplectic Iwahori Whittaker functions and supersymmetric lattice models, 2020, arXiv:2012.15778.

[7] B. Brubaker, D. Bump, and S. Friedberg. Matrix coefficients and Iwahori-Hecke algebra modules. *Adv. Math.*, 299:247–271, 2016.

[8] D. Bump and S. Naprienko. Colored bosonic models and matrix coefficients, 2022, arXiv:2211.15850.

[9] S.-J. Cheng and W. Wang. Dualities for Lie superalgebras. In *Lie theory and representation theory*, volume 2 of *Surv. Mod. Math.*, pages 1–46. Int. Press, Somerville, MA, 2012.

[10] L. Corwin, Y. Ne'eman, and S. Sternberg. Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry). *Rev. Modern Phys.*, 47:573–603, 1975.

[11] V. G. Kac. Characters of typical representations of classical Lie superalgebras. *Comm. Algebra*, 5(8):889–897, 1977.

[12] T. Kojima. Diagonalization of transfer matrix of supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ chain with a boundary. *J. Math. Phys.*, 54(4), 2013.

[13] J.-H. Kwon. Crystal bases of $q$-deformed Kac modules over the quantum superalgebras $U_q(\mathfrak{gl}(m|n))$. *Int. Math. Res. Not. IMRN*, (2):512–550, 2014.

[14] I. M. Musson. *Lie superalgebras and enveloping algebras*, volume 131 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[15] J. H. H. Perk and C. L. Schultz. New families of commuting transfer matrices in $q$-state vertex models. *Phys. Lett. A*, 84(8):407–410, 1981.

[16] L. E. Ross. Representations of graded Lie algebras. *Trans. Amer. Math. Soc.*, 120:17–23, 1965.

[17] H. Yamane. Quantized enveloping algebras associated with simple Lie superalgebras and their universal $R$-matrices. *Publ. Res. Inst. Math. Sci.*, 30(1):15–87, 1994.

[18] H. Zhang. Fundamental representations of quantum affine superalgebras and $R$-matrices. *Transform. Groups*, 22(2):559–590, 2017, arxiv:1506.06093.