Lecture 17

A standard method in analysis, going back to Hilbert and Schmidt, and much earlier to Green, for studying operators is to find a larger commuting family of operators. A simple example is the Laplacian, an unbounded self-adjoint operator. This commutes with the integral operators $F \longmapsto F * \phi$, where $\phi \in C_c^\infty(\mathbb{R}^n)$, which are Hilbert-Schmidt operators, hence compact. This method is used extensively in the theory of automorphic forms, for example in the Selberg trace formula.

Baxter's approach to the six- and eight-vertex models was to embed the row transfer matrix into a larger commuting family of row transfer matrices. Another application of the same idea led to the solution of a problem in one-dimensional quantum mechanics, his analysis of the Heisenberg spin chains, a model of ferromagnetism [5]. Baxter was able to introduce the theory of elliptic functions by finding a commuting family of row transfer matrices from the eight-vertex model.

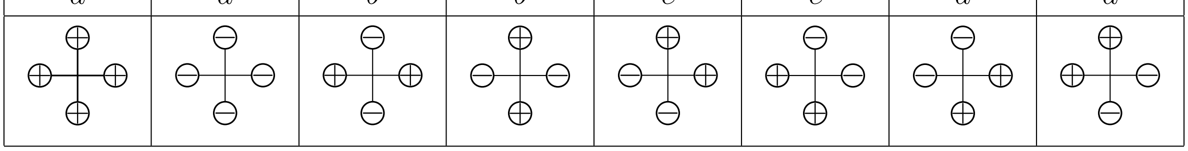
Baxter [1, 2] knew that two operators arising from physical problems were related to each other. (This fact was also observed by Sutherland [6].) The operators are:

- The row transfer matrices from the field-free 6 or 8 vertex models
- Hamiltonians for Heisenberg spin chains, called the XXZ and XYZ Hamiltonians.

Using the Yang-Baxter equation, the row transfer matrices can be organized into commuting families. This means that the row transfer matrix contains a parameter that can be differentiated, and roughly the Hamiltonian is the logarithmic derivative of the row transfer matrix. Equivalently, the row transfer matrix is an exponentiated Hamiltonian. As a consequence, the Hamiltonian also commutes with this family of row transfer matrices.

The field-free eight vertex model can be solved similarly to the six vertex model, using a parametrized Yang-Baxter equation. Let a, b, c, d be the Boltzmann weights, thus:

a	a	b	b	c	c	d	d



Define

$$\Delta = \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}, \qquad \Gamma = \frac{ab - cd}{ab + cd}. \tag{1}$$

If a', b', c', d' are another set of Boltzmann weights, and Δ', Γ' are defined like Δ, Γ, the condition for the Yang-Baxter equation to have a solution is

$$\Delta = \Delta', \qquad \Gamma = \Gamma'.$$

(See [2], Chapter 10.) Now with Γ and Δ fixed, the solutions $a : b : c : d$ to (1) form an elliptic curve, and indeed, the relevant Yang-Baxter equation is a parametrized Yang-Baxter equation with this curve as its parameter group. The relevant quantum group is an elliptic quantum group ([4]).

Baxter's work solving the eight-vertex model was carried out on a ship, where he took over the chart room for his calculations. Baxter [2] wrote in Chapter 10:

> Sutherland (1970) showed directly that the transfer matrix of any zero-field eight-vertex model commutes with an XYZ operator X. They therefore have the same eigenvectors. I was not aware of Sutherland's result when I solved the eight-vertex model (I did much of the work in the writing room of the P & O liner *Arcadia*, in the Atlantic and Indian Oceans. This was good for concentration, but not for communication). It should be obvious from Sections 10.4-10.6 that such commutation relations are closely linked with the solution of the problem.

This theory is outside the scope of these lectures, but we will consider the simpler case where $d = 0$, where the field-free six-vertex model is related to the XXZ Hamiltonian. As we know, the relevant quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$.

For the free-fermionic six vertex model (where the quantum group is $U_q(\widehat{\mathfrak{gl}}(1|1))$) a similar result was obtained by Brubaker and Schultz [3]. See also [7].

1 Heisenberg Spin Chains

In classical mechanics, *observables* are functions A on the *phase space*, which is a parameter space representing the state of a physical system, including the positions and momenta of all particles. Given a state of the system, every observable thus has a definite value.

In quantum mechanics, by contrast, it is possible for the system to be in a state where a given observable does not have a definite value. The state of the system is represented by a vector in a Hilbert space $\mathfrak{H}$, and the classical observable A is replaced by a Hermitian (self-adjoint) operator $\hat{A} : \mathfrak{H} \longrightarrow \mathfrak{H}$ (or an unbounded operator defined on a dense subspace). If $\Psi \in \mathfrak{H}$ represents the state of the system, the observable A has a definite value λ if $\hat{A}\Psi = \lambda\Psi$.

For simplicity let us assume that $\hat{A}$ has a discrete spectrum. By the spectral theorem, the state Ψ may be expanded as a "Fourier series"

$$\Psi = \sum a_i \Psi_i$$

where Ψ_i are eigenfunctions of $\hat{A}$. If we normalize Ψ so that $|\Psi| = 1$, then the "amplitudes" a_i have a probabilistic interpretation: if the observable f is measured, a definite value λ_i is returned, and the "wave function" Ψ collapses to the state Ψ_i. The probability of this happening is $|a_i|^2$. By the Plancherel theorem $\sum_i |a_i|^2 = 1$, and so this scheme gives a probability distribution on the spectrum of $\hat{A}$.

In quantum mechanics, two observables A and B can be measured simultaneously if and only if the corresponding operators $\hat{A}$ and $\hat{B}$ commute. In this case, the eigenfunctions Ψ_i can be chosen to be simultaneous eigenfunctions of $\hat{A}$ and $\hat{B}$.

A particular observable is *energy*, and the corresponding operator is the *Hamiltonian*. It determines the evolution of the system in time, through Schrödinger's equation.

An examples of a collection of observables that cannot be measured simultaneously are electron spin in different directions. A 2 dimensional Hilbert space $\mathfrak{H} = \mathbb{C}^2$ is sufficient to represent a particle such as the electron with spin $\frac{1}{2}$. If the spin is measured along the z axis, it will be found in one of two states, up or down. The spin operator is therefore represented by the matrix

$$\sigma^z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

On the other hand, if the spin is measured along the x or y axes, it will again be found in one of two possible states. The corresponding operators do not commute with σ^z, and with respect to the same basis, are represented by the matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The three matrices σ^x, σ^y, σ^z are called the *Pauli spin matrices*. They are both Hermitian and unitary. We have an alternative labeling

$$\sigma^1 = \sigma^x, \qquad \sigma^2 = \sigma^y, \qquad \sigma^3 = \sigma^z, \qquad \sigma^4 = I_2. \tag{2}$$

Heisenberg [5] proposed a quantum mechanical model of ferromagnetism. We consider a sequence of N magnetic atoms such as iron at adjacent sites. We will assume that the sites of the spin chain are arranged in a ring. Consequently the boundary conditions for the six-vertex model will also be periodic, as in Lecture 2.

Each atom is a magnetic dipole whose dipole moment is proportional to the spin. Since the spin module is 2-dimensional, it is represented by a vector in a 2-dimensional space. The Hilbert space of a single magnetic atom is $\mathbb{C}^2$. Therefore the Hilbert space $\mathfrak{H}$ of N atoms is $\otimes^N \mathbb{C}^2$. We let σ_j^x, σ_j^y and σ_j^z denote the Pauli matrices acting on the j-th site, and as the identity operator on all other sites.

To give the simplest formulation, we will assume that the chain is periodic, so $\sigma_{N+1}^x = \sigma_1^x$ etc. Adjacent dipoles tend to align in the same direction, which partly explains the form of the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{N} (J_x \sigma_j^x \otimes \sigma_{j+1}^x + J_y \sigma_j^y \otimes \sigma_{j+1}^y + J_z \sigma_j^z \otimes \sigma_{j+1}^z), \tag{3}$$

for suitable positive constants J_x, J_y, J_z. Due to the assumed periodicity, $\sigma_{N+1} = \sigma_N$. If $J_x = J_y$, this is called the XXZ Hamiltonian. It is an endomorphism of $\otimes^N \mathbb{C}^2$.

On the other hand, we can fix Boltzmann weights a, b, c, d and consider the row transfer matrix $T_{a,b,c,d}(\alpha, \beta)$ as follows. We will denote the standard basis of $\mathbb{C}^2$ as

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A basis of $\otimes^N \mathbb{C}^2$ consists of vectors v_α where $\alpha = (\alpha_1, \cdots, \alpha_N) \in \{+, -\}^N$, where

$$v_\alpha = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}.$$

We may thus regard both the Hamiltonian H and the row transfer matrices for the 8 vertex model as endomorphisms of the same Hilbert space, $\otimes^N \mathbb{C}^2$. Baxter proved:

- The XXZ Hamiltonian commutes with a family of 6-vertex model row transfer matrices.
- The XYZ Hamiltonian commutes with a family of 8-vertex model row transfer matrices.

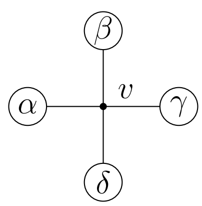
We will review the relationship of the XXZ Hamiltonian with the row transfer matrices for the field-free 6-vertex model. We will partly investigate the 8-vertex model, but we will specialize to the 6-vertex model before long, and prove the second statement. See Baxter [1] for the 8-vertex model case, which requires some elliptic and theta functions.

2 Preliminaries

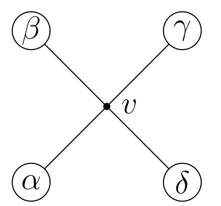
We will make use of the Pauli spin matrices with respect to this basis, and if $\alpha, \beta \in \{+, -\}$, and if σ is one of the Paul spin matrices, we will denote by $\sigma_{\alpha,\beta}$ the corresponding matrix entry. Thus $\sigma^y_{-+} = i$ and $\sigma^y_{+-} = -i$. Let

$$p_1 = \frac{1}{2}(b + d), \qquad p_2 = \frac{1}{2}(b - d), \qquad p_3 = \frac{1}{2}(a - c), \qquad p_4 = \frac{1}{2}(a + c). \tag{4}$$

Let v be a vertex type. We will denote by $R^{\gamma\delta}_{\alpha\beta}(v)$ the Boltzmann weight



or



Lemma 2.1. *Let $\alpha, \beta, \gamma, \delta \in \{+, -\}$. Then*

$$R^{\gamma\delta}_{\alpha\beta} = \sum_{k=1}^{4} p_k \sigma^k_{\beta\gamma} \sigma^k_{\alpha\delta}. \tag{5}$$

Proof. This can be checked by case-by-case consideration. There are 16 choices for $\alpha, \beta, \gamma, \delta$, but only eight give a nonzero result. Let us consider for example $(\alpha, \beta, \gamma, \delta) = (+, -, +, -)$. Since σ^k_{-+} and σ^k_{+-} are nonzero only for $k = 1, 2$, there are two terms:

$$\frac{1}{2}(b + d)\sigma^1_{\beta\gamma}\sigma^1_{\alpha\delta} + \frac{1}{2}(b - d)\sigma^2_{\beta\gamma}\sigma^2_{\alpha\delta} = b.$$

The remaining cases are similar. □

There is a similar identity

$$R^{\gamma\delta}_{\alpha\beta} = \sum_{k=1}^{4} w_k \sigma^k_{\beta\delta} \sigma^k_{\gamma\alpha}$$

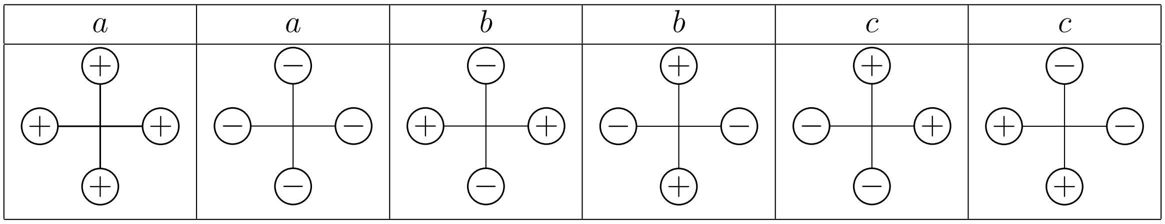
where

$$w_1 = \frac{1}{2}(c + d), \qquad w_2 = \frac{1}{2}(-c + d), \qquad w_3 = \frac{1}{2}(a - b), \qquad w_4 = \frac{1}{2}(a + b).$$

We won't need this but mention it for completeness.

3 Six-vertex model and the XXZ Hamiltonian

Now we specialize to the six-vertex model, referring to [1] for the general case. Thus now $d = 0$, and as a consequence of this simplification we will have $J_x = J_y$ in the Hamiltonian.



Let $T_{a,b,c}(\alpha, \beta)$ be the corresponding row transfer matrix.

We saw in Lecture 4 that if Δ is fixed, then we have a parametrized Yang-Baxter equation involving a, b, c such that

$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta.$$

Let q be such that $\Delta = \frac{1}{2}(q + q^{-1})$. We may parametrize the solutions by a map

$$R_\Delta : \mathbb{C}^\times \longrightarrow \{\text{field free Boltzmann weights } a, b, c\}$$

given by

$$R_\Delta(x) = (a, b, c) = \left(\frac{xq - (xq)^{-1}}{q - q^{-1}}, \frac{x - x^{-1}}{q - q^{-1}}, 1 \right).$$

These are the Boltzmann weights from Theorem 5.2 in Lecture 4, divided by the constant $\frac{1}{2}(q - q^{-1})$. We saw that this gives a parametrized Yang-Baxter equation. (Dividing by a constant does not affect this since both sides of the Yang-Baxter equation are divided by the same constant.)

Then by Theorem 1.1 of Lecture 2, the row transfer matrices $T_{a,b,c}(\alpha, \beta)$ form a commuting family. We choose fixed χ so that $e^{i\chi} = q$. Then we choose variable θ so that $e^{i\theta} = xq$. We slightly modify the notation, omitting Δ from the notation R_Δ because it is fixed, and regarding R as a function of θ instead of x. We will use the notation $R(\theta)_{\alpha\beta}^{\gamma\delta}$ as explained in Section 2.

Lemma 3.1. *When $\theta = \chi$, we have*

$$R(\chi)_{\alpha\beta}^{\gamma\delta} = \begin{cases} 1 & \text{if } \alpha = \delta \text{ and } \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that when $\theta = \chi$ we have $(a, b, c) = (1, 0, 1)$. So this follows from the definition of the Boltzmann weights. □

Let $T_\theta(\alpha, \beta)$ be the row transfer matrix $T_{a,b,c}(\alpha, \beta)$ with this parametrization.

Remark 1. At the special point $\theta = \chi$, since $b = 0$, we are in a 5-vertex model case in which the particles are allowed to move to the right but not straight down. In fact $T_\chi(\alpha, \beta)$ is the right shift operator, moving each particle one step to the right. Obviously T_χ is invertible, the inverse being the left shift operator.

We may differentiate the operator T_θ with respect to θ. The derivative T'_θ commutes with T_θ, and we may consider the logarithmic derivative at $\theta = \chi$.

$$\mathcal{L} = \frac{1}{2} T_\chi^{-1} T'_\chi.$$

It is at this point $\theta = \chi$ that there is a relationship between the XXZ Hamiltonian and the six-vertex model. Regard the p_i in (4) as functions of θ.

$$J_x = \frac{1}{2} p'_1(\chi), \qquad J_y = \frac{1}{2} p'_2(\chi), \qquad J_z = \frac{1}{2} p'_3(\chi). \tag{6}$$

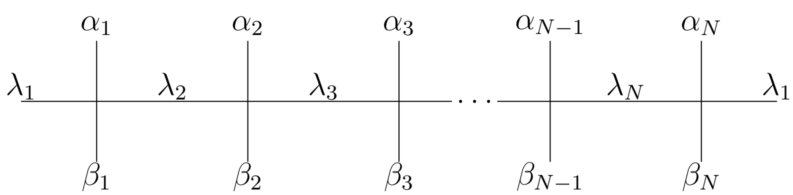
Since $d = 0$, we have $J_x = J_y$. Let H be the XXZ Hamiltonian (3).

Theorem 3.2. *With these notations, we have*

$$\mathcal{L} = H + cI_{\otimes^N \mathbb{C}^2},$$

where c is an explicit constant. The operator H commutes with the 6-vertex row transfer matrices T_θ.

Proof. We will label the interior edges of the single-layer grid whose partition function is T_θ by $\lambda_1, \cdots, \lambda_N$, thus:



Due to the periodic boundary conditions, $\lambda_{N+1} = \lambda_1$. Thus

$$T_\theta(\alpha, \beta) = \sum_\lambda \prod_{i=1}^N R(\theta)_{\lambda_i \alpha_i}^{\lambda_{i+1} \beta_i}.$$

Differentiating with respect to θ and setting $\theta = \chi$,

$$T'_\chi(\alpha, \beta) = \sum_{j=1}^N \sum_\lambda \left[\frac{d}{d\theta} R(\theta)_{\lambda_j \alpha_j}^{\lambda_{j+1} \beta_j} \right]_{\theta = \chi} \prod_{i \neq j} R(\chi)_{\lambda_i \alpha_i}^{\lambda_{i+1} \beta_i}.$$

By Lemma 3.1, if $i \neq j$ then $R(\chi)_{\lambda_i \alpha_j}^{\lambda_{i+1} \beta_i} = 1$ provided $\lambda_i = \beta_i$ and $\alpha_i = \lambda_{i+1}$, and is zero otherwise. Therefore the j-th term only contributes if $\beta_i = \alpha_{i-1}$ when $i \neq j, j+1$. Assuming this, since we are summing over λ, we may omit these factors and take $\lambda_j = \alpha_{j-1}$, $\lambda_{j+1} = \beta_{j+1}$ to obtain

$$T'_\chi(\alpha, \beta) = \sum_{j=1}^N \frac{d}{d\theta} R(\theta)_{\alpha_{j-1} \alpha_j}^{\beta_{j+1} \beta_j} |_{\theta = \chi} .$$

Now we substitute (5) to obtain

$$T'_\chi(\alpha,\beta) = \sum_{k=1}^{4} p'_k(\chi) \sum_{j=1}^{N} \sigma^k_{\alpha_j\beta_{j+1}} \sigma^k_{\alpha_{j-1}\beta_j}.$$

But now we remember that it is not T'_χ that we are trying to compute, but $\frac{1}{2}T_\chi^{-1}T'_\chi$, and T_χ^{-1} is the left-shift operator by Remark 1. Thus

$$\mathcal{L} = \frac{1}{2}\sum_{k=1}^{4} p'_k(\chi) \sum_{j=1}^{N} \sigma^k_{\alpha_j\beta_j} \sigma^k_{\alpha_{j-1}\beta_{j-1}}.$$

The first three terms produce the XXZ Hamiltonian, with $J_x = J_y = p'_1(\chi)$ and $J_z = p'_3(\chi)$. The last term produces $cI_{\otimes^N \mathbb{C}^2}$ with the constant $c = \frac{1}{2}Np'_4(\chi)$. $\square$

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