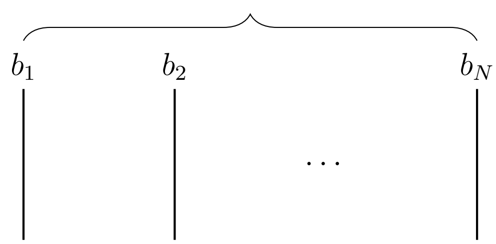
Lecture 16

1 Fusion

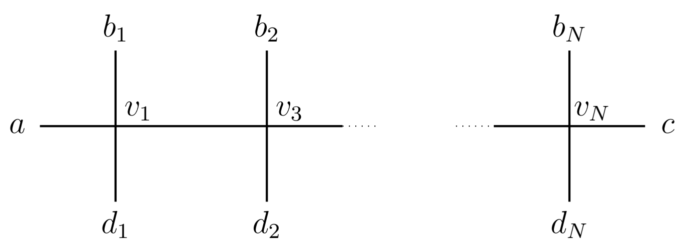
Roughly the fusion operation in lattice models corresponds to the tensor product of modules. But just as the tensor product has different applications in representation theory, so there are different kinds of thing that are called fusion.

Suppose that we have a sequence of vertices as follows. We assume that a sequence of vertical edges labeled $b_1, \cdots, b_N$ have spinsets $\Sigma_1, \cdots, \Sigma_N$.

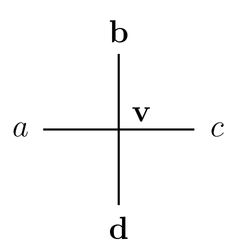


We may associate with these a single vertical edge with spinset $\Sigma_1 \times \cdots \times \Sigma_N$. Assigning spins $b_i \in \Sigma_i$ for $i = 1, \cdots, N$ is equivalent to assigning a spin $\mathbf{b} = (b_1, \cdots, b_N)$ to the fused edge.

If there are vertices on these edges, we may also fuse these into a single vertex. Thus:

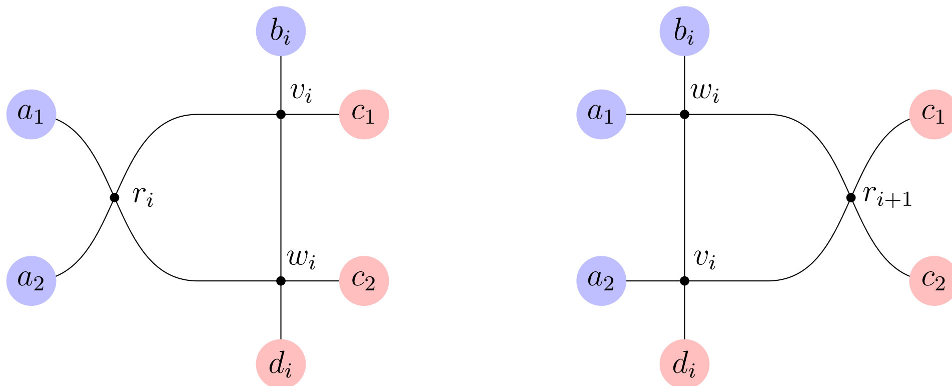


becomes:

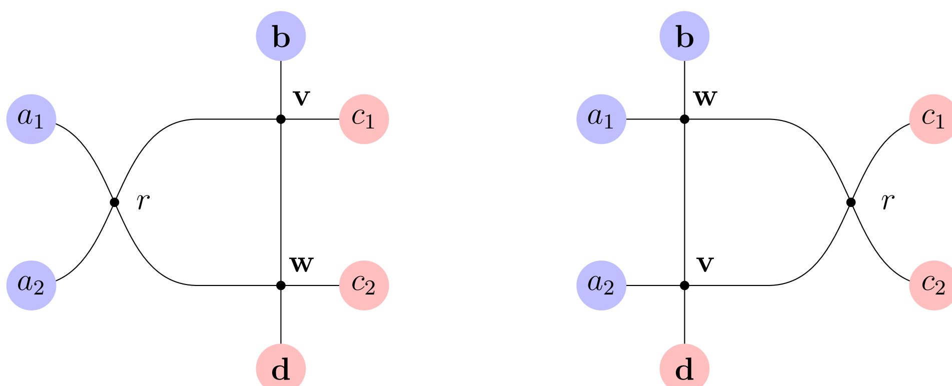


We will call the vertices $v_1, \cdots, v_N$ *unfused* and the vertex $\mathbf{v}$ *fused*.

Now we will encounter exotic versions of the Yang-Baxter equations in which the R-matrix changes when it moves past the vertex. If we are dealing with N unfused vertices $v_1, \cdots, v_N$ and $w_1, \cdots, w_N$, we may encounter a Yang-Baxter equation that looks like this:



Assuming the periodicity $r_{N+1} = r_1$, and denoting this vertex as just r, we obtain the usual kind of Yang-Baxter equation for the fused vertices:



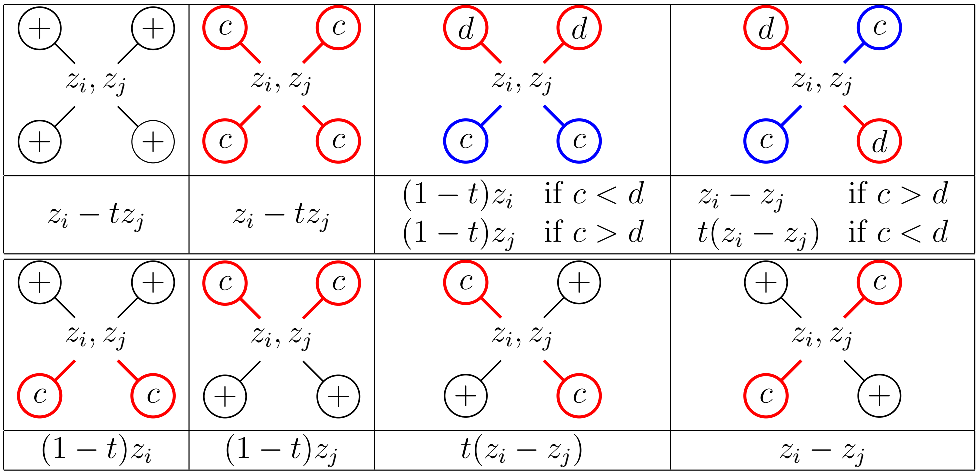
It is expected when this happens that the *fused* edges should have a quantum group interpretation.

2 Example

We know several examples of this factorization phenomenon. The ones we describe are in [3]. More general colored models in [1] do not factorize this way. The fermionic models in [2] also have such a factorization.

$c_1 > c_2 > \cdots > c_N$. The unfused vertical edges are *monochrome* in that each is only allowed to carry a single color. They are *bosonic* in that each vertical edge is allowed to carry multiple instances of its designated color. The partition functions are nonsymmetric

Hall-Littlewood polynomials. Before we describe the fused vertices, here is the R-matrix:



This tells us that the quantum group is $U_q(\widehat{\mathfrak{gl}}_{N+1})$ or equivalently (for this purpose) $U_q(\widehat{\mathfrak{sl}}_{N+1})$.

The horizontal edges in this model are only allowed to carry one color c_i, or no color, designated $+$. Thus the spinset of the horizontal edges is $\{c_1, \cdots, c_N, +\}$.

The states of the fused vertical edges can be described in terms of N bosons, one of each color. The spinset of the fused vertical edges is thus $\mathbb{N}^N$, where where $\mathbb{N} = \{0, 1, 2, \cdots\}$. If $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{N}^N$, we think of this as a state in which the edge carries k_i bosons of color c_i.

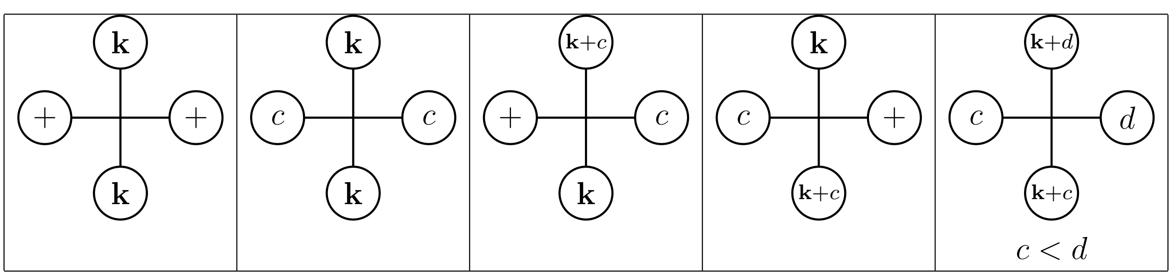
To describe the admissible states, let us introduce this notation. Let $\mathbf{k} \in \mathbb{N}^N$, which is the vertical edge spinset. By $\mathbf{k} + c_i$ we mean

$$(k_1, k_2, \cdots, k_i + 1, \cdots k_N).$$

Here the k_i component, which is interpreted as the number of bosons of color c_i, is increased by 1. Similarly

$$\mathbf{k} - c_i = (k_1, k_2, \cdots, k_i - 1, \cdots k_N).$$

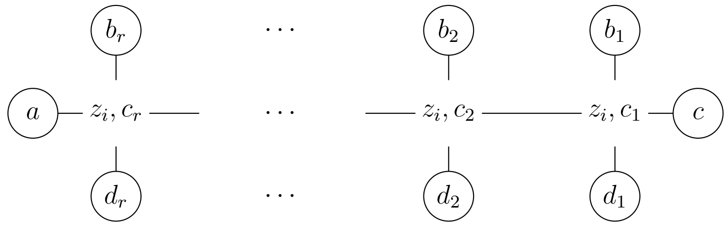
Here are the admissible states.



The striking thing to note here is that the last state is only allowed if $c < d$. More general models in [1] do not have this property.

Now the vertical edges can be obtained by fusion according to the following scheme. We have weight labeled by a spectral parameter z and a color c We fuse the vertices in order

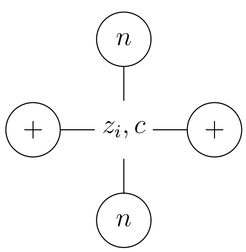
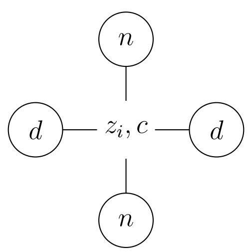
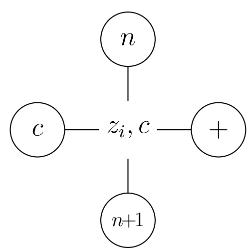
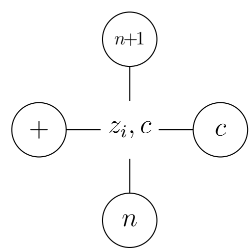
$c_N, c_{N-1}, \cdots, c_1$:



The vertical edges are also labeled by the colors c_i. *The vertical edge labeled c_i is only allowed to carry that color* and no others. For this reason, we call the vertex and color labeled c_i *monochrome.*

Remark 1. From this, we can see why we the last state is is forbidden if $c > d$. The reason is that if $c > d$, the horizontal edges between the c column, which is to the left of the d column, would have to carry both colors, and a horizontal edge is only allowed to carry one color.

Here are the weights of the monochrome vertices:

$A(n)$	$B(n)$	$C(n)$	$D(n)$
			
1	$1 \quad c < d$ $z_i \quad c = d$ $t^n \quad c > d$	$z_i(1 - t^{n+1})$	1

The auxiliary R-matrix depends on the color of the monochrome edge to the right of the vertex:

$\begin{matrix} + & & + \\ & z_i, z_j, c & \\ + & & + \end{matrix}$	$\begin{matrix} d & & d \\ & z_i, z_j, c & \\ d & & d \end{matrix}$	$\begin{matrix} e & & d \\ & z_i, z_j, c & \\ d & & e \end{matrix}$	$\begin{matrix} e & & e \\ & z_i, z_j, c & \\ d & & d \end{matrix}$
$z_i - tz_j$	$z_i - tz_j$	$t(z_i - z_j)$ $e > d$ $z_i - z_j$ $d > e$	$(1-t)z_j$ $e > d > c$ $c > d > e$ $d > e > c$ $(1 - tz_i)$ $d > c > e$ $c > e > d$ $e > d > c$
$\begin{matrix} c & & c \\ & z_i, z_j, c & \\ d & & d \end{matrix}$	$\begin{matrix} d & & d \\ & z_i, z_j, c & \\ c & & c \end{matrix}$	$\begin{matrix} d & & + \\ & z_i, z_j, c & \\ + & & d \end{matrix}$	$\begin{matrix} + & & d \\ & z_i, z_j, c & \\ d & & + \end{matrix}$
$(1-t)z_j$	$(1-t)z_i$	$t(z_i - z_j)$	$z_i - z_j$
$\begin{matrix} + & & + \\ & z_i, z_j, c & \\ d & & d \end{matrix}$	$\begin{matrix} d & & d \\ & z_i, z_j, c & \\ + & & + \end{matrix}$		
$(1-t)z_i$	$(1-t)z_j$		

We do not show the Yang-Baxter equation, but as in the last section, it changes when the R-matrix moves past the vertex. After moving past all the vertices, the R-matrix is restored to its original state. See [3] for further information.

3 Explanation in terms of Verma modules

The claims in this section are undoubtedly true but haven't been verified.

The Lie algebra $\mathfrak{g} = \mathfrak{gl}_{N+1}$ has a parabolic subalgebra $\mathfrak{p}$ that is the semidirect product of $\mathfrak{m} = \mathfrak{gl}_N \oplus \mathfrak{gl}_1$ with the nilpotent subalgebra $\mathfrak{u}_+$ supported on the last column. If $N = 3$:

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}, \qquad \mathfrak{m} = \left\{ \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \qquad \mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Let $\mathfrak{u}_-$ be the complementary nilpotent subalgebra that is the transpose of $\mathfrak{u}$ in matrix form. We then have $\mathfrak{gl}_{N+1} = \mathfrak{u}_- \oplus \mathfrak{p}$. By the PBW theorem,

$$U(\mathfrak{gl}_{N+1}) \cong U(\mathfrak{u}_-) \otimes_{\mathbb{C}} U(\mathfrak{p}).$$

Let us take any one-dimensional representation ψ of $\mathfrak{p}$, afforded by the module $\mathbb{C}_\psi$. The induced module $V_\psi = U(\mathfrak{gl}_{N+1}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\psi$ is then isomorphic to $U(\mathfrak{u}_-)$ as a vector space. Since $\mathfrak{u}_-$ is abelian, the enveloping algebra $U(\mathfrak{u}_-)$ is just the symmetric algebra $\mathrm{Sym}(\mathfrak{u}_-)$.

If $\beta \in \Phi$, we are regarding β as an element of $\mathfrak{h}^*$, where $\mathfrak{h}$ is the diagonal Cartan subalgebra. There is a unique (up to scalar) element $X_\beta \in \mathfrak{g}$ such that

$$[H, X_\beta] = \beta(H) X_\beta.$$

There are N negative roots $\beta_1, \cdots, \beta_N$ such that $X_{\beta_i} \in \mathfrak{u}_-$. We order these so that X_{β_i} has its nonzero entry in the $N+1-i$ column. Thus for $N=3$:

$$X_{\beta_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_{\beta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{\beta_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $U(\mathfrak{u}_-) = \mathbb{C}[X_{\beta_1}, \cdots, X_{\beta_N}]$ is a polynomial ring.

Conjecture 3.1. *The quantized version of V_ψ for suitable ψ is the $U_q(\widehat{\mathfrak{gl}}_{N+1})$-module associated with the vertical edges in the model described in Section 2. The more general models of [1], the module would be a Borel Verma module as in Lecture 14.*

For this it is likely important that the nilpotent subalgebra $\mathfrak{u}_-$ is abelian.

How should we view the monochrome edges and vertices? Only the fused edges correspond to $U_q(\widehat{\mathfrak{sl}}_{N+1})$ modules. However the constitutent unfused edges, each of which can carry only one color, resembles the $U_q(\widehat{\mathfrak{sl}}_2)$ vertex.

So there is an embedding of $\mathfrak{sl}_2 \longrightarrow \mathfrak{gl}_{N+1}$ along the positive root $-\beta_i$, namely

$$\langle X_{-\beta}, X_\beta \rangle \cong \mathfrak{sl}_2. \tag{1}$$

And the Verma module

$$V_\psi \cong U(\mathfrak{u}_-) \cong \bigotimes_{i=1}^{N} U(\mathbb{C}X_{\beta_i}).$$

Each factor $U(\mathbb{C}X_{\beta_i})$ is an $\mathfrak{sl}_2$ Verma module, for the copy (1) of $\mathfrak{sl}_2$. The last isomorphism shows that the $\mathfrak{sl}_{N+1}$ parabolic Verma module is a tensor product of these N $\mathfrak{sl}_2$ Verma modules, and this fact is reflected in the factorization of the edges into monochrome edges.

References

[1] A. Borodin and M. Wheeler. Colored stochastic vertex models and their spectral theory. *Astérisque*, (437):ix+225, 2022, arXiv:1808.01866.

[2] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Colored vertex models and Iwahori Whittaker functions, 2019, arXiv:1906.04140.

[3] D. Bump and S. Naprienko. Colored bosonic models and matrix coefficients, 2022, arXiv:2211.15850.