Lecture 15

1 Bosonic Models

Let us return to the bosonic models in Lecture 8. The R-matrix tells us that the quantum group is $U_q(\mathfrak{sl}_2)$ or $U_q(\mathfrak{gl}_2)$. The horizontal edges in the model correspond to 2-dimensional modules $V_z$ where $z \in \mathbb{C}^\times$.

The vertical edges, however, have spinset $N$. From the point of view of Kulish [4], where the bosonic models first appeared, these spins correspond to the energy levels of the quantum mechanical harmonic oscillator, or rather, a $q$-deformation of that. But from the point of view we are taking, these edges should correspond to a module of $U_q(\mathfrak{sl}_2)$. This quantum group module is a Verma module.

We will not discuss Verma modules for quantized enveloping algebras, but at least we will look at Verma modules for $U(g)$. The theory is standard. The books [2] and [3] Chapter 9 are good references.

To get R-matrices out of Verma modules, one must extend this theory to $U_q(g)$. For this, see [5]. A paper where quantum Verma modules are used to compute R-matrices is [1].

2 Verma modules continued

We continue from Lecture 14, where we introduced the PBW theorem. We will review a few ideas about highest weight modules and the BGG Category $O$. See [3] Chapter 9 for more information about these topics.

We make use of the tensor product for noncommutative rings. This is a topic omitted in Lang’s Algebra but as a reference see Mac Lane’s Homology, Section 5.1. If $R$ is a noncommutative ring, and $M$ is a right $R$-module and $N$ is a left $R$-module, and if $T$ is an abelian group, a map $\beta : M \times N \rightarrow T$ is called balanced if it is $\mathbb{Z}$-bilinear and $\beta(ma, n) = \beta(m, an)$ for $a \in A, m \in M$ and $n \in N$. Then $M \otimes_A N$ is defined to be an abelian group with a balanced map $\otimes : M \times N \rightarrow M \otimes_A N$ such that any balanced map $\beta : M \times N \rightarrow T$ factors uniquely through $M \otimes_A N$. We naturally write $m \otimes n$ instead of $\otimes(m, n)$.

There is no natural way to make $M \otimes N$ to an $A$-module. However a common special case is where $M$ is a bi-module. If $B$ and $A$ are rings, a $(B,A)$-bimodule is an $M$ that is simultaneously a left $B$-module and a right $A$-module, such that these actions commute: $b(ma) = (bm)a$ for $m \in M, b \in B$ and $a \in A$. In this case, if $N$ is a left $A$-module then $M \otimes N$ becomes a left $B$-module.
As an example, suppose that $A$ is a ring and $B$ a ring containing $A$. Then $B$ is a left $B$-module and a right $A$-module, so it is a bimodule and

$$ N \mapsto B \otimes_A N $$

is a functor from the category of left $A$-modules to left $B$-modules. This functor is called extension of scalars.

We now return to the setting at the end of Lecture 14. Let $\mathfrak{g}$ be a simple complex Lie algebra such as $\mathfrak{sl}_n$. We saw that it has a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The Cartan subalgebra $\mathfrak{h}$ is abelian, so any simple $\mathfrak{h}$-module is one-dimensional.

In the Lie algebra setting, weights are elements of $\mathfrak{h}^*$, which we equip with a $W$-invariant inner product. The root system $\Phi$ can then be characterized as the set of nonzero $\alpha \in \mathfrak{h}^*$ such that

$$ \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} | [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{h} \} $$

is nonzero. In this case $\mathfrak{g}_\alpha$ is one-dimensional. Let $X_\alpha$ be a generator. For the simple roots $\alpha_1, \ldots, \alpha_r$ we denote $X_{\alpha_i} = E_i$ and $X_{-\alpha_i} = F_i$.

If $V$ is any module, and $\mu \in \mathfrak{h}^*$ let

$$ V_{\mu} = \{ v \in V | Hv = \mu(H)v \text{ for all } H \in \mathfrak{h} \} $$

be the corresponding weight space. We will always assume that $V$ is the direct sum of its weight spaces.

The Lie algebra $\mathfrak{g}$ is itself a $\mathfrak{g}$-module with respect to the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) = \text{End}_C(\mathfrak{g})$, where $\text{ad}(X)$ is the endomorphism $\text{ad}(X)Y = [X,Y]$. Then the roots are just the nonzero weights in the adjoint representation, and the definition (1) is seen to be a special case of the definition (2).

**Lemma 2.1.** We have $X_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$.

**Proof.** If $H \in \mathfrak{h}$ and $v \in V_\lambda$ then

$$ HX_\alpha v = [H, X_\alpha]v + X_\alpha Hv = \alpha(H)X_\alpha v + X_\alpha \lambda(H)v $$

$$ = (\alpha + \lambda)(H)X_\alpha v. $$

We will call elements of $\mathfrak{h}$ such that $V_\mu \neq 0$ the weights of the representation. A weight $\lambda$ is integral if

$$ \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} $$

for all $\alpha \in \Phi$. The set of integral weights is the weight lattice $\Lambda$. If $\mathfrak{g}$ is the Lie algebra of a simply-connected complex Lie group $G$, this weight lattice can be identified with the weight lattice of $G$.

**Definition 1.** Let $V$ be a module. A vector $v \in V$ is a highest weight vector with weight $\lambda \in \mathfrak{h}^*$ if $v \in V_\lambda$ and $\mathfrak{n}_+ v = 0$. If $V$ is generated by $v$, then $V$ is called a highest weight module for the weight $\lambda$. 

2
For example, if \( V \) is a finite-dimensional irreducible representation, then by the Weyl theory \( V \) has a highest weight vector that is up to scalar multiple for a unique \( \lambda \), which is a dominant integral weight.

**Lemma 2.2.** If \( V \) is a highest weight module for \( \lambda \), then \( V = U(n_-)v \).

**Proof.** By the PBW theorem we have

\[
U(g) = U(n_-)U(b).
\]

Although we do not need this fact, the PBW theorem actually implies that the multiplication map \( U(n_-) \times U(b) \rightarrow U(g) \) induces a vector space isomorphism \( U(n_-) \otimes U(b) \rightarrow U(g) \). Then \( V = U(n_-)U(b)v \) and we can discard the \( U(b) \) since clearly \( U(b)v = v \).

**Theorem 2.3.** Let \( \lambda \in h^* \). Then \( g \) has a universal highest weight module \( M(\lambda) \), with a highest weight vector \( m_\lambda \), such that if \( V \) is any module and \( v \in V \) is a highest weight vector with weight \( \lambda \), then there is a unique homomorphism \( M(\lambda) \rightarrow V \) taking \( m_\lambda \) to \( v \).

**Proof.** Let \( \mathbb{C}_\lambda \) be the \( \mathbb{C} \) equipped with the \( h \)-module structure affording the character \( \lambda \). We can extend this character of \( b = h \oplus n_+ \) by letting \( n_+ \) act by zero. This gives us a \( U(b) \)-module. Now let

\[
M(\lambda) = U(g) \otimes_{U(b)} \mathbb{C}_\lambda.
\]

It is easy to see that \( M(\lambda) \) is a highest weight module with \( m_\lambda = 1_{U(g)} \otimes 1_{\mathbb{C}_\lambda} \). To check the universal property, note that the map \( \beta : U(g) \otimes \mathbb{C}_\lambda \rightarrow V \) defined by \( \beta(\xi \otimes a) = \xi a v \) is balanced, hence induces a unique map \( U(g) \otimes_{U(b)} \mathbb{C}_\lambda \rightarrow V \). This is the unique homomorphism.

If \( \lambda \in h^* \) let \( e^\lambda \) be a formal symbol such that \( e^\lambda e^\mu = e^{\lambda + \mu} \). In this setting the “exponential” \( e^\lambda \) is just a formal device for writing the weight lattice multiplicatively. The character of a module \( V \) is

\[
\chi_V = \sum_{\mu \in h^*} \dim(V_\mu) e^\mu.
\]

**Proposition 2.4.** Let \( \lambda \in h^* \). Then \( \xi \mapsto \xi m_\lambda \) is a vector space isomorphism \( U(n_-) \rightarrow M(\lambda) \). The character of \( M(\lambda) \) is

\[
e^\lambda \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.
\]

It is understood that we expand the geometric series and collect the terms:

\[
\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1} = \prod_{\alpha \in \Phi^+} \sum_{k_\alpha = 0}^{\infty} e^{-\sum k_\alpha \alpha} = \sum_{\mu} \varphi(\mu) e^{-\mu}
\]

where \( \varphi(\mu) \) is the number of ways of writing \( \mu = \sum_{\alpha \in \Phi^+} k_\alpha \alpha \) for some vector \( (k_\alpha | \alpha \in \Phi^+) \) of nonnegative integers. The function \( \varphi \) is called the Kostant partition function.

*https://en.wikipedia.org/wiki/Kostant_partition_function*
Proof. This is a stronger statement than Lemma 2.2, which asserts that the map $\xi \mapsto \xi m_\lambda$ is surjective $U(n_-) \rightarrow M(\lambda)$. For this, standard isomorphisms give

$$M(\lambda) = U(n_-) \otimes_C U(b) \otimes_{U(b)} C_{\lambda} \cong U(n_-) \otimes_C C_{\lambda} \cong U(n_-)$$

as a vector space.

We want to show that the character of $U(n_-)$ as an $\mathfrak{h}$-module is (3). By the PBW Theorem a basis of $U(n_-)$ consists of elements of the form

$$\prod_{\alpha \in \Phi^+} X_{-\alpha}^{k_{\alpha}} \in U(n_-)$$

and the weight of this is $-\sum k_{\alpha} \alpha$. \qed

References


