Lecture 15

1 Bosonic Models

Let us return to the bosonic models in Lecture 8. The R-matrix tells us that the quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$ or $U_q(\widehat{\mathfrak{gl}}_2)$. The horizontal edges in the model correspond to 2-dimensional modules $V_{\mathbf{z}}$ where $\mathbf{z} \in \mathbb{C}^\times$.

The vertical edges, however, have spinset $\mathbb{N}$. From the point of view of Kulish [4], where the bosonic models first appeared, these spins correspond to the energy levels of the quantum mechanical harmonic oscillator, or rather, a q-deformation of that. But from the point of view we are taking, these edges should correspond to a module of $U_q(\widehat{\mathfrak{sl}}_2)$. This quantum group module is a Verma module.

We will not discuss Verma modules for quantized enveloping algebras, but at least we will look at Verma modules for $U(\mathfrak{g})$. The theory is standard. The books [2] and [3] Chapter 9 are good references.

To get R-matrices out of Verma modules, one must extend this theory to $U_q(\mathfrak{g})$. For this, see [5]. A paper where quantum Verma modules are used to compute R-matrices is [1].

2 Verma modules continued

We continue from Lecture 14, where we introduced the PBW theorem. We will review a few ideas about highest weight modules and the BGG Category $\mathcal{O}$. See [3] Chapter 9 for more information about these topics.

We make use of the tensor product for noncommutative rings. This is a topic omitted in Lang's *Algebra* but as a reference see Mac Lane's *Homology*, Section 5.1. If R is a noncommutative ring, and M is a *right* R-module and N is a *left* R-module, and if T is an abelian group, a map $\beta : M \times N \longrightarrow T$ is called *balanced* if it is $\mathbb{Z}$-bilinear and $\beta(ma, n) = \beta(m, an)$ for $a \in A$, $m \in M$ and $n \in N$. Then $M \otimes_A N$ is defined to be an abelian group with a balanced map $\otimes : M \times N \longrightarrow M \otimes_A N$ such that any balanced map $\beta : M \times N \longrightarrow T$ factors uniquely through $M \otimes_A N$. We naturally write $m \otimes n$ instead of $\otimes(m, n)$.

There is no natural way to make $M \otimes N$ to an A-module. However a common special case is where M is a bi-module. If B and A are rings, a (B, A)-*bimodule* is an M that is simultaneously a left B-module and a right A-module, such that these actions commute: $b(ma) = (bm)a$ for $m \in M$, $b \in B$ and $a \in A$. In this case, if N is a left A-module then $M \otimes N$ becomes a left B-module.

As an example, suppose that A is a ring and B a ring containing A. Then B is a left B-module and a right A-module, so it is a bimodule and

$$N \mapsto B \otimes_A N$$

is a functor from the category of left A-modules to left B-modules. This functor is called *extension of scalars*.

We now return to the setting at the end of Lecture 14. Let $\mathfrak{g}$ be a simple complex Lie algebra such as $\mathfrak{sl}_n$. We saw that it has a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The Cartan subalgebra $\mathfrak{h}$ is abelian, so any simple $\mathfrak{h}$-module is one-dimensional.

In the Lie algebra setting, weights are elements of $\mathfrak{h}^*$, which we equip with a W-invariant inner product. The root system Φ can then be characterized as the set of nonzero $\alpha \in \mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{h}\} \tag{1}$$

is nonzero. In this case $\mathfrak{g}_\alpha$ is one-dimensional. Let X_α be a generator. For the simple roots $\alpha_1, \cdots, \alpha_r$ we denote $X_{\alpha_i} = E_i$ and $X_{-\alpha_i} = F_i$.

If V is any module, and $\mu \in \mathfrak{h}^*$ let

$$V_\mu = \{v \in V | Hv = \mu(H)v \text{ for all } H \in \mathfrak{h}\} \tag{2}$$

be the corresponding weight space. We will always assume that V is the direct sum of its weight spaces.

The Lie algebra $\mathfrak{g}$ is itself a $\mathfrak{g}$-module with respect to the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$, where $\mathrm{ad}(X)$ is the endomorphism $\mathrm{ad}(X)Y = [X, Y]$. Then the roots are just the nonzero weights in the adjoint representation, and the definition (1) is seen to be a special case of the definition (2).

Lemma 2.1. *We have $X_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$.*

Proof. If $H \in \mathfrak{h}$ and $v \in V_\lambda$ then

$$\begin{aligned} HX_\alpha v = [H, X_\alpha]v + X_\alpha Hv &= \alpha(H)X_\alpha v + X_\alpha \lambda(H)v \\ &= (\alpha + \lambda)(H)X_\alpha v. \end{aligned}$$

□

We will call elements of $\mathfrak{h}$ such that $V_\mu \neq 0$ the *weights* of the representation. A weight λ is *integral* if

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

for all $\alpha \in \Phi$. The set of integral weights is the *weight lattice* Λ. If $\mathfrak{g}$ is the Lie algebra of a simply-connected complex Lie group G, this weight lattice can be identified with the weight lattice of G.

Definition 1. Let V be a module. A vector $v \in V$ is a *highest weight vector* with weight $\lambda \in \mathfrak{h}^*$ if $v \in V_\lambda$ and $\mathfrak{n}_+ v = 0$. If V is generated by v, then V is called a *highest weight module* for the weight λ.

For example, if V is a finite-dimensional irreducible representation, then by the Weyl theory V has a highest weight vector that is up to scalar multiple for a unique λ, which is a dominant integral weight.

Lemma 2.2. *If V is a highest weight module for λ, then $V = U(\mathfrak{n}_-)v$.*

Proof. By the PBW theorem we have

$$U(\mathfrak{g}) = U(\mathfrak{n}_-)U(\mathfrak{b}).$$

Although we do not need this fact, the PBW theorem actually implies that the multiplication map $U(\mathfrak{n}_-) \times U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$ induces a vector space isomorphism $U(\mathfrak{n}_-) \otimes U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$. Then $V = U(\mathfrak{n}_-)U(\mathfrak{b})v$ and we can discard the $U(\mathfrak{b})$ since clearly $U(\mathfrak{b})v = v$. $\square$

Theorem 2.3. *Let $\lambda \in \mathfrak{h}^*$. Then $\mathfrak{g}$ has a universal highest weight module $M(\lambda)$, with a highest weight vector m_λ, such that if V is any module and $v \in V$ is a highest weight vector with weight λ, then there is a unique homomorphism $M(\lambda) \longrightarrow V$ taking m_λ to v.*

Proof. Let $\mathbb{C}_\lambda$ be the $\mathbb{C}$ equipped with the $\mathfrak{h}$-module structure affording the character λ. We can extend this character of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero. This gives us a $U(\mathfrak{b})$-module. Now let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

It is easy to see that $M(\lambda)$ is a highest weight module with $m_\lambda = 1_{U(\mathfrak{g})} \otimes 1_{\mathbb{C}_\lambda}$. To check the universal property, note that the map $\beta : U(\mathfrak{g}) \times \mathbb{C}_\lambda \longrightarrow V$ defined by $\beta(\xi \otimes a) = \xi\, a\, v$ is balanced, hence induces a unique map $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \longrightarrow V$. This is the unique homomorphism. $\square$

If $\lambda \in \mathfrak{h}^*$ let e^λ be a formal symbol such that $e^\lambda e^\mu = e^{\lambda+\mu}$. In this setting the "exponential" e^λ is just a formal device for writing the weight lattice multiplicatively. The *character* of a module V is

$$\chi_V = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, e^\mu.$$

Proposition 2.4. *Let $\lambda \in \mathfrak{h}^*$. Then $\xi \mapsto \xi m_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$. The character of $M(\lambda)$ is*

$$e^\lambda \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.$$

It is understood that we expand the geometric series and collect the terms:

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1} = \prod_{\alpha \in \Phi^+} \sum_{k_\alpha = 0}^{\infty} e^{-\sum k_\alpha \alpha} = \sum_\mu \wp(\mu) e^{-\mu} \tag{3}$$

where $\wp(\mu)$ is the number of ways of writing $\mu = \sum_{\alpha \in \Phi^+} k_\alpha \alpha$ for some vector $(k_\alpha | \alpha \in \Phi^+)$ of nonnegative integers. The function $\wp$ is called the *Kostant partition function.*

- `https://en.wikipedia.org/wiki/Kostant_partition_function`

Proof. This is a stronger statement than Lemma 2.2, which asserts that the map $\xi \mapsto \xi m_\lambda$ is surjective $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$. For this, standard isomorphisms give

$$M(\lambda) = U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda \cong U(\mathfrak{n}_-)$$

as a vector space.

We want to show that the character of $U(\mathfrak{n}_-)$ as an $\mathfrak{h}$-module is (3). By the PBW Theorem a basis of $U(\mathfrak{n}_-)$ consists of elements of the form

$$\prod_{\alpha \in \Phi^+} X_{-\alpha}^{k_\alpha} \in U(\mathfrak{n}_-)$$

and the weight of this is $-\sum k_\alpha \alpha$. $\square$

References

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[2] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

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[4] P. P. Kulish. Contraction of quantum algebras, and q-oscillators. *Teoret. Mat. Fiz.*, 86(1):157–160, 1991.

[5] K. S. Nirov and A. V. Razumov. Quantum groups, Verma modules and q-oscillators: general linear case. *J. Phys. A*, 50(30):305201, 19, 2017.