Lecture 14

1 A survey of some possibilities

The spinsets of lattice models that we care about usually correspond to modules of various quantum groups. It is not necessary to use the quantum group theory such as the universal R-matrix to compute the R-matrices, since this can be done by other methods. (Using a computer is sometimes useful.) However knowing that the edges of the grid correspond to modules of a quantum group seems an important point, and understanding this fact has predictive power.

Our goal is to survey some of the various spinsets that we encounter and explain how these are related to various modules of particular quantum groups. Here is an overview of what we want to cover. We will cover none of these topics in any depth.

- We have encountered several examples of parametrized Yang-Baxter equations with parameter $\mathbb{C}^\times$. Although we were able to obtain such a parametrized Yang-Baxter equation from $U_q(\mathfrak{sl}_2)$ in Lecture 13, it is better to regard this as the R-matrix for the *affine* quantum group $U_q(\widehat{\mathfrak{sl}}_2)$.

- Affine Lie algebras come with Weyl groups, which are Coxeter groups that are also associated with Hecke algebras. We have seen that the Hecke algebra of $\mathfrak{gl}_n$ acts on partition functions of colored models, and these actions can be extended to the affine Hecke algebra.

- We have briefly discussed bosonic models, in which the spinsets are infinite. These often correspond to *Verma modules*, which are usually infinite-dimensional.

- In addition to quantized enveloping algebras of Lie algebras, we encounter enveloping algebras of Lie superalgebras such as $\mathfrak{gl}(m|n)$. So we want to touch on this topic.

- Lie superalgebras have a special kind of Verma module called *Kac modules* that are finite-dimensional. We believe these to be important for this topic. For example, $U_q(\mathfrak{gl}(m|n))$ has Kac modules that have dimension 2^{mn}. In particular the Kac modules for $U_q(\mathfrak{gl}(1|1))$ are 2-dimensional modules that differ from the 2-dimensional standard modules. These account for the vertical edges in the Tokuyama models.

In this and the next lectures we will briefly introduce each of these topics.

2 Affine Lie algebras

We have seen that quantum groups are sources of solutions to the Yang-Baxter equation. From the quantum group $U_q(\mathfrak{sl}_2)$ we obtained two R-matrices R' and R'' and obtained a parametrized Yang-Baxter equation by taking a linear combination of these. This is an *ad hoc* procedure that works for $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, but which would require modification for other Cartan types.

An alternative, better approach is to work with the (untwisted) *affine* Lie algebra $\hat{\mathfrak{g}}$, for any complex reductive Lie algebra $\mathfrak{g}$. Since appear in a great deal of mathematics, it is worth digressing to introduce them. They are special cases of *Kac-Moody* Lie algebras, for which the standard work is [7].

Affine Lie algebras and more general Kac-Moody Lie algebras were only discovered as recently as the 1970's. Now however they are everywhere. For us, they underlie the most important parametrized Yang-Baxter equations we have seen, and so we will spend a lecture on them.

Kac-Moody Lie algebras have much in common with simple complex Lie algebras. They have a Weyl group, a weight lattice, and for the integrable highest weight representations, an analog of the Weyl character formula. The characters of affine Lie algebras turn out to be modular forms.

Like the finite-dimensional simple Lie algebras, affine Lie algebras are a very special case of the more general Kac-Moody Lie algebras, and it is worth while treating them separately. Every finite dimensional simple Lie algebra $\mathfrak{g}$ gives rise to an (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$.

This has two different descriptions. First, it can be described by generators and relations. But an alternative description in Chapter 6 of [7] shows the relationship between $\mathfrak{g}$ and $\hat{\mathfrak{g}}$.

If $\mathfrak{g}$ is a Lie algebra and A is an associative algebra then $A \otimes \mathfrak{g}$ is naturally a Lie algebra with bracket

$$[a \otimes X, b \otimes Y] = ab \otimes [X, Y].$$

So we may construct the Lie algebra

$$\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g},$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring. If $\mathfrak{g}$ is simple this has a central extension by a one-dimensional abelian Lie algebra spanned by K. Thus we have a Lie algebra $\hat{\mathfrak{g}}'$ with a short-exact sequence

$$0 \longrightarrow \mathbb{C} \cdot K \longrightarrow \hat{\mathfrak{g}}' \longrightarrow \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \longrightarrow 0.$$

It is possible to enlarge this one more time by adjoining a derivation d of $\hat{\mathfrak{g}}'$, so that

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C}d.$$

The main subtlety is in constructing a cocycle that produces the central extension $\hat{\mathfrak{g}}'$. The difference between $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{g}}$ is important, but we can ignore it for our purposes.

If V is any irreducible $\hat{\mathfrak{g}}$-module, then by Schur's Lemma the central element K acts by a scalar. The algebra $\hat{\mathfrak{g}}$ has an important family of infinite-dimensional representations, the integrable highest weight representations, in which K acts by a nonzero scalar. As far as I know these have not been used in lattice models but maybe they should be. One particular integrable highest weight representation, called the *basic representation* is of particular importance, showing up in diverse places such as string theory and the modular representations of the symmetric group.

If V is any $\mathfrak{g}$-module, and if $z \in \mathbb{C}^\times$, then V becomes a $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ module in which t acts by the scalar z. We can then pull this back to $\hat{\mathfrak{g}}'$ and obtain a family of modules V_z in which K acts by zero. At least when $\mathfrak{g}$ is a classical group, the R-matrices for these were computed by Jimbo [6]. For $\widehat{\mathfrak{sl}}_2$, this gives the parametrized R-matrices that were computed in Lecture 13.

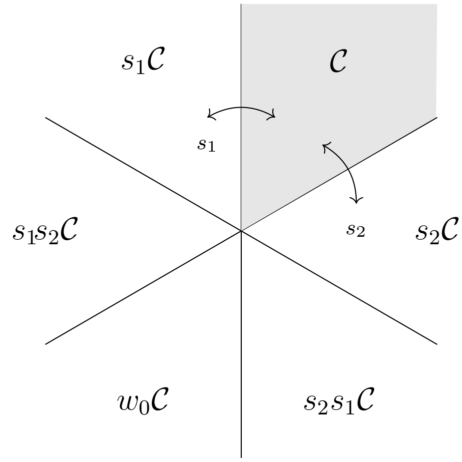
3 Affine Weyl group

The affine Lie algebra $\hat{\mathfrak{g}}$ has a Weyl group W_{aff} that is a Coxeter group. As before, let $\mathfrak{g}$ be a complex simple Lie algebra, with Weyl group W, root lattice Λ and root system Φ. The weight lattice can be embedded in a Euclidean space, that is, a real vector space V with a positive definite inner product that is W-invariant. The lattice Λ_{root} is of finite index in Λ. The weight lattice Λ can be characterized as

$$\left\{ \lambda \in V | \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for } \alpha \in \Phi \right\}.$$

Let α_i be the simple positive roots, and $s_i \in W$ the corresponding simple reflections. If $\alpha \in \Phi$, let H_α be the hyperplane through the origin orthogonal to α. The set $V - \bigcup H_\alpha$ is disconnected, and the connected components are called *Weyl chambers*. One particular one is $\mathcal{C}_+^\circ = \{x \in V | \langle x, \alpha \rangle > 0 \text{ for } \alpha \in \Phi^+\} = \{x \in V | \langle x, \alpha \rangle > 0 \text{ for } \alpha \in \Phi^+\}$. Let $\mathcal{C}_+$ be the closure of $\mathcal{C}_+^\circ$. It is a fundamental domain for the action of W on V, in that every orbit of W intersects $\mathcal{C}_+$ in a unique point.

Then W can be defined as the group generated by the reflections in the hyperplanes H_α ($\alpha \in \Phi$). The simple reflection s_i is the reflection in H_{α_i}. The group W is actually generated by the subset $\{s_1, \cdots, s_r\}$ generated by the simple reflections. The hyperplanes H_{α_i} are just the walls of $\mathcal{C}_+$. For $\mathfrak{g} = \mathfrak{sl}_3$, here is a picture of the six Weyl chambers, with $\mathcal{C}_+$ shaded.



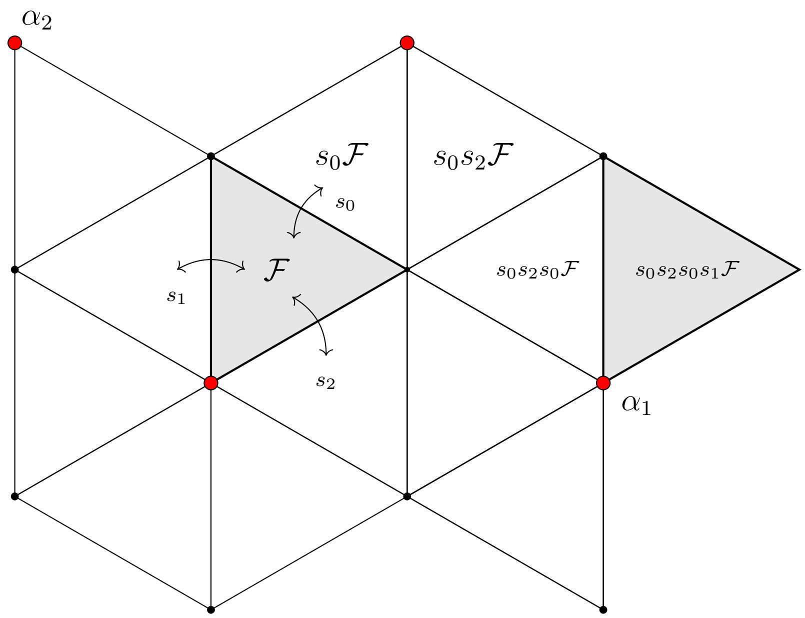
We now turn to W_{aff}.

If $k \in \mathbb{Z}$ let $H_{\alpha,k} = \{x \in V | \langle x, \alpha \rangle = k\}$. Again we may consider the complement of $\bigcup H_{\alpha,k}$. The closure of one connected component of this complement is called an *alcove*. In particular

$$\mathcal{F} = \{x \in V | \langle x, \alpha_i \rangle \geqslant 0, \langle x, \alpha_\ell \rangle \leqslant 1\},$$

where α_ℓ is the highest root is called the *fundamental alcove*. The *affine reflection* s_0 is the reflection in the hyperplane $H_{\alpha_\ell,1}$.

For $\mathfrak{g} = \mathfrak{sl}_3$, $\alpha_\ell = \alpha_1 + \alpha_2$. Here is a picture showing some of the alcoves.



This figure also shows weight lattice (small black dots at the corners of the alcove) and some elements of the root lattice (larger red dots).

The group W_{aff} can be defined as the group generated by the reflections in the hyperplanes $H_{\alpha,k}$. But actually it is generated by $\langle s_0, s_1, \cdots, s_r \rangle$, and it is a Coxeter group with these generators.

The group W_{aff} contains the subgroup Λ_{root} of translations by elements of the root lattice. In the above picture, it is shown that $s_0 s_2 s_0 s_1$ takes the fundamental alcove $\mathcal{F}$ into $\mathcal{F} + \alpha_1$. Indeed there is an isomorphism Θ of Λ_{root} into W, and W_{aff} is the semidirect product of W with the normal subgroup $\Theta(\Lambda_{\text{root}})$.

The group $W_{\text{aff}} \cong W \ltimes \Theta(\Lambda_{\text{root}})$ can be expanded by adding the group of translations by Λ. This expanded group is called the *extended affine Weyl group*.

The Hecke algebra $\mathcal{H}(W_{\text{aff}})$ of W_{aff} has generators $T_0, T_1, \cdots, T_r$ subject to the quadratic relations

$$T_i^2 = (q-1)T_i + q \tag{1}$$

and the braid relations. It has an alternative presentation, due to Bernstein, that is generated by $T_1, \cdots, T_r$ and an abelian subalgebra isomorphic to Λ_{root}.

To describe the Bernstein presentation, we make use of a complex torus T such that the group of rational characters of T is identified with the weight lattice Λ. If $\mathbf{z} \in T$, let $\mathbf{z}^\lambda$ be the character λ evaluated at $\mathbf{z}$. Let $\mathcal{H}(W) = \langle T_1, \cdots, T_r \rangle$ be the finite Hecke algebra, with generators omitting T_0, subject to the quadratic relations and braid relations (which may be read off from the Dynkin diagram). We omit T_0 in this presentation. Now we consider the algebra $\mathcal{H}(W) \otimes \mathbf{z}^\Lambda$, where the generators of $\mathcal{H}(W)$ commute with the weight lattice by the *Bernstein relation*

$$\mathbf{z}^\lambda T_i - T_i \mathbf{z}^{s_i \lambda} = \frac{q-1}{1-\mathbf{z}^{-\alpha_i}}(\mathbf{z}^\lambda - \mathbf{z}^{s_i\lambda}).$$

Just as the affine Weyl group is smaller than the extended affine Weyl group, the algebra $\mathcal{H}(W) \otimes \mathbf{z}^\Lambda$ is alsoslightly bigger than the Coxeter group $\langle T_0, \cdots, T_r \rangle$. It is called the *extended affine Hecke algebra*. To recover the Coxeter group, we restrict the elements $\mathbf{z}^\lambda$ to the root lattice.

Theorem 3.1 (Bernstein, Zelevinsky, Lusztig)**.** *The subalgebra $\mathcal{H}(W) \otimes \mathbf{z}^{\Lambda_{\text{root}}}$ is isomorphic to the Coxeter group $\mathcal{H}(W_{\text{aff}})$.*

See [10, 2, 3] for more information.

In Lecture 12, Theorem 1.1 we saw that there is an action of $\mathcal{H}(W)$ on $\mathcal{O}(T)$ in which $T_1, \cdots, T_r$ act by Demazure-Lusztig operators. In the special case where $W = S_n$ is the Weyl group of $\mathrm{GL}(n)$ we applied this to study the partition functions of colored lattice models.

Theorem 3.2 (Lusztig [9])**.** *This action extends to the affine Hecke algebra $\mathcal{H}(W) \otimes \mathbf{z}^\Lambda$.*

In this action we let $\mathbf{z}^\lambda$ act by its inverse $\mathbf{z}^{-\lambda}$. To prove this, one must check the Bernstein relation.

4 The Poincaré-Birkhoff-Witt theorem

As we saw in the last lecture, many examples of the Yang-Baxter equation come from quantum groups. It is also possible to work backwards from the Yang-Baxter equation and produce a quantum group ([13] or [8] Section 8.6). If we understand the term "quantum group" to mean a quasitriangular Hopf algebra, many instances turn out to be quantized enveloping algebras. Recall the $H = U_q(\mathfrak{g})$ is actually not quasitriangular (though if q is a root of unity it has a quasitriangular quotient), but it is "morally" quasitriangular, meaning that there is a universal R-matrix, but it is not in $H \otimes H$ but in a completion which might be denoted $H \hat{\otimes} H$. There are various ways of handling this difficulty.

The notion of Hopf algebra is self-dual, but quasitriangularity is not, so there are also dual quasitriangular Hopf algebras ([11, 8]). Quantized function algebras are dual quasitriangular. For the purpose of investigating the Yang-Baxter equation, whether to work with quasitriangular or dual quasitriangular Hopf algebra is a matter of taste.

In preparation for discussing Verma modules, we will introduce here a tool, the *Poincaré-Birkhoff-Witt* (PBW) theorem.

We work with $U(\mathfrak{g})$, not $U_q(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. We assume that $\mathfrak{g}$ is finite-dimensional, though this hypothesis is easily lifted. Let $X_1, \cdots, X_d$ be a basis of $\mathfrak{g}$. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Theorem 4.1 (PBW)**.** *A basis of $U(\mathfrak{g})$ as a vector space consists of the elements $X_1^{k_1} \cdots X_d^{k_d}$ as $\mathbf{k} = (k_1, \cdots, k_d)$ runs through $\mathbb{N}^d$.*

Proof. See [4] Section 17.3. ☐

As an application, let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, which we recall is $\mathrm{Mat}_n(\mathbb{C})$ with the bracket operation $[X, Y] = XY - YX$ (matrix multiplication). This Lie algebra has 3 subalgebras, the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, and the subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ nilpotent upper triangular and lower triangular matrices, respectively.

Proposition 4.2. *The multiplication map* $U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \longrightarrow U(\mathfrak{g})$ *is a vector space isomorphism.*

Proof. This follows from the PBW theorem by choosing the basis $X_1, \cdots, X_n$ so that the first $\frac{1}{2}n(n-1)$ elements are in $\mathfrak{n}_-$, the next n are in $\mathfrak{h}$ and the last $\frac{1}{2}n(n-1)$ are in $\mathfrak{n}_+$. Then every basis element of $\mathfrak{g}$ is uniquely the product of basis elements of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$, from which the statement is clear. ☐

In Lecture 15 we will use this to describe certain infinite-dimensional representations of $\mathfrak{g}$ called *Verma modules*. The natural habitat for these is the Bernstein-Gelfand-Gelfand *Category* $\mathcal{O}$ ([1, 5], [7] Chapter 9). They are not integrable, meaning that they do not lift to representations of $\mathrm{GL}(n, \mathbb{C})$. However Verma modules are still important for us because they do have analogs for quantum groups, and these have applications to lattice models. See [12].

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