1 Examples of colored models

In Lecture 12, we considered the following R-matrix:

\[
\begin{array}{cccc}
\text{z, w} & \text{c} & \text{c} & \text{d} \\
\text{c} & \text{d} & \text{z, w} & \text{c} \\
\text{c} & \text{d} & \text{z, w} & \text{c} \\
\text{z, w} & \text{c} & \text{d} & \text{c} \\
\text{z, w} & \text{c} & \text{d} & \text{c} \\
\end{array}
\]

We mentioned that this satisfies a Yang-Baxter equation as follows:

\[
\begin{array}{cccc}
z - qw & z - qw & (1 - q)z & z - w \\
(1 - q)w & (1 - q)w & (1 - q)w & q(z - w) \\
\end{array}
\]

We will refer to this as the RRR equation since it involves three copies of the R-matrix. We described this as a parametrized Yang-Baxter equation, but it requires a bit of explanation why this is an instance of a parametrized Yang-Baxter equation. We recall that this requires a group \( \Gamma \) and a map \( R \) from \( \Gamma \) to the set of Boltzmann weights such that the following
Yang-Baxter equation holds:

\[
R(\gamma \delta) R(\gamma) R(\gamma \delta) = R(\gamma) R(\delta) R(\gamma)
\]

One way to interpret our Yang-Baxter equation as a parametrized one is to divide the Boltzmann weights by \( z - qw \), and use these weights instead:

\[
\frac{z}{w} + \frac{1}{1 - q} z - qw \text{ if } c < d \\
\frac{z}{w} + \frac{1}{1 - q} z - qw \text{ if } c > d \\
\frac{z - w}{z - tw} + \frac{1}{1 - t} z - tw
\]

This change does not affect the validity of the Yang-Baxter equation since it divides both sides by the same constant \((z_1 - qz_2)(z_1 - qz_3)(z_2 - qz_3)\). But with this change the Boltzmann weights only depend on \( z/w \) and we have indicated this in the notation by labeling the \( R \)-matrix with \( z/w \in \mathbb{C}^\times \). We then recognize the Yang-Baxter equation as a parametrized Yang-Baxter equation with parameter group \( \mathbb{C}^\times \).

In Lecture 12 we considered partition functions assuming we have a Yang-Baxter equation as follows:

We will refer to this as the RTT equation, which can be written symbolically as \( RTT = TTR \). The letter \( T \) refers to the vertex types labeled \( z \) and \( w \). We did not specify the Boltzmann weights at the “T” vertices except to remark that there are multiple possibilities.
And if we form the partition function of a system $Z(z; d)$ with boundary conditions as in the open models, then these satisfy a recursion

$$Z(z; s_j d) = L_j Z(z; d)$$

where $L_j$ is the Demazure-Lusztig operator, assuming $d_j > d_{j+1}$.

Let us investigate some choices for the $T$ weights.

**Example 1.1.** First, we can just use the same vertex types as with the R-matrix, but rotated by $45^\circ$ (clockwise).

To explain this, we rotate the R-matrix and replace the parameter $w$ by a new parameter $\alpha$ which can depend on the column, and obtain these weights:

<table>
<thead>
<tr>
<th>$z/\alpha$</th>
<th>$z/\alpha$</th>
<th>$z/\alpha$</th>
<th>$z/\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$z/\alpha$</td>
<td>$z/\alpha$</td>
<td>$z/\alpha$</td>
<td>$z/\alpha$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$(1-q)z$</td>
<td>$(1-q)\alpha$</td>
<td>$q(z-\alpha)$</td>
<td>$z-\alpha$</td>
</tr>
<tr>
<td>$z-q\alpha$</td>
<td>$z-q\alpha$</td>
<td>$z-q\alpha$</td>
<td>$z-q\alpha$</td>
</tr>
</tbody>
</table>

Here $\alpha$ can be arbitrary but in the partition function $\alpha$ must be constant in the column. Note that the RRR parametrized Yang-Baxter equation is equivalent to the RTT equation.

**Example 1.2.** Another possibility, and an interesting one, is the bosonic models used in [3], which are special cases of more general ones in [1]. In these models, every vertical edge can carry an arbitrary number of bosons for every color. Thus if $c_1, \cdots, c_n$ are the colors, the spinset of the vertical edges is $\mathbb{N}^n$ where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and if $\mu \in \mathbb{N}^n$ we may write $c_\mu$ for the spin with $\mu_i$ bosons of color $c_i$, where $c_0 = (c_1, \cdots, c_n)$ is the standard flag. We will not describe the Boltzmann weights here, but see [3] for details. The partition functions are nonsymmetric Hall-Littlewood polynomials, and in [1] there are similar bosonic models whose partition functions are more general nonsymmetric Macdonald polynomials.

Our point is that there are multiple choices for the edges in the models for a very good reason. In the paradigm we are considering, every edge of the model corresponds to an object in a braided category. In this case, we will see (later) that this category is the category of $U_{\sqrt{q}(\widehat{sl}_n+1)}$-modules. And if $U, V$ are any two objects of this category, then there is a braiding $c_{U,V} : U \otimes V \rightarrow V \otimes U$, and these all satisfy the Yang-Baxter equation (Lecture 4).
2 Back to quantum groups

The theory of quantum groups gives an explanation of where the Yang-Baxter equation comes from, and what instances we may expect. Our goal is to give a taste of this.

Please review Lecture 12. We saw that a vector space $H$ over a field $F$ (for us usually $\mathbb{C}$) equipped with map $\mu : H \otimes H \rightarrow H$ and $\varepsilon : F \rightarrow H$ satisfying the associativity and unit axioms is the same as an associative algebra, with multiplication $x \cdot y = \mu(x \otimes y)$ and identity element $\varepsilon(1_F)$. Similarly, a vector space $H$ equipped with a linear map $\Delta : H \rightarrow H \otimes H$ (called comultiplication) and $\eta : H \rightarrow F$ satisfying the coassociativity and counit axioms is called a coalgebra. A Hopf algebra is thus both an algebra and a coalgebra.

If $A$ and $B$ are algebras, so is $A \otimes B$ and the Hopf axiom can be interpreted as saying that $\Delta : H \rightarrow H \otimes H$ is an algebra homomorphism. So is the counit $\eta : H \rightarrow F$. It is equivalent to say that $\mu : H \otimes H \rightarrow H$ is a homomorphism of coalgebras.

**Proposition 2.1.** Let $H$ be a Hopf algebra. Then the category of $H$-modules is monoidal.

**Proof.** For an associative algebra $A$, if $V$ and $W$ are $A$-modules, then $V \otimes W$ is not naturally an $A$-module. It is, however, very naturally an $A \otimes A$-module.

Now let $V$ and $W$ be $H$-modules. We need to put an $H$-module structure on $V \otimes W$. For this, we use the comultiplication, which is an algebra homomorphism $H \rightarrow H \otimes H$.

There are two important and related types of Hopf algebras that have deformations into “quantum groups.” Let $G$ be a reductive algebraic group over $\mathbb{C}$ such as $GL(n)$. Let $\mathcal{O}(G)$ be the ring of polynomial functions on $G$. This algebra is of course commutative. The multiplication map $G \times G \rightarrow G$ is a morphism hence induces an algebra homomorphism $\mathcal{O}(G) \rightarrow \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. This is the comultiplication, making $\mathcal{O}(G)$ into a Hopf algebra. A deformation of this will be called a deformed function algebra.

On the other hand, let us recall the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. This is an associative algebra $\mathfrak{U}(\mathfrak{g})$ that contains a copy of $\mathfrak{g}$ as a vector subspace, such that if $X, Y \in \mathfrak{g}$ then

$$[X, Y] = X \cdot Y - Y \cdot X. \quad (\cdot \text{ = multiplication in } \mathfrak{U}(\mathfrak{g})\text{ )}$$

It has the universal property that if $f : \mathfrak{g} \rightarrow A$ of $\mathfrak{g}$ into an associative algebra $A$ such that

$$f([X, Y]) = f(X)f(Y) - f(Y)f(X),$$

then $f$ extends uniquely to an algebra homomorphism $\mathfrak{U}(\mathfrak{g}) \rightarrow A$. Then $\mathfrak{U}(\mathfrak{g})$ is a cocommutative Hopf algebra whose comultiplication satisfies

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad (X \in \mathfrak{g}).$$

What Drinfeld and Jimbo showed ([4, 6]) was that it is possible to deform the enveloping algebra $\mathfrak{U}(\mathfrak{g})$, after expanding it slightly to include some group-like elements. The deformation $\mathfrak{U}_q(\mathfrak{g})$, with $q$ a complex parameter, is called a quantized enveloping algebra.
A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $F$ with a bilinear “bracket” operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, for which we use the notation $[X,Y]$, that satisfies

$$ [Y,X] = -[X,Y], \quad [[X,Y], Z] + [[Y,Z], X] + [[Z,X], Y] = 0. $$

The second relation is called the *Jacobi relation*. The Lie algebra $\mathfrak{gl}_n$ is $\text{Mat}_n(C)$ with the bracket operation

$$ [X,Y] = XY - YX. \tag{1} $$

It can be easily checked that this is a Lie algebra. Alternatively, if $V$ is a vector space, $\mathfrak{gl}(V)$ is the endomorphism ring of $V$ with bracket operation $[\cdot, \cdot]$. The Lie algebra $\mathfrak{sl}_n$ is the vector subspace $\mathfrak{gl}_n$ consisting of matrices of trace zero.

**Definition 1.** A **representation** of the Lie algebra $\mathfrak{g}$ is a homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. Thus it is a linear map to $\text{End}(V)$ that satisfies

$$ \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X). $$

**Example 2.2.** If $\pi : \text{GL}(n, C) \to \text{GL}(V)$ is a representation, then we obtain a representation $d\pi : \mathfrak{gl}_n(C) \to \mathfrak{gl}(V)$ by differentiating. Thus

$$ d\pi(X)v = \frac{d}{dt}e^{tx}v|_{t=0}. $$

It can be checked that this is a representation ([2], Proposition 7.2).

The **universal enveloping algebra** $U(\mathfrak{g})$ is the algebra generated by $\mathfrak{g}$ subject to relations

$$ X \cdot Y - Y \cdot X = [X,Y]. \tag{2} $$

This resembles (1) but note that in (1) the multiplication is matrix multiplication and in (2) the multiplication is the multiplication in $U(\mathfrak{g})$. Now if $\pi : \mathfrak{g} \to \text{End}(V)$ is a representation, then since by the definition of a representation the relations (2) are satisfied by $\pi(X), \pi(Y)$ and $\pi([X,Y])$, the linear map $\pi$ extends to an algebra homomorphism $U(\mathfrak{g}) \to \text{End}(V)$.

To summarize:

- Representations of a Lie group $G$ become representations of its Lie algebra $\mathfrak{g}$, by differentiating. A representation of $\mathfrak{g}$ that comes from a representation of $G$ is called **integrable**.

- Representations of a Lie algebra $\mathfrak{g}$ extend to representations of the associative algebra $U(\mathfrak{g})$.

So the enveloping algebra captures the representations of a Lie group or Lie algebra. We caution that the Lie algebra of a Lie group has representations that are not integrable, such as Verma modules, so its representation theory is slightly richer than $G$. Quantum versions of these “non-integrable” representations can still figure in the Yang-Baxter equation. For example, Verma modules of $U_{\sqrt{q}}(\mathfrak{sl}_{n+1})$ underlie Example [1.2]
Proposition 2.3. The enveloping algebra $U(g)$ is a Hopf algebra with comultiplication satisfying

\[ \Delta(X) = X \otimes 1 + 1 \otimes X \quad (X \in g). \]

Proof. We take $\Delta : g \rightarrow g \otimes g \subset U(g) \otimes U(g)$ to be defined by (2.3) when $X \in g$. We must show that this definition extends to $U(g)$. First let us note that if $X, Y \in g$ then

\[ \Delta(X) \Delta(Y) - \Delta(Y) \Delta(X) = XY \otimes 1 - YX \otimes 1 + 1 \otimes XY - 1 \otimes YX. \]

Indeed, expanding the left-hand side gives eight terms but four cancel in pairs. In $U(g) \otimes U(g)$ we therefore have

\[ \Delta(X) \Delta(Y) - \Delta(Y) \Delta(X) = \Delta([X,Y]). \]

The elements $\Delta(X)$ in $U(g) \otimes U(g)$ thus satisfy the generating relations of $U(g)$, which was defined by generators $X \in g$ and relations [2]. It follows that they extend to an algebra homomorphism $U(g) \rightarrow U(g) \otimes U(g)$. As for the counit $\eta : U(g) \rightarrow F$, this is obtained by extending the zero map $g \rightarrow F$ to an algebra homomorphism $U(g) \rightarrow F$.

The antipode is an antimultiplicative map $U(g) \rightarrow U(g)$ that satisfies $S(X) = -X$ for $X \in g$. To see that this map exists, if $U(g)^{opp}$ is the opposite ring then the generators $-X$ satisfy the defining relations for $U(g)$, so there is a homomorphism $S : U(g) \rightarrow U(g)^{opp}$ that sends $X$ to $-X$, and this is the antipode.

We leave checking the axioms to the reader. \hfill \square

3 $U_q(\mathfrak{sl}_2)$

The very simplest and most important case is $g = \mathfrak{sl}_2$. It has a basis consisting of:

\[ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

with

\[ [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \]

Thus the enveloping algebra $U(\mathfrak{sl}_2)$ is a noncommutative polynomial ring with generators $E, F, H$ modulo the ideal generated by the relations

\[ HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H. \]

The comultiplication, we have already seen, is

\[ \Delta X = X \otimes 1 + 1 \otimes X, \quad X \in g, \]

and the antipode satisfies $S(X) = -X$ for $X \in g$.

Now let us explain how to deform $U(g)$. Let $q \in \mathbb{C}$. We will first define $U_q(g)$ as an associative algebra, then prove it has a comultiplication. In place of $H$ we make use of a “grouplike” element $K$ which we can think of as the matrix

\[ \begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}. \]
We can express \( H = (q - q^{-1})^{-1}(K - K^{-1}) \) and so we do not need \( H \) among the generators. The algebra is then generated by \( E, F \) and \( K \) with relations
\[
KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - FE = (q - q^{-1})^{-1}(K - K^{-1}).
\]
We should also take \( K^{-1} \) among the generators of \( U_q(\mathfrak{sl}_2) \) with obvious relations.

**Proposition 3.1.** The ring \( U_q(\mathfrak{g}) \) admits a comultiplication \( \Delta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) such that
\[
\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.
\]
There is also an antipode \( S \) that satisfies
\[
S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1},
\]
and a counit satisfying \( \eta(F) = \eta(E) = 0 \), so \( U_q(\mathfrak{g}) \) is a Hopf algebra.

*Proof.* The proof consists of showing that the elements \( K \otimes K, E \otimes K + 1 \otimes E \) and \( F \otimes 1 + K^{-1} \otimes F \) satisfy the same relations as \( K, E \) and \( F \). We will omit this verification, or the verification of the antipode. \( \Box \)

## 4 R-matrices

Drinfeld [4] defined the notion of a *quasitriangular Hopf algebra*. This is a Hopf algebra \( H \) with an invertible element \( R \in H \otimes H \) satisfying certain axioms. The first axiom is that for \( h \in H \) we have
\[
\tau(\Delta h) = R(\Delta h)R^{-1},
\]
where \( \tau : H \otimes H \rightarrow H \otimes H \) is the flip map \( \tau(x \otimes y) = y \otimes x \). It is not hard to check that this implies that if \( U, V \) are \( H \)-modules, then the map
\[
\tau(U \otimes V) \rightarrow V \otimes U \quad \tau(U \otimes V) \rightarrow V \otimes U
\]
is an \( H \)-module homomorphism \( U \otimes V \rightarrow V \otimes U \). Then there are two more axioms that guarantee that this map is a braiding. See [8] Chapter 5 for further details. The element \( R \) of \( H \otimes H \) is called the universal R-matrix.

**Theorem 4.1.** Assume that \( q \) is not a root of unity. The category of finite-dimensional modules a quantized enveloping algebra such as \( U_q(\mathfrak{sl}_2) \) is braided.

*Proof.* Unfortunately \( H = U_q(\mathfrak{g}) \) is not a quasitriangular Hopf algebra. There is a universal R-matrix, but it is given by an infinite series and so it does not live in \( H \otimes H \) but rather in a completion. There are various ways of avoiding this difficulty. One way is to work with a quantized function algebra that is in duality with \( H \), and show that this Hopf algebra is dual quasitriangular. \( \Box \)
So even though $U_q(\mathfrak{sl}_2)$ is not quasitriangular, it is almost as good. But rather than try to work with the universal R-matrix, it is usually possible to work directly with equations to find the braiding. So let us see how that works in this particular case.

Let $V = \mathbb{C}^2$ be the two-dimensional standard module, with basis $\{x, y\}$ such that $E, F$ and $K$ are represented by the matrices

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
q & 0 \\
0 & q^{-1}
\end{pmatrix}.
$$

We will begin by determining the endomorphisms of $V \otimes V$. The tensor product module is not irreducible, but splits into two irreducible submodules, of dimensions 1 and 3. So the endomorphism ring will turn out to be two dimensional.

We recall that the action of $H$ on $V \otimes V$ is via the comultiplication. In particular $\Delta K = K \otimes K$, so

$$
K \cdot (x \otimes y) = Kx \otimes Ky.
$$

Hence the eigenspaces of $K$ corresponding to the eigenvalues $q^2$, 1 and $q^{-2}$ have bases $\{x \otimes x\}$, $\{x \otimes y, y \otimes x\}$ and $\{y \otimes y\}$. These must be invariant by any endomorphism $\phi$ of $V \otimes V$, so with respect to the basis $x \otimes x, x \otimes y, y \otimes x, y \otimes y$, the matrix of $\phi$ has the form

$$
\begin{pmatrix}
* & * & * & * \\
* & * & * & *
\end{pmatrix}.
$$

Assuming that $\phi$ is invertible, we may scale it so that

$$
\phi(x \otimes x) = x \otimes x, \quad \phi(x \otimes y) = ax \otimes y + cy \otimes x, \quad \phi(y \otimes x) = bx \otimes y + dy \otimes x, \quad \phi(y \otimes y) = \lambda y \otimes y
$$

for some nonzero constant $y$.

**Lemma 4.2.** We have

$$
a + qc = 1, \quad b + dq = q, \quad q^{-1}a + b = q^{-1}, \quad q^{-1}c + d = 1.
$$

Moreover $b = c, \lambda = 1$.

**Proof.** From $\Delta E = E \otimes K + 1 \otimes E$ we have $E(x \otimes y) = Ex \otimes Ky + x \otimes Ey = x \otimes x$ and similarly $E(y \otimes x) = qx \otimes x$. Then

$$
x \otimes x = \phi(x \otimes x) = \phi(E(x \otimes y)) = E\phi(x \otimes y) = aE(x \otimes y) + cE(y \otimes x) = (a + cq)x \otimes x,
$$

proving that $a + cq = 1$. Similarly

$$
qx \otimes x = \phi(E(y \otimes x)) = E\phi(y \otimes x) = bE(x \otimes y) + dE(y \otimes x) = (b + dq)x \otimes x,
$$

8
proving that \( b + dq = q \). We have proved

Starting with \( \phi(x \otimes x) = x \otimes x \) and noting that \( F(x \otimes x) = q^{-1} x \otimes y + y \otimes x \) we get

\[
q^{-1} x \otimes y + y \otimes x = F(x \otimes x) = \phi(x \otimes x) = \phi F(x \otimes x) = \phi(q^{-1} x \otimes y + y \otimes x).
\]

Expanding this using (3) and (4), then comparing coefficients gives the identities (6). Comparing (5) and (6) gives \( b = c \).

Proceeding similarly but starting with \( y \otimes y \) instead of \( x \otimes x \) gives the same identities (5) and (6) but contingent on \( \lambda = 1 \).

**Theorem 4.3.** There are two \( U_q(\mathfrak{sl}_2) \)-module endomorphisms \( R \) and \( R' \) of \( V \otimes V \) that satisfy the Yang-Baxter equation in the form

\[
R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}. \tag{7}
\]

They are the endomorphisms with matrices

\[
R = \begin{pmatrix}
1 & 0 \\
1 - q^2 & q \\
q & 0 \\
1 & 1
\end{pmatrix}, \quad R' = \begin{pmatrix}
1 & 0 & q^{-1} \\
0 & q^{-1} & 1 - q^{-2} \\
q^{-1} & 1 - q^{-2} & 1
\end{pmatrix}.
\]

**Remark 1.** The notation is as follows: if \( R \in \text{End}(V \otimes V) \) then \( R_{i,j} \in \text{End}(V \otimes V \otimes V) \) is \( R \) applied to the \( i \)- and \( j \)-th components of \( V \otimes V \otimes V \). The Yang-Baxter equation is often written

\[
R_{12} R_{13} R_{22} = R_{23} R_{12} R_{23}. \tag{8}
\]

The relationship between the two versions is that if \( R \) satisfied (8), then \( \tau R \) satisfies (7), where as usual \( \tau(x \otimes y) = y \otimes x \).

**Proof of Theorem 4.3.** We have seen in the Lemma that every invertible \( H \)-module homomorphism \( V \otimes V \rightarrow V \otimes V \) is a scalar multiple of one of the form

\[
\begin{pmatrix}
1 & a & b \\
a & b & d \\
b & d & 1
\end{pmatrix}
\]

with \( a + qb = 1 \) and \( b + dq = q \). Such a matrix is a linear combination of two standard ones. With \( d = 0 \), we have \( b = q \) and hence \( a = 1 - q^2 \). On the other hand, with \( a = 0 \), we have \( b = q^{-1} \) and so \( d = 1 - q^{-2} \). These give \( R \) and \( R' \) as a basis of the two-dimensional vector space \( \text{End}_H(V \otimes V) \).

Now, for the Yang-Baxter equation, we can take a linear combination \( tR + uR' \) and check whether it satisfies the Yang-Baxter equation. This can be checked using a computer. We find three solutions, but one is the scalar matrix \( qR' - q^{-1}R = (q - q^{-1}) I_{V \otimes V} \). The other solutions of the Yang-Baxter equation are just \( R \) and \( R' \) (or constant multiples). \( \square \)
5 Parametrized Yang-Baxter equations

Theorem 5.1. Let $q \in \mathbb{C}^\times$ be fixed. Let $R$ and $R'$ be as in Theorem 4.3. For $z \in \mathbb{C}^\times$ let

$$R(z) = R - zq^2R'.$$

Then we have a parametrized Yang-Baxter equation

$$R(z)_{12}R(zw)_{23}R(w)_{12} = R(w)_{23}R(zw)_{12}R(z)_{23}.$$

Proof. This can be checked by hand, or by computer (see sl2param.sage, posted on the class web page).

This parametrized Yang-Baxter equation is equivalent to the one in Lecture 4. The colored equation from Lecture 12 (and the beginning of this lecture) is a generalization due to Jimbo [7]. (To compare them replace $q \rightarrow \sqrt{q}$ in Theorem 5.1.)

Jimbo [7] also gave generalizations to the other classical Cartan types. These Yang-Baxter equations come from the quantized enveloping algebras of affine Lie algebras, which we will consider briefly in future lectures. The Lie algebra $\hat{sl}_2$ or $U_q(\hat{sl}_2)$, at least when $q$ is not a root of unity, has one two-dimensional irreducible representation $V_z$ for each $z \in \mathbb{C}^\times$. In one way imitating the proof of Theorem 4.3 is actually simpler in the affine case, for if $z$ and $w$ are in general position, the representation $V_z \otimes V_w$ is irreducible, so the R-matrix $V_z \otimes V_w \rightarrow V_w \otimes V_z$ is determined up to scalar multiple. See [5] Proposition 9.2.4.
References


