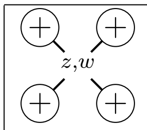
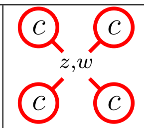
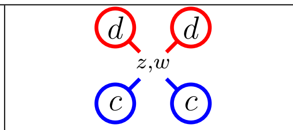
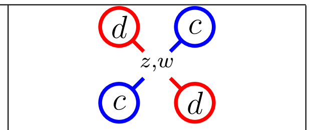
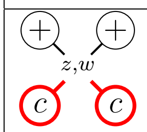
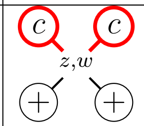
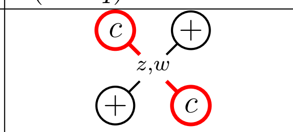
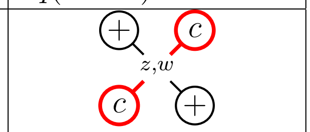
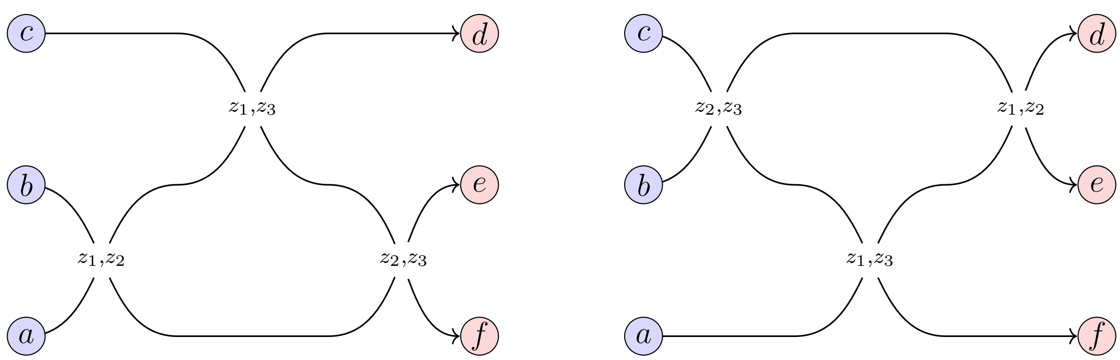
Lecture 13

1 Examples of colored models

In Lecture 12, we considered the following R-matrix:

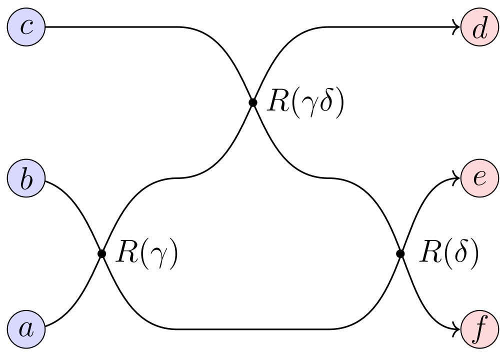
			
$z - qw$	$z - qw$	$(1-q)z$ if $c < d$ $(1-q)w$ if $c > d$	$z - w$ if $c > d$ $q(z-w)$ if $c < d$
			
$(1-q)z$	$(1-q)w$	$q(z-w)$	$z - w$

We mentioned that this satisfies a Yang-Baxter equation as follows:

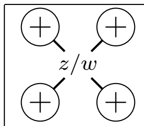
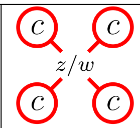
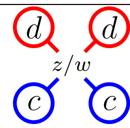
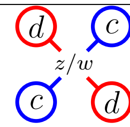
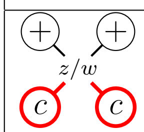
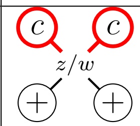
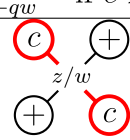
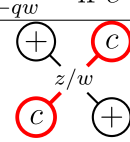


We will refer to this as the RRR equation since it involves three copies of the R-matrix.We described this as a parametrized Yang-Baxter equation, but it requires a bit of explanation why this is an instance of a parametrized Yang-Baxter equation. We recall that this requires a group Γ and a map R from Γ to the set of Boltzmann weights such that the following

Yang-Baxter equation holds:

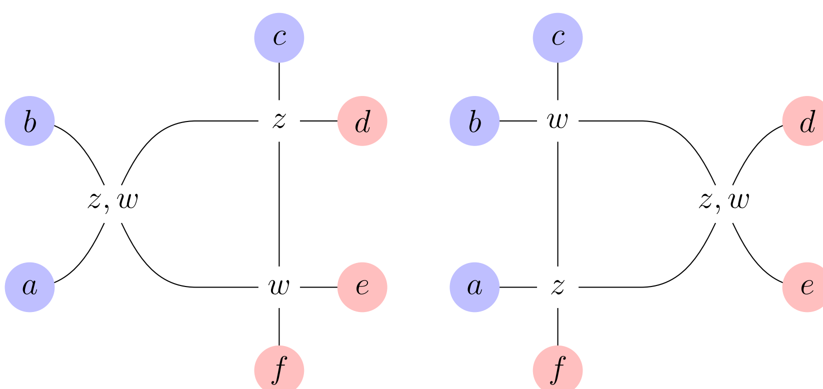


One way to interpret our Yang-Baxter equation as a parametrized one is to divide the Boltzmann weights by $z - qw$, and use these weights instead:

			
1	1	$\frac{(1-q)z}{z-qw}$ if $c < d$ $\frac{(1-q)w}{z-qw}$ if $c > d$	$\frac{z-w}{z-qw}$ if $c > d$ $\frac{(1-q)w}{z-qw}$ if $c < d$
			
$\frac{(1-t)z}{z-tw}$	$\frac{(1-t)w}{z-tw}$	$\frac{t(z-w)}{z-tw}$	$\frac{z-w}{z-tw}$

This change does not affect the validity of the Yang-Baxter equation since it divides both sides by the same constant $(z_1 - qz_2)(z_1 - qz_3)(z_2 - qz_3)$. But with this change the Boltzmann weights only depend on z/w and we have indicated this in the notation by labeling the R-matrix with $z/w \in \mathbb{C}^\times$. We then recognize the Yang-Baxter equation as a parametrized Yang-Baxter equation with parameter group $\mathbb{C}^\times$.

In Lecture 12 we considered partition functions assuming we have a Yang-Baxter equation as follows:



We will refer to this as the RTT equation, which can be written symbolically as RTT=TTR. The letter T refers to the vertex types labeled z and w. We did not specify the Boltzmann weights at the "T" vertices except to remark that there are multiple possibilities.

And if we form the partition function of a system $Z(\mathbf{z};\mathbf{d})$ with boundary conditions as in the open models, then these satisfy a recursion

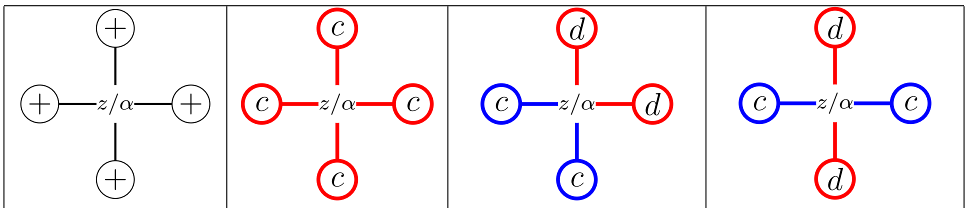
$$Z(\mathbf{z}; s_i\mathbf{d}) = \mathcal{L}_i Z(\mathbf{z};\mathbf{d})$$

where $\mathcal{L}_i$ is the Demazure-Lusztig operator, assuming $d_i > d_{i+1}$.

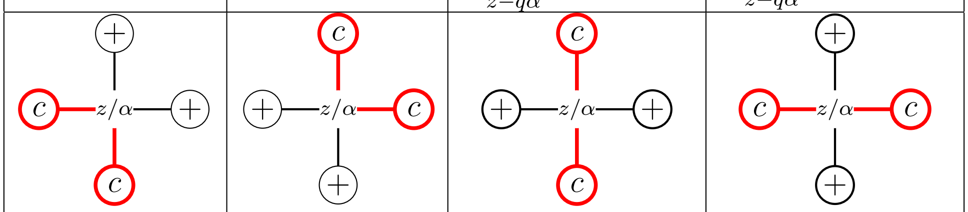
Let us investigate some choices for the T weights.

Example 1.1. First, we can just use the same vertex types as with the R-matrix, but rotated by 45° (clockwise).

To explain this, we rotate the R-matrix and replace the parameter w by a new parameter α which can depend on the column, and obtain these weights:



1	1	$\frac{(1-q)z}{z-q\alpha}$ if $c<d$; $\frac{(1-q)\alpha}{z-q\alpha}$ if $c>d$	$\frac{z-\alpha}{z-q\alpha}$ if $c>d$; $\frac{(1-q)\alpha}{z-q\alpha}$ if $c<d$



$\frac{(1-q)z}{z-q\alpha}$	$\frac{(1-q)\alpha}{z-q\alpha}$	$\frac{q(z-\alpha)}{z-q\alpha}$	$\frac{z-\alpha}{z-q\alpha}$

Here α can be arbitrary but in the partition function α must be constant in the column. Note that the RRR parametrized Yang-Baxter equation is equivalent to the RTT equation.

Example 1.2. Another possibility, and an interesting one, is the bosonic models used in [3], which are special cases of more general ones in [1]. In these models, every vertical edge can carry an arbitrary number of bosons for every color. Thus if $c_1, \cdots, c_n$ are the colors, the spinset of the vertical edges is $\mathbb{N}^n$ where $\mathbb{N} = \{0,1,2,\cdots\}$ and if $\mu \in \mathbb{N}^n$ we may write $\mathbf{c}_0^\mu$ for the spin with μ_i bosons of color c_i, where $\mathbf{c}_0 = (c_1,\cdots,c_n)$ is the standard flag. We will not describe the Boltzmann weights here, but see [3] for details. The partition functions are nonsymmetric Hall-Littlewood polynomials, and in [1] there are similar bosonic models whose partition functions are more general nonsymmetric Macdonald polynomials.

Our point is that there are multiple choices for the edges in the models for a very good reason. In the paradigm we are considering, every edge of the model corresponds to an object in a braided category. In this case, we will see (later) that this category is the category of $U_{\sqrt{q}}(\widehat{\mathfrak{sl}}_{n+1})$-modules. And if U, V are any two objects of this category, then there is a braiding $c_{U,V} : U \otimes V \longrightarrow V \otimes U$, and these all satisfy the Yang-Baxter equation (Lecture 4).

2 Back to quantum groups

The theory of quantum groups gives an explanation of where the Yang-Baxter equation comes from, and what instances we may expect. Our goal is to give a taste of this.

Please review Lecture 12. We saw that a vector space H over a field F (for us usually $\mathbb{C}$) equipped with map $\mu : H \otimes H \longrightarrow H$ and $\varepsilon : F \longrightarrow H$ satisfying the associativity and unit axioms is the same as an associative algebra, with multiplication $x \cdot y = \mu(x \otimes y)$ and identity element $\varepsilon(1_F)$. Similarly, a vector space H equipped with a linear map $\Delta : H \longrightarrow H \otimes H$ (called comultiplication) and $\eta : H \longrightarrow F$ satisfying the coassociativity and counit axioms is called a coalgebra. A Hopf algebra is thus both an algebra and a coalgebra.

If A and B are algebras, so is $A \otimes B$ and the Hopf axiom can be interpreted as saying that $\Delta : H \longrightarrow H \otimes H$ is an algebra homomorphism. So is the counit $\eta : H \longrightarrow F$. It is equivalent to say that $\mu : H \otimes H \longrightarrow H$ is a homomorphism of coalgebras.

Proposition 2.1. *Let H be a Hopf algebra. Then the category of H-modules is monoidal.*

Proof. For an associative algebra A, if V and W are A-modules, then $V \otimes W$ is not naturally an A-module. It is, however, very naturally an $A \otimes A$-module.

Now let V and W be H-modules. We need to put an H-module structure on $V \otimes W$. For this, we use the comultiplication, which is an algebra homomorphism $H \longrightarrow H \otimes H$. □

There are two important and related types of Hopf algberas that have deformations into "quantum groups." Let G be a reductive algebraic group over $\mathbb{C}$ such as $\mathrm{GL}(n)$. Let $\mathcal{O}(G)$ be the ring of polynomial functions on G. This algebra is of course commutative. The multiplication map $G \times G \longrightarrow G$ is a morphism hence induces an algebra homorphism $\mathcal{O}(G) \longrightarrow \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. This is the comultiplication, making $\mathcal{O}(G)$ into a Hopf algebra. A deformation of this will be called a *deformed function algebra*.

On the other hand, let us recall the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. This is an associative algebra $U(\mathfrak{g})$ that contains a copy of $\mathfrak{g}$ as a vector subspace, such that if $X, Y \in \mathfrak{g}$ then

$$[X, Y] = X \cdot Y - Y \cdot X. \qquad (\cdot = \text{multiplication in } U(\mathfrak{g}))$$

It has the universal property that if $f : \mathfrak{g} \longrightarrow A$ of $\mathfrak{g}$ into an associative algebra A such that

$$f([X, Y]) = f(X)f(Y) - f(Y)f(X),$$

then f extends uniquely to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow A$. Then $U(\mathfrak{g})$ is a cocommutative Hopf algebra whose comultiplication satisfies

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad (X \in \mathfrak{g}).$$

What Drinfeld and Jimbo showed ([4, 6]) was that it is possible to deform the enveloping algebra $U(\mathfrak{g})$, after expanding it slightly to include some group-like elements. The deformation $U_q(\mathfrak{g})$, with q a complex parameter, is called a *quantized enveloping algebra*.

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field F with a bilinear "bracket" operation $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, for which we use the notation $[X, Y]$, that satisfies

$$[Y, X] = -[X, Y], \qquad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

The second relation is called the *Jacobi relation*. The Lie algebra $\mathfrak{gl}_n$ is $\mathrm{Mat}_n(\mathbb{C})$ with the bracket operation

$$[X, Y] = XY - YX. \tag{1}$$

It can be easily checked that this is a Lie algebra. Alternatively, if V is a vector space, $\mathfrak{gl}(V)$ is the endomorphism ring of V with bracket operation (1). The Lie algebra $\mathfrak{sl}_n$ is the vector subspace $\mathfrak{gl}_n$ consisting of matrices of trace zero.

Definition 1. A *representation* of the Lie algebra $\mathfrak{g}$ is a homomorphism $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Thus it is a linear map to $\mathrm{End}(V)$ that satsifies

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

Example 2.2. If $\pi : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(V)$ is a representation, then we obtain a representation $d\pi : \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ by differentiating. Thus

$$d\pi(X)v = \frac{d}{dt} e^{tX} v|_{t=0}.$$

It can be checked that this is a representation ([2], Proposition 7.2).

The *universal enveloping algebra* $U(\mathfrak{g})$ is the algebra generated by $\mathfrak{g}$ subject to relations

$$X \cdot Y - Y \cdot X = [X, Y]. \tag{2}$$

This resembles (1) but note that in (1) the multiplication is matrix multiplication and in (2) the multiplication is the multiplication in $U(\mathfrak{g})$. Now if $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a representation, then since by the definition of a representation the relations (2) are satisfied by $\pi(X)$, $\pi(Y)$ and $\pi([X, Y])$, the linear map π extends to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$.

To summarize:

- Representations of a Lie group G become representations of its Lie algebra $\mathfrak{g}$, by differentiation. A representation of $\mathfrak{g}$ that comes from a representation of G is called *integrable*.

- Representations of a Lie algebra $\mathfrak{g}$ extend to representations of the associative algebra $U(\mathfrak{g})$.

So the enveloping algebra captures the representations of a Lie group or Lie algebra. We caution that the Lie algebra of a Lie group has representations that are not integrable, such as Verma modules, so its representation theory is slightly richer than G. Quantum versions of these "non-integrable" representations can still figure in the Yang-Baxter equation. For example, Verma modules of $U_{\sqrt{q}}(\mathfrak{sl}_{n+1})$ underlie Example 1.2.

Proposition 2.3. *The enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with comultiplication satisfying*

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad (X \in \mathfrak{g}).$$

Proof. We take $\Delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$ to be defined by (2.3) when $X \in \mathfrak{g}$. We must show that this definition extends to $U(\mathfrak{g})$. First let us note that if $X, Y \in \mathfrak{g}$ then

$$\Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) = XY \otimes 1 - YX \otimes 1 + 1 \otimes XY - 1 \otimes YX.$$

Indeed, expanding the left-hand side gives eight terms but four cancel in pairs. In $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ we therefore have

$$\Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) = \Delta([X, Y]).$$

The elements $\Delta(X)$ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ thus satisfy the generating relations of $U(\mathfrak{g})$, which was defined by generators $X \in \mathfrak{g}$ and relations (2). It follows that they extend to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$. As for the counit $\eta : U(\mathfrak{g}) \longrightarrow F$, this is obtained by extending the zero map $\mathfrak{g} \longrightarrow F$ to an algebra homomorphism $U(\mathfrak{g}) \longrightarrow F$.

The antipode is an antimultiplicative map $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ that satisfies $S(X) = -X$ for $X \in \mathfrak{g}$. To see that this map exists, if $U(\mathfrak{g})^{\text{opp}}$ is the opposite ring then the generators $-X$ satisfy the defining relations for $U(\mathfrak{g})$, so there is a homomorphism $S : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\text{opp}}$ that sends X to $-X$, and this is the antipode.

We leave checking the axioms to the reader. □

3 $U_q(\mathfrak{sl}_2)$

The very simplest and most important case is $\mathfrak{g} = \mathfrak{sl}_2$. It has a basis consisting of:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$[H, E] = 2E, \quad [H, F] = -2F, \qquad [E, F] = H.$$

Thus the enveloping algebra $U(\mathfrak{sl}_2)$ is a noncommutative polynomial ring with generators E, F, H modulo the ideal generated by the relations

$$HE - EH = 2E, \qquad HF - FH = -2F, \qquad EF - FE = H.$$

The comultiplication, we have already seen, is

$$\Delta X = X \otimes 1 + 1 \otimes X, \qquad X \in \mathfrak{g},$$

and the antipode satsifies $S(X) = -X$ for $X \in \mathfrak{g}$.

Now let us explain how to deform $U(\mathfrak{g})$. Let $q \in \mathbb{C}$. We will first define $U_q(\mathfrak{g})$ as an associative algebra, then prove it has a comultiplication. In place of H we make use of a "grouplike" element K which we can think of as the matrix

$$\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}.$$

We can express $H = (q - q^{-1})^{-1}(K - K^{-1})$ and so we do not need H among the generators. The algebra is then generated by E, F and K with relations

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F, \qquad EF - FE = (q - q^{-1})^{-1}(K - K^{-1}).$$

We should also take K^{-1} among the generators of $U_q(\mathfrak{sl}_2)$ with obvious relations.

Proposition 3.1. *The ring $U_q(\mathfrak{g})$ admits a comultiplication $\Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ such that*

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

There is also an antipode S that satisfies

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1},$$

and a counit satisfying $\eta(F) = \eta(E) = 0$, so $U_q(\mathfrak{g})$ is a Hopf algebra.

Proof. The proof consists of showing that the elements $K \otimes K$, $E \otimes K + 1 \otimes E$ and $F \otimes 1 + K^{-1} \otimes F$ satisfy the same relations as K, E and F. We will omit this verification, or the verification of the antipode. □

4 R-matrices

Drinfeld [4] defined the notion of a *quasitriangular Hopf algebra*. This is a Hopf algebra H with an invertible element $R \in H \otimes H$ satisfying certain axioms. The first axiom is that for $h \in H$ we have

$$\tau(\Delta h) = R(\Delta h)R^{-1},$$

where $\tau : H \otimes H \longrightarrow H \otimes H$ is the flip map $\tau(x \otimes y) = y \otimes x$. It is not hard to check that this implies that if U, V are H-modules, then the map

$$u \otimes v \longmapsto \tau(R(u \otimes v))$$

is an H-module homorphism $U \otimes V \longrightarrow V \otimes U$. Then there are two more axioms that guarantee that this map is a braiding. See [8] Chapter 5 for further details. The element R of $H \otimes H$ is called the *universal R-matrix*.

Theorem 4.1. *Assume that q is not a root of unity. The category of finite-dimensional modules a quantized enveloping algebra such as $U_q(\mathfrak{sl}_2)$ is braided.*

Proof. Unfortunately $H = U_q(\mathfrak{g})$ is *not* a quasitriangular Hopf algebra. There *is* a universal R-matrix, but it is given by an infinite series and so it does not live in $H \otimes H$ but rather in a completion. There are various ways of avoiding this difficulty. One way is to work with a quantized function algebra that is in duality with H, and show that this Hopf algebra is dual quasitriangular. □

So even though $U_q(\mathfrak{sl}_2)$ is not quasitriangular, it is almost as good. But rather than try to work with the universal R-matrix, it is usually possible to work directly with equations to find the braiding. So let us see how that works in this particular case.

Let $V = \mathbb{C}^2$ be the two-dimensional standard module, with basis $\{x, y\}$ such that E, F and K are represented by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}.$$

We will begin by determining the endomorphisms of $V \otimes V$. The tensor product module is not irreducible, but splits into two irreducible submodules, of dimensions 1 and 3. So the endomorphism ring will turn out to be two dimensional.

We recall that the action of H on $V \otimes V$ is via the comultiplication. In particular $\Delta K = K \otimes K$, so

$$K \cdot (x \otimes y) = Kx \otimes Ky.$$

Hence the eigenspaces of K corresponding to the eigenvalues q^2, 1 and q^{-2} have bases $\{x \otimes x\}$, $\{x \otimes y, y \otimes x\}$ and $\{y \otimes y\}$. These must be invariant by any endomorphism ϕ of $V \otimes V$, so with respect to the basis $x \otimes x, x \otimes y, y \otimes x, y \otimes y$, the matrix of ϕ has the form

$$\begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix}.$$

Assuming that ϕ is invertible, we may scale it so that

$$\phi(x \otimes x) = x \otimes x,$$

$$\phi(x \otimes y) = ax \otimes y + cy \otimes x, \tag{3}$$

$$\phi(y \otimes x) = bx \otimes y + dy \otimes x, \tag{4}$$

$$\phi(y \otimes y) = \lambda y \otimes y$$

for some nonzero constant y.

Lemma 4.2. *We have*

$$a + qc = 1, \qquad b + dq = q, \tag{5}$$

$$q^{-1}a + b = q^{-1}, \qquad q^{-1}c + d = 1. \tag{6}$$

Moreover $b = c$, $\lambda = 1$.

Proof. From $\Delta E = E \otimes K + 1 \otimes E$ we have $E(x \otimes y) = Ex \otimes Ky + x \otimes Ey = x \otimes x$ and similarly $E(y \otimes x) = qx \otimes x$. Then

$$x \otimes x = \phi(x \otimes x) = \phi(E(x \otimes y)) = E\phi(x \otimes y) = aE(x \otimes y) + cE(y \otimes x) = (a + cq)x \otimes x,$$

proving that $a + cq = 1$. Similarly

$$qx \otimes x = \phi(E(y \otimes x)) = E\phi(y \otimes x) = bE(x \otimes y) + dE(y \otimes x) = (b + dq)x \otimes x,$$

proving that $b + dq = q$. We have proved

Starting with $\phi(x \otimes x) = x \otimes x$ and noting that $F(x \otimes x) = q^{-1}x \otimes y + y \otimes x$ we get

$$q^{-1}x \otimes y + y \otimes x = F(x \otimes x) = F\phi(x \otimes x) = \phi F(x \otimes x) = \phi(q^{-1}x \otimes y + y \otimes x).$$

Expanding this using (3) and (4), then comparing coefficients gives the identities (6). Comparing (5) and (6) gives $b = c$.

Proceeding similarly but starting with $y \otimes y$ instead of $x \otimes x$ gives the same identities (5) and (6) but contingent on $\lambda = 1$. □

Theorem 4.3. *There are two $U_q(\mathfrak{sl}_2)$-module endomorphisms R and R' of $V \otimes V$ that satisfy the Yang-Baxter equation in the form*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{7}$$

They are the endomorphisms with matrices

$$R = \begin{pmatrix} 1 & & & \\ & 1-q^2 & q & \\ & q & 0 & \\ & & & 1 \end{pmatrix}, \qquad R' = \begin{pmatrix} 1 & & & \\ & 0 & q^{-1} & \\ & q^{-1} & 1-q^{-2} & \\ & & & 1 \end{pmatrix}.$$

Remark 1. The notation is as follows: if $R \in \mathrm{End}(V \otimes V)$ then $R_{i,j} \in \mathrm{End}(V \otimes V \otimes V)$ is R applied to the i- and j-th components of $V \otimes V \otimes V$. The Yang-Baxter equation is often written

$$R_{12}R_{13}R_{22} = R_{23}R_{13}R_{12}. \tag{8}$$

The relationship between the two versions is that if R satisfied (8), then τR satisfies (7), where as usual $\tau(x \otimes y) = y \otimes x$.

Proof of Theorem 4.3. We have seen in the Lemma that every invertible H-module homomorphism $V \otimes V \longrightarrow V \otimes V$ is a scalar multiple of one of the form

$$\begin{pmatrix} 1 & & & \\ & a & b & \\ & b & d & \\ & & & 1 \end{pmatrix}$$

with $a + qb = 1$ and $b + dq = q$. Such a matrix is a linear combination of two standard ones. With $d = 0$, we have $b = q$ and hence $a = 1 - q^2$. On the other hand, with $a = 0$, we have $b = q^{-1}$ and so $d = 1 - q^{-2}$. These give R and R' as a basis of the two-dimensional vector space $\mathrm{End}_H(V \otimes V)$.

Now, for the Yang-Baxter equation, we can take a linear combination $tR + uR'$ and check whether it satisfies the Yang-Baxter equation. This can be checked using a computer. We find three solutions, but one is the scalar matrix $qR' - q^{-1}R = (q - q^{-1})I_{V \otimes V}$. The other solutions of the Yang-Baxter equation are just R and R' (or constant multiples). □

5 Parametrized Yang-Baxter equations

Theorem 5.1. *Let $q \in \mathbb{C}^\times$ be fixed. Let R and R' be as in Theorem 4.3. For $z \in \mathbb{C}^\times$ let*

$$R(z) = R - zq^2 R'.$$

Then we have a parametrized Yang-Baxter equation

$$R(z)_{12}R(zw)_{23}R(w)_{12} = R(w)_{23}R(zw)_{12}R(z)_{23}.$$

Proof. This can be checked by hand, or by computer (see `sl2param.sage`, posted on the class web page). ☐

This parametrized Yang-Baxter equation is equivalent to the one in Lecture 4. The colored equation from Lecture 12 (and the beginning of this lecture) is a generalization due to Jimbo [7]. (To compare them replace $q \to \sqrt{q}$ in Theorem 5.1.)

Jimbo[7] also gave generalizations to the other classical Cartan types. These Yang-Baxter equations come from the quantized enveloping algebras of affine Lie algebras, which we will consider briefly in future lectures. The Lie algebra $\widehat{\mathfrak{sl}}_2$ or $U_q(\widehat{\mathfrak{sl}}_2)$, at least when q is not a root of unity, has one two-dimensional irreducible representation V_z for each $z \in \mathbb{C}^\times$. In one way imitating the proof of Theorem 4.3 is actually simpler in the affine case, for if z and w are in general position, the representation $V_z \otimes V_w$ is irreducible, so the R-matrix $V_z \otimes V_w \longrightarrow V_w \otimes V_z$ is determined up to scalar multiple. See [5] Proposition 9.2.4.

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