Lecture 12

1 Hecke algebras and Demazure-Lusztig operators

Let $W = S_n$ or more generally, let W be any Coxeter group. The theory in this section works in generality.

The Demazure operators ∂_w span a $|W|$-dimensional algebra $\mathcal{H}_0$, generated by the ∂_i. The ∂_w° give another basis of this ring, with generators $\partial_i^\circ = \partial_i - 1$. The generators satisfy the braid relations (so $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ and $\partial_w^\circ = \partial_{i_1}^\circ \cdots \partial_{i_k}^\circ$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression) and quadratic relations $\partial_i^2 = \partial_i$.

A more general ring $\mathcal{H}(W)$ depends on a parameter q. This ring has generators T_i that satisfy the braid relations and quadratic relations

$$T_i^2 = (q-1)T_i + q.$$

These first appeared in the work of Iwahori and Matsumoto [4] determining the Iwahori Hecke algebra of a p-adic group, for example $G = \mathrm{GL}(n, \mathbb{Q}_p)$. For this, we there is a Coxeter group $\tilde{W}$ called the *affine Weyl group*. The main result of Iwahori and Matsumoto is that $\mathcal{H}(\tilde{W})$ can be realized as a convolution ring of functions on G. Actually there is a slightly larger ring than $\mathcal{H}(\tilde{W})$ that can be realized this way, the *extended* affine Weyl group, but we will not discuss that.

In addition to the appearance of the Iwahori Hecke algebra as a convolution ring of functions on the p-adic group G, the same Hecke algebra appears in the theory in a seemingly different way in the theory of intertwining operators between different induce representations on the group [11].

Lusztig [9] realized that the same Hecke algebra appears in a different context, namely in the equivariant K-theory of the complex flag varieties. (See [2] for context.) Kazhdan and Lusztig [8] exploited the fact that the affine Iwahori Hecke algebra $\mathcal{H}(\tilde{W})$ appears in two different contexts to translate statements about representations of p-adic groups into algebraic geometry, where they could be proved.

The Hecke algebra $\mathcal{H}(W)$ has the following representation on functions. For definiteness we will work with just $W = S_n$ but everything would work if W is the Weyl group of any reductive algebraic group over $\mathbb{C}$. As in earlier lectures, let $T = (\mathbb{C}^\times)^n$ and let $\mathcal{O}(T)$ be the ring of functions on T spanned by the functions $\mathbf{z}^\mu$. Let $\mathcal{L}_i$ be the operator

$$\mathcal{L}_i = (\mathbf{z}^{\alpha_i} - 1)^{-1}[(1 - s_i) - q(1 - \mathbf{z}^{\alpha_i})s_i].$$

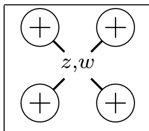
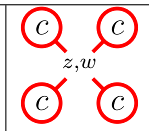
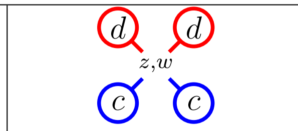
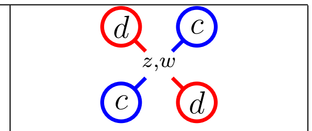
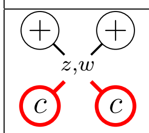
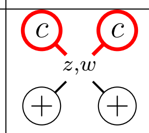
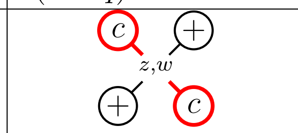
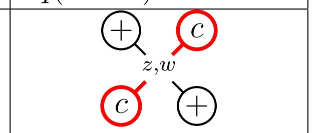
Theorem 1.1 (Lusztig)**.** *The operators $\mathcal{L}_i$ satisfy the braid and quadratic relations*

$$\mathcal{L}_i^2 = (q-1)\mathcal{L}_i + q,$$

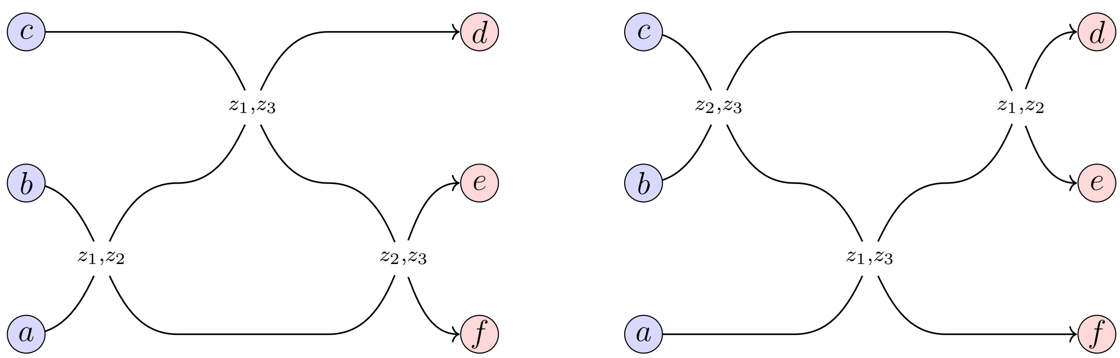
hence generate an algebra isomorphic to $\mathcal{H}(W)$.

Proof. This is just a calculation. The braid relation is somewhat tedious to check [9]. The quadratic relation is Exercise 10. □

But let us show how these operators may appear in a lattice model. We will use the following R-matrix (due to Jimbo [6] in the notation of [1])

			
$z-qw$	$z-qw$	$(1-q)z$ if $c<d$ $(1-q)w$ if $c>d$	$z-w$ if $c>d$ $q(z-w)$ if $c<d$
			
$(1-q)z$	$(1-q)w$	$q(z-w)$	$z-w$

This satisfies a parametrized Yang-Baxter equation thus:



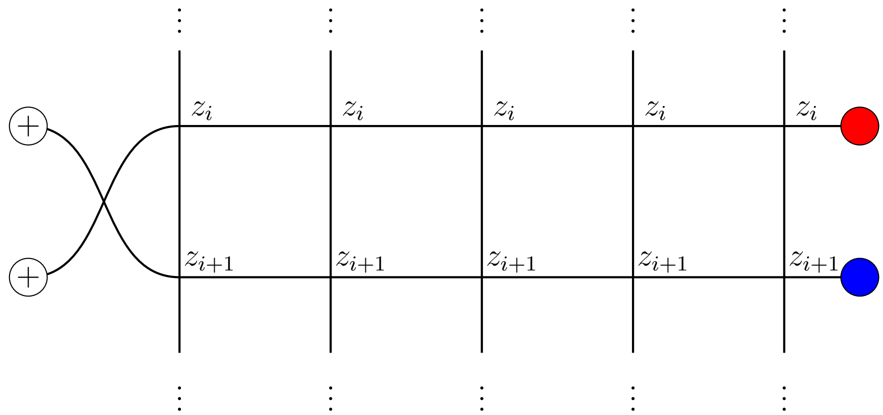
We will postulate a lattice model that uses this R-matrix. We do not need to specify the Boltzmann weights for the model itself, and indeed there are multiple choices. But let $Z(\mathbf{z};\mathbf{d})$ be the partition function, where as in our discussion of the open model, $\mathbf{d} = (d_1, \cdots, d_n)$ is a flag describing the boundary conditions on the right edge.

Proposition 1.2. *If $d_i > d_{i+1}$ then*

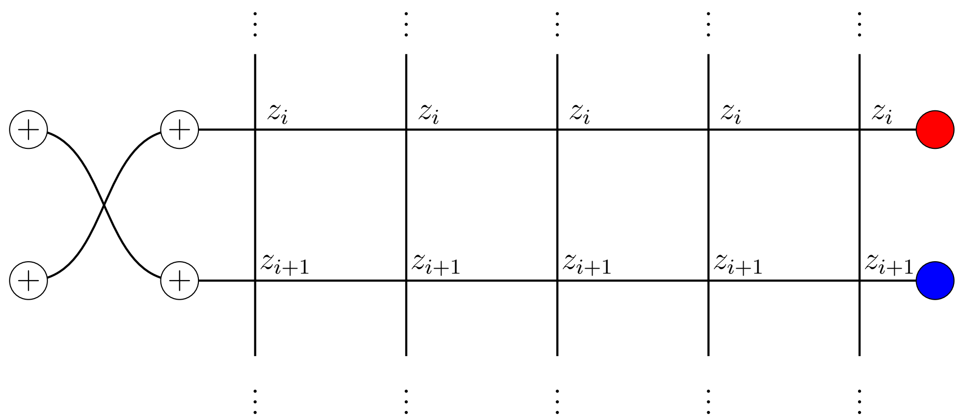
$$Z(\mathbf{z}; s_i\mathbf{d}) = \mathcal{L}_i Z(\mathbf{z};\mathbf{d}),$$

where $\mathcal{L}_i$ is the Demazure-Lusztig operator.

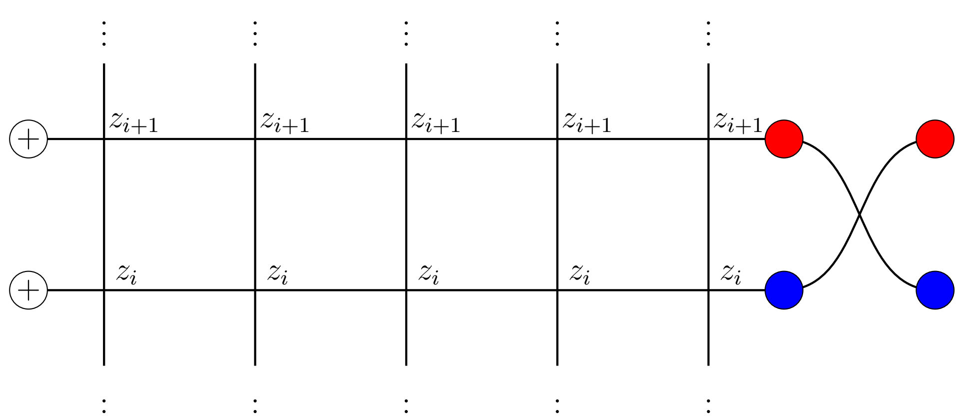
Proof. Imitating the argument in Lecture 8, Let us attach the R-matrix to the left:



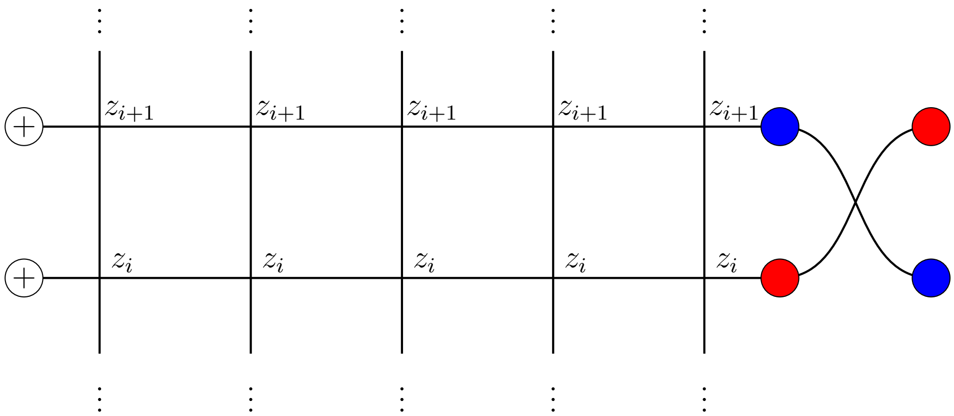
Given the spins $+$, $+$ on the left edge the spins on the R-matrix can only be all $+$, so we may assume that the configuration is as follow:



the partition function of this system is $Z(\mathbf{z}; \mathbf{d})$ times the value $z_i - qz_{i+1}$ of the R-matrix. (We are using $z = z_i$ and $w = z_{i+1}$ in the table.) Running the train argument, it turns out there are two possible configurations on the right-hand side, namely



and



Inserting the values of the R-matrices for these two configurations gives the identity

$$(z_i - qz_{i+1})Z(\mathbf{z};\mathbf{d}) = (1-q)z_i Z(s_i\mathbf{z};\mathbf{d}) + (z_i - z_{i+1})Z(s_i\mathbf{z}; s_i\mathbf{d}).$$

It will be convenient to replace $\mathbf{z} \to s_i\mathbf{z}$ so $z_i \leftrightarrow z_{i+1}$ and rewrite this identity

$$(z_{i+1} - qz_{i+1})Z(s_i\mathbf{z};\mathbf{d}) = (1-q)z_{i+1} Z(\mathbf{z};\mathbf{d}) + (z_{i+1} - z_i)Z(\mathbf{z}; s_i\mathbf{d}).$$

Reorganizing,

$$Z(\mathbf{z}; s_i\mathbf{d}) = \frac{(1-q)z_j Z(\mathbf{z};\mathbf{d}) - (z_{i+1} - qz_i)Z(s_i\mathbf{z};\mathbf{d})}{z_i - z_{i+1}} = \mathcal{L}_i Z(\mathbf{z};\mathbf{d}).$$

□

We see that the same Iwahori Hecke algebra, by its representation by Demazure-Lusztig operators appears in at least three completely different places: the representation theory of p-adic groups (Iwahori-Matsumoto) the equivariant K-theory of the complex flag variety (Kazhdan-Lusztig) and now the theory of solvable lattice models.

2 Groups

Solutions to the Yang-Baxter equation have at least two types of applications:

- Solvable lattice models;
- Knot invariants, such as the Jones and Alexander polynomials.

Therefore it is of interest that there is a mechanism that produces a variety of solutions to the Yang-Baxter equation, underlying most of the examples that we need. This is the theory of *quantum groups*, invented by Drinfeld [3] and Jimbo [5]. For modern treatments [7, 10].

Quantum groups are actually *Hopf algebras*. In this lecture we will introduce Hopf algebras and show how groups can produce Hopf algebras.

Please review the notions of *monoidal category* and *braided category* in Lecture 4.

The notion of a *Hopf algebra* is extremely similar to the notion of a group, but more flexible. Quantum groups, as defined by Drinfeld, are actually Hopf algebras, so we start with those.

To motivate the definition, let us formulate the axioms of a group "categorically." We work in the category of sets, which has products and a terminal object, namely the set $I = \{1_I\}$ with one element. The category of sets is a monoidal category with $\times$ its operation and I its unit element. It is actually a *symmetric* monoidal category, which is a special case of a braided monoidal category. In a braided category, there are morphisms $c_{A,B} : A \times B \longrightarrow B \times A$ for objects A and B. In a symmetric monoidal category, we assume further that the composition

$$A \times B \stackrel{c_{A,B}}{\longrightarrow} B \times A \stackrel{c_{B,A}}{\longrightarrow} A \times B$$

is the identity map.

The reason for this digression into the definition of a group, the idea is that if we formulate the notion correctly we can apply it in other symmetric monoidal categories, particularly the category of vector spaces. We will find that if we take the definition of a group, formulated categorically, and apply it in the category of vector spaces, we get a useful notion, that of a Hopf algebra.

Let G be a group. Let $\varepsilon : I \longrightarrow G$ be the map that sends 1_I to 1_G. Let $\mu : G \times G \longrightarrow G$ be multiplication, and $S : G \longrightarrow G$ the inverse map. We will also need to make use of the diagonal map $\Delta : G \longrightarrow G \times G$ that sends g to (g, g), and the map $\eta : G \longrightarrow I$ that sends $g \mapsto 1_I$.

Then the following properties are satisfied.

The Associative Law: The following diagram is commutative.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times 1} & G \times G \\ \Big\downarrow{\scriptstyle 1 \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\quad \mu \quad} & G \end{array}$$

The Unit axiom: The following diagrams is commutative. The maps $I \times G \cong G$ and $G \times I \cong G$ are the obvious ones. These maps are part of the data making the category of sets into a monoidal category.

$$\begin{array}{ccc} I \times G & \xrightarrow{\varepsilon \times 1} & G \times G \\ \Big\downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \mu} & \\ G & & \end{array} \qquad\qquad \begin{array}{ccc} G \times I & \xrightarrow{1 \times \varepsilon} & G \times G \\ \Big\downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \mu} & \\ G & & \end{array}$$

The diagonal map Δ and counit η have dual properties to the associative law and unit axiom.

Coassociativity: The following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\Delta} & G \times G \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\times\Delta} \\ G \times G & \xrightarrow{\Delta\times 1} & G \times G \times G \end{array}$$

Counit: The following diagrams are commutative.

$$\begin{array}{ccc} I \otimes G & \xleftarrow[\eta\times 1]{} & G \times G \\ \downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle\Delta} & \\ G & & \end{array} \qquad \begin{array}{ccc} G \otimes I & \xleftarrow[1\times\eta]{} & G \times G \\ \downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle\Delta} & \\ G & & \end{array}$$

The group law requires $g \cdot S(g) = S(g) \cdot g = 1_G$, and we may formulate these diagramatically thus:

Antipode: The following diagram is commutative.

$$\begin{array}{ccccc} G \times G & & \xrightarrow{1\times S} & & G \times G \\ {\scriptstyle\Delta}\uparrow & & & & \downarrow{\scriptstyle\mu} \\ G & \xrightarrow{\eta} & I & \xrightarrow{\varepsilon} & G \end{array} \qquad \begin{array}{ccccc} G \times G & & \xrightarrow{S\times 1} & & G \times G \\ {\scriptstyle\Delta}\uparrow & & & & \downarrow{\scriptstyle\mu} \\ G & \xrightarrow{\eta} & I & \xrightarrow{\varepsilon} & G \end{array}$$

There is one more property that is needed, and this requires the "flip" map $\tau : G \times G \longrightarrow G \times G$ that sends $(x, y) \mapsto (y, x)$. This is just a new notation for $c_{G,G}$ in the definition of a symmetric monoidal category.

Hopf: The following diagram commutes.

$$\begin{array}{ccccc} G \times G & \xrightarrow{\Delta\times\Delta} & G \times G \times G \times G & \xrightarrow{1\times\tau\times 1} & G \times G \times G \times G \\ \downarrow{\scriptstyle\mu} & & & & \downarrow{\scriptstyle\mu\times\mu} \\ G & & \xrightarrow{\Delta} & & G \times G \end{array}$$

Indeed both compositions are the map $(g, h) \longmapsto (gh, gh)$.

3 Hopf Algebras

The same axioms but applied in the category of vector spaces produces the notion of a *Hopf algebra*. We work over a field F. Then category of F-vector spaces is a monoidal category with unit element F and monoidal operation $\otimes$ (tensor product).

But let us start with just the first two axioms. We need a vector space H with a vector space homomorphism $\mu : H \otimes H \longrightarrow H$ and a homomorphism $\varepsilon : F \longrightarrow H$. Note that these data are equivalent to a bilinear operation $H \times H \longrightarrow H$ and a distinguished element $1_H := \varepsilon(1_F)$.

Associative:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\mu\otimes 1} & H \otimes H \\ \downarrow{\scriptstyle 1\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ H \otimes H & \xrightarrow{\mu} & H \end{array}$$

Unit:

$$\begin{array}{ccc} \mathbb{C} \otimes H & \xrightarrow{\varepsilon\otimes 1} & H \otimes H \\ \downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \mu} & \\ H & & \end{array} \qquad \begin{array}{ccc} H \otimes I & \xrightarrow{1\otimes\varepsilon} & H \otimes H \\ \downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \mu} & \\ H & & \end{array}$$

The isomorphisms $H \cong F \otimes H \cong H \otimes F$ are part of the data making the category of vector spaces into a monoidal category.

Given such μ we can define a bilinear composition law $H \times H \longrightarrow H$ by $x \cdot y = \mu(x \otimes y)$. The two axioms mean that H is an associative F-algebra.

We return to generalizing the group axioms. The role of the diagonal map is played by a lnear map $\Delta : H \longrightarrow H \otimes H$, and we also need a *counit* which is a linear map $\eta : H \longrightarrow F$. We need:

Coassociative:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ H \otimes H & \xrightarrow{\Delta\otimes 1} & H \otimes H \otimes H \end{array}$$

Counit:

$$\begin{array}{ccc} I \otimes H & \xleftarrow[\eta\otimes 1]{} & H \otimes H \\ \downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \Delta} & \\ H & & \end{array} \qquad \begin{array}{ccc} H \otimes I & \xleftarrow[1\otimes\eta]{} & H \otimes H \\ \downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \Delta} & \\ H & & \end{array}$$

A vector space with maps Δ and η satisfying these two axioms is called a *coalgebra*.

Now we need a linear map $S : H \longrightarrow H$ and two more axioms:

Antipode:

$$\begin{array}{ccccc} H \otimes H & & \xrightarrow{1\otimes S} & & H \otimes H \\ {\scriptstyle \Delta}\uparrow & & & & \downarrow{\scriptstyle \mu} \\ H & \xrightarrow{\eta} & I & \xrightarrow{\varepsilon} & H \end{array} \qquad \begin{array}{ccccc} H \otimes H & & \xrightarrow{S\otimes 1} & & H \otimes H \\ {\scriptstyle \Delta}\uparrow & & & & \downarrow{\scriptstyle \mu} \\ H & \xrightarrow{\eta} & I & \xrightarrow{\varepsilon} & H \end{array}$$

Hopf:

$$\begin{array}{ccccc}
H \otimes H & \xrightarrow{\Delta\otimes\Delta} & H\otimes H\otimes H\otimes H & \xrightarrow{1\otimes\tau\otimes 1} & H\otimes H\otimes H\otimes H \\
\downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu\otimes\mu} \\
H & & \xrightarrow{\qquad\qquad\Delta\qquad\qquad} & & H\otimes H
\end{array}$$

If A, B are algebras, so is $A \otimes B$. If A, B are co-algebras, so is $A \otimes B$. The Hopf axiom can be expressed as saying either that the comultiplication $\Delta : H \longrightarrow H \otimes H$ is an algebra homomorphism, or (equivalently) that the multiplication $\mu : H \otimes H \longrightarrow H$ is a coalgebra homomorphism.

Let us give some examples of Hopf algebras that arise from groups. First, let G be a finite group, and let $\mathbb{C}[G]$ be its group algebra. This is a Hopf algebra. To define the comultiplication, we extend the diagonal map $\Delta : G \longrightarrow G \times G$ to a map $\mathbb{C}[G] \longrightarrow \mathbb{C}[G \times G] \cong \mathbb{C}[G] \otimes \mathbb{C}[G]$ by linearity. The counit is the augmentation map $\mathbb{C}[G] \longrightarrow \mathbb{C}$. The multiplication of G is encoded in the multiplication of $\mathbb{C}[G]$.

On the other hand, let $\mathcal{O}(G)$ be the commutative algebra of functions on G. The multiplication is just pointwise multiplication. For the comultiplication, let $\delta_g \in \mathcal{O}(G)$ be the basis of $\mathcal{O}(G)$ defined for $g \in G$ by

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta(\delta_g) = \bigoplus_{\substack{(x,y)\in G \\ xy=g}} \delta_x \otimes \delta_y.$$

These two Hopf algebras are in duality. This means we have a dual pairing $\mathbb{C}[G] \otimes \mathcal{O}(G) \longrightarrow \mathbb{C}$ defined by $g \otimes f \longmapsto f(g)$, which make the multiplication in $\mathbb{C}[G]$ dual to the comultiplication in $\mathcal{O}(G)$.

Similarly if G is a complex Lie group, such as $\mathrm{GL}(n, \mathbb{C})$, we have two types of Hopf algebras. On the one hand, there is the universal enveloping algebra $U(\mathfrak{g})$ where $\mathfrak{g} = \mathrm{Lie}(G)$. On the other hand, regarding G as an affine algebraic group, there is the commutative algebra $\mathcal{O}(G)$ of polynomial functions on G. These are Hopf algebras, and they are related to each other by a dual pairing.

Both $U(\mathfrak{g})$ and $\mathcal{O}(G)$ have deformations (depending on a parameter q). Denoting these as $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, the category of modules over $U_q(\mathfrak{g})$ is braided, or the category of comodules for $\mathcal{O}_q(G)$. As a variant, instead of the finite-dimensional algebra $U_q(\mathfrak{g})$ we may use $U_q(\hat{\mathfrak{g}})$ where $\hat{\mathfrak{g}}$ is an affine Lie algebra.

This gives rise to many examples of the Yang-Baxter equation. For example, the R-matrix that we ended the last section is associated with pairs of representations of $U_q(\widehat{\mathfrak{gl}}(n+1))$.

An important use of the comultiplication in a Hopf algebra is that the category of finite-dimensional representations is a monoidal category. That is, given H-modules V and W we can give $V \otimes W$ the structure of an H-module.

Naturally $V \otimes W$ is a module over the algebra $H \otimes H$, and this may be true for modules over any associative algebra. But we want it to be a module over H, and we may do this using an algebra homomorphism $H \to H \otimes H$, and for this we use the comultiplication.

Drinfeld's accomplishment was to show how to define a class of Hopf algebras called *quasitriangular*. The beauty of quasitriangular Hopf algebras is that the module category is not just monoidal, it is braided. The braiding is described thus: there is, in $H \otimes H$ an invertible element R called the *universal R-matrix* which has the property that if U, V are H-modules then the map $c_{U,V} : U \otimes V \to V \otimes U$ defined by τR is a braiding.

The Hopf algebra $U_q(\mathfrak{g})$ (which we have not yet defined) is not quasitriangular, since the universal R-matrix is given by an infinite sum and hence lives not in $H \otimes H$ but in a completion. This is not really a problem since there are various workarounds, so $U_q(\mathfrak{g})$ is "morally quasitriangular," and the module category is braided.

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