Lecture 11

1 Open models and Demazure atoms

Let $\mathfrak{S}_\lambda(\mathbf{z}; q)$ be the Tokuyama models as in Lectures 5 and 6. We proved that the partition function $Z_\lambda(\mathbf{z}; q)$ of these models equals

$$Z_\lambda(\mathbf{z}; q) = \prod_{i<j}(z_i - qz_j)s_\lambda(\mathbf{z})$$

Let $Y_\lambda(\mathbf{z}; q) = \mathbf{z}^{-\rho} Z_\lambda(\mathbf{z}; q)$. Dividing the last equation by $\mathbf{z}^\rho = z_1^{n-1} z_2^{n-2} \cdots$ gives

$$Y_\lambda(\mathbf{z}; q) = \prod_{\alpha \in \Phi^+} (1 - q\mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z}).$$

We are specializing to the case $q = 0$ in order to view some phenomena concerned with colored models in their simplest cases. These are:

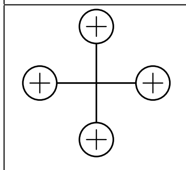
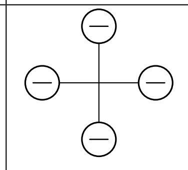
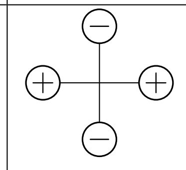
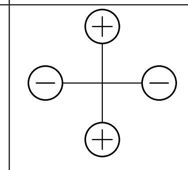
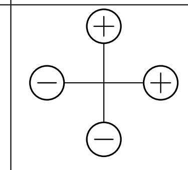
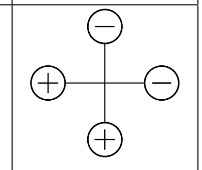
- There is a relationship between the colored and uncolored models;
- In a family of colored models there are some that are *monostatic* meaning that the system has only one state, and the partition function is therefore easy to evaluate;
- The Yang-Baxter equation gives us a recursion relations between partition functions of models in the category involving Demazure operators.

Such phenomena can be seen in other models such as [2, 3, 1, 7] and many other examples. This is therefore an important paradigm.

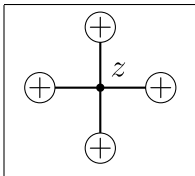
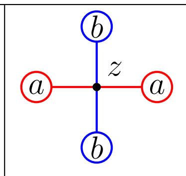
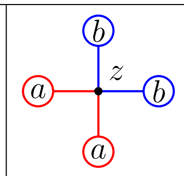
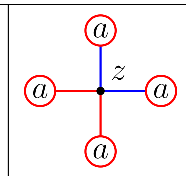
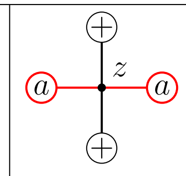
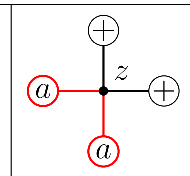
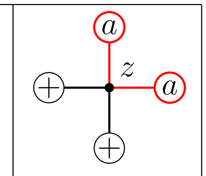
The Tokuyama models have at least two variants, which we are calling the open and closed models. The open models are investigated in [5], and the closed models are (as far as I know) not in any published literature.

Let $\mathfrak{S}^\circ_\lambda(\mathbf{z}; \mathbf{d})$ be the colored model, where $\mathbf{d}$ is a flag of colors, which we covered in Lectures 8 and 9. We may also write $\mathbf{d} = w\mathbf{c}_0$ where $\mathbf{c}_0 = (c_1, \cdots, c_n)$ is the "standard flag" of colors in decreasing order: $c_1 > \ldots > c_n$, and $w \in W = S_n$ is a permutation. We recall the results that we have already proved. We will denote $Y^\circ_\lambda(\mathbf{z}; \mathbf{d}) = \mathbf{z}^{-\rho} Z^\circ_\lambda(\mathbf{z}; \mathbf{d})$.

For convenience, here are the Boltzmann weights of the two types of systems. For the Tokuyama with $q = 0$:

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
					
1	z_i	0	z_i	z_i	1

For the closely related open models:

						
1	$z \quad a > b$ $0 \quad a < b$	$0 \quad a > b$ $z \quad a < b$	z	z	z	1

One relationship between the two models is easy to see: if we take the open model but only use a single color, we recover the Tokuyama models with $q = 0$.

But we will show a *deeper* relationship, by showing that the partition function of the uncolored Tokuyama $q = 0$ model can be decomposed into a sum (over flags $\mathbf{d}$) of partition functions of the uncolored model.

Proposition 1.1. *The model $\mathfrak{S}_\lambda(\mathbf{z}; \mathbf{c}_0)$ has only one state. The normalized partition function*

$$Y^\circ_\lambda(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^\lambda.$$

Proof. This is Proposition 3.1 in Lecture 9. □

Lemma 1.2. *If $s_i w > w$ then*

$$Y_\lambda(\mathbf{z}; s_i w \mathbf{c}_0) = \partial^\circ_i Y_\lambda(\mathbf{z}; w\mathbf{c}_0)$$

Proof. This is proved (using the Yang-Baxter equation) as Proposition 3.2 in Lecture 9, with the assumption $d_i > d_{i+1}$. This assumption is equivalent to $s_i w > w$ by Exercise 8(i). □

Proposition 1.3. *If $s_i w > w$, then*

$$Y^\circ_\lambda(\mathbf{z}; \mathbf{d}) = \partial^\circ_w \mathbf{z}^\lambda.$$

Proof. This follows by induction from the last two propositions. □

The polynomial $\partial^\circ_w \mathbf{z}^\lambda$ is called a *Demazure atom.*

Proposition 1.4. *We have*

$$s_\lambda(\mathbf{z}) = \sum_{w \in W} \partial^\circ_w \mathbf{z}^\lambda. \tag{1}$$

Proof. Take $w = w_0$ (the longest element in W) in Theorem 2.1 of Lecture 10. Since $w \leqslant w_0$ for all $w \in W$, this gives

$$\sum_{w \in W} \partial_w^\circ \mathbf{z}^\lambda = \partial_{w_0} \mathbf{z}^\lambda.$$

This equals the Schur polynomial $s_\lambda(\mathbf{z})$ by the Demazure character formula ([6] Theorem 25.3). $\square$

Now in equation (1) the left-hand side is the partition function of the uncolored $q = 0$ model by Proposition 1.1, and the right hand side is the *sum* of the partition functions of the open models. This shows that there is a relationship between the open and uncolored models. Let us prove this directly.

Theorem 1.5. *We have*

$$Z_\lambda(\mathbf{z}; 0) = \sum_{w \in W} Z_\lambda^\circ(\mathbf{z}; w\mathbf{c}_0).$$

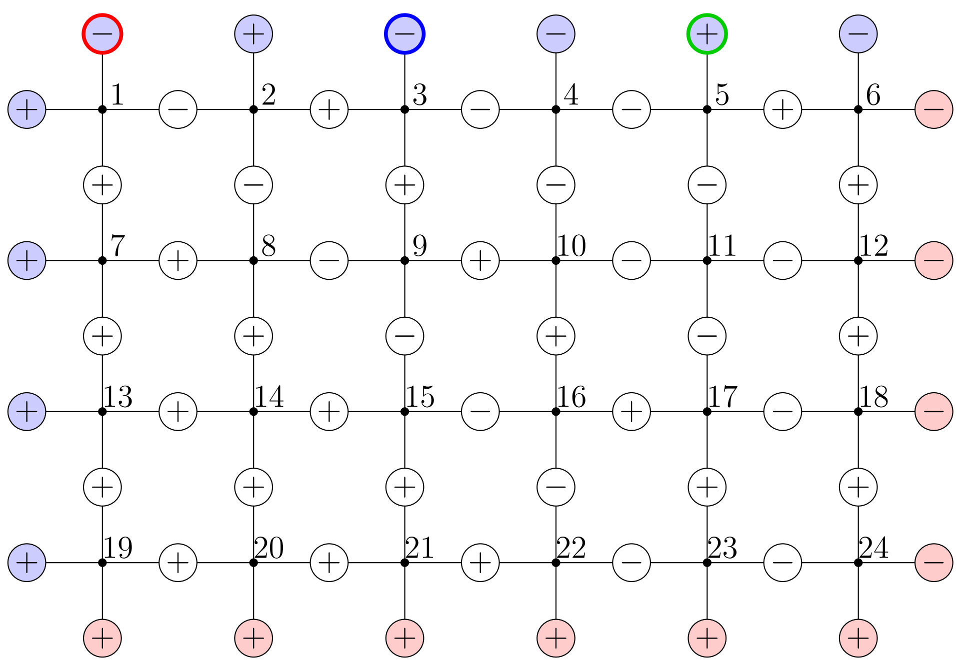
Proof. We will show that there is a bijection

$$\mathfrak{S}_\lambda(\mathbf{z}; 0) \quad \longleftrightarrow \quad \bigsqcup_{\mathbf{d}} \mathfrak{S}_\lambda^\circ(\mathbf{z}; \mathbf{d})$$

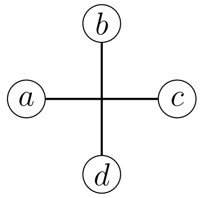
in which corresponding states have the same Boltzmann weights. The existence of maps $\phi_{\mathbf{d}} : \mathfrak{S}_\lambda^\circ(\mathbf{z}; \mathbf{d}) \longrightarrow \mathfrak{S}_\lambda(\mathbf{z}; 0)$ is easy: we just take a state $\mathfrak{s}^\circ$ of any one of the models $\mathfrak{S}_\lambda^\circ(\mathbf{z}; \mathbf{d})$ and replace every colored spin by $-$ to obtain a state of $\mathfrak{S}_\lambda(\mathbf{z}; 0)$.

So what we need to show is that every state $\mathfrak{s}$ of $\mathfrak{S}_\lambda(\mathbf{z}; 0)$ is $\phi_{\mathbf{d}}(\mathfrak{s}^\circ)$ for a unique state $\mathfrak{s}^\circ$ of one of the systems $\mathfrak{S}_\lambda^\circ(\mathbf{z}; \mathbf{d})$. About the desired $\mathfrak{s}^\circ$ we know which edges will be colored. Of the boundary edges, we know the colors of the edges on the top, since the boundary conditions put color c_i in the $\lambda_i + n - i$ column, and $+$ elsewhere. We do not know the colors of the edges at the right, since we do not know for which $\mathbf{d}$ we will have $\mathfrak{s} \in \mathfrak{S}_\lambda(\mathbf{z}; \mathbf{d})$.

Our procedure will be to "color" the uncolored state $\mathfrak{s}$ by replacing the $-$ spins in order by colors. We order the edges of the grid from left to right and from top to bottom as follows.



visiting each vertex in this order, let the boundary spins be labeled in this order:



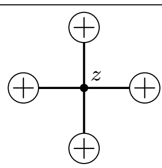
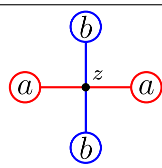
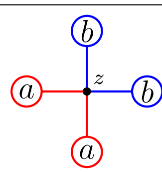
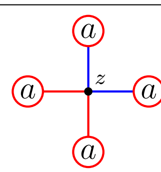
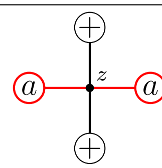
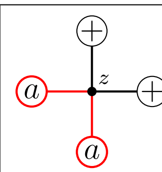
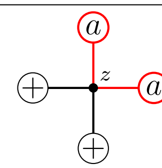
Because we have already considered all prior vertices, the spins a and b in the colored state $\mathfrak{s}^\circ$ are already determined. As for the colored spins c and d, they are not determined, but we will argue that they have a unique possible assignment. Because we know the uncolored state $\mathfrak{s}$, we know whether c and d are colors or $+$, and we know what colors (or $+$) a and b are. From the Boltzmann weights, there is a unique assignment of colors (or $+$) to c and d. For example, if a and b are both colors, c and d will be the same two colors, and c will be the larger of the two.

Visiting all the colors in order, we find there is a unique colored spin $\mathfrak{s}^\circ$ such that $\phi(\mathfrak{s}^\circ) = \mathfrak{s}$. We can read off the flag $\mathbf{d}$ from the colors on the right edge. $\square$

Remark 1. Theorem 1.5 implies Proposition 1.4, which was not used in its proof. Thus we have a proof of the Demazure character formula for $\mathrm{GL}(n)$.

2 Closed models

Here for convenience are the Boltzmann weights for the closed models.

						
1	$z \quad a > b$ $0 \quad a < b$	$z \quad a > b$ $0 \quad a < b$	z	z	z	1

Boundary conditions are the same as for the open model. Let $\mathfrak{S}^\bullet_\lambda(\mathbf{z};\mathbf{d})$ be the closed model, and let $Z^\bullet_\lambda(\mathbf{z};\mathbf{d})$ be its partition function. We will again denote by

$$Y^\bullet_\lambda(\mathbf{z};\mathbf{d}) = \mathbf{z}^{-\rho} Z^\bullet_\lambda(\mathbf{z};\mathbf{d})$$

the normalized partition function.

Proposition 2.1. *We have*

$$Y_\lambda(\mathbf{z};\mathbf{d}) = \partial_w \mathbf{z}^\lambda.$$

Proof. This is Exercise 9. $\square$

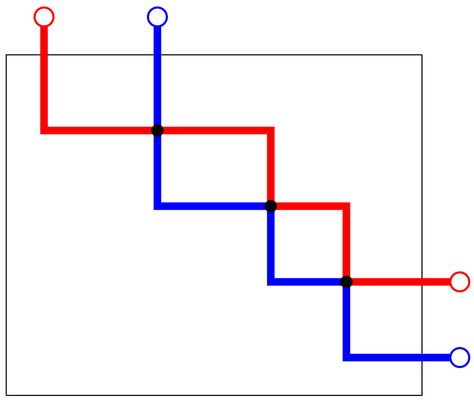
The polynomial $\partial_w \mathbf{z}^\lambda$ is called a *Demazure character* or *key polynomial*. In Lecture 10 we proved

$$\partial_w \mathbf{z}^\lambda = \sum_{y \leqslant w} \partial_y^\circ \mathbf{z}^\lambda,$$

expressing the Demazure character as a sum of Demazure atoms. In the extreme case that $w = w_0$ is the long element, the Demazure character formula ([6] Theorem 25.3) asserts that $\partial_{w_0} \mathbf{z}^\lambda = s_\lambda(\mathbf{z})$, so the closed model and the Tokuyama $q = 0$ model have the same partition function.

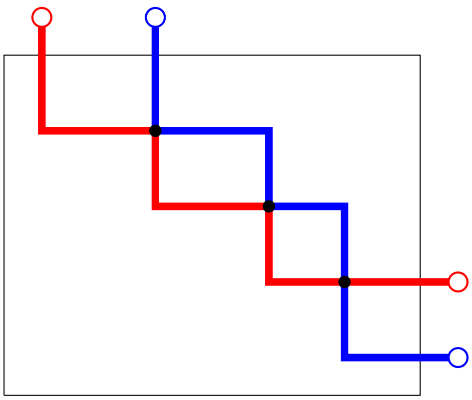
We can get an intimation of the reason for the difference between the open and closed models from Exercises 5 and 6. Let us consider two colored paths in the open models. Let us assume that the paths meet several times, as in Exercise 5.

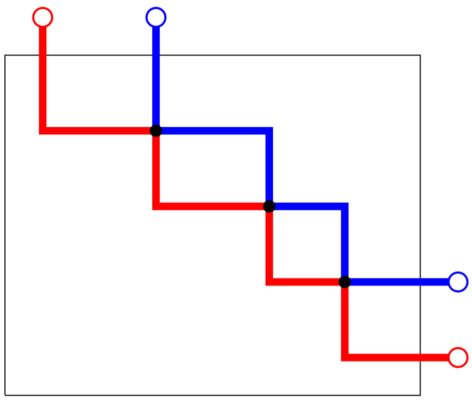
For the open model, the paths must be colored as follows (with red $>$ blue).



Thus in the open model paths must cross the first time they meet, but may not cross again. This may be seen easily from consideration of the Boltzmann weights.

Now in the closed models, there are two possibilities (Exercise 5).





The paths may cross, or not cross, but if they do cross, it must be the *last* time they meet. This extra flexibility causes the difference between the open and closed models.

3 Generalizations

The open and closed models may be described as refinements of the Tokuyama $q = 0$. Refinements of the general Tokuyama model for general q are described in [3].

We recall that the normalized partition function $Y(\mathbf{z}; q) = \mathbf{z}^{-\rho} Z(\mathbf{z}; q)$ equals

$$\prod_{\alpha \in \Phi^+} (1 - q^{-1} \mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z}). \tag{2}$$

This product appears in a very different context: a famous formula of Casselman and Shalika [8] for Whittaker functions on p-adic groups (particularly $\mathrm{GL}(n)$, where the formula was found earlier by Shintani) gives (2) as the values of the spherical Whittaker function.

The models in [3] are open models in that they break this function into pieces $Y(\mathbf{z}; q; w\mathbf{c}_0)$ whose sum equals (2). There are some unexpected phenomena.

- Some vertical edges may carry more than one color; that is, two colored paths may travel along the same vertical edge. However the models are still fermionic since each vertical edge may not carry more than one instance of the same color.

- There are some unexpected cancellations: terms coming from different w may appear as "twins" with opposite sign that cancel out in the sum (2).

- Generalizing the Casselman-Shalika formula, the partition functions can be identified with the values of Iwahori Whittaker functions on $\mathrm{GL}(n, F)$ where F is a p-adic field.

Although there are new phenomena, the general framework for understanding these models is the same as for the open models that we have devoted a few lectures to. That is, if $\mathbf{d} = \mathbf{c}_0$ the systems are monostatic, and the partition functions are easy to evaluate; and in the general case, they satisfy Demazure recursion that can be proved using the Yang-Baxter equation. These are sufficient to evaluate the partition functions in a useful way.

Other related fermionic models appear in [4, 1]. There are other related models including bosonic variants ([2, 7]) and models with column parameters.

4 Open questions about closed models

Question 1. Are there also closed models that are related to the Tokuyama models for general q? This is not known if $q \neq 0$.

Question 2. The proof of Theorem 1.5 gives maps $\mathfrak{S}^\circ_\lambda(\mathbf{z}; w\mathbf{c}_0) \longrightarrow \mathfrak{S}_\lambda(\mathbf{z}; 0)$ and similarly we can find maps $\mathfrak{S}^\bullet_\lambda(\mathbf{z}; w\mathbf{c}_0) \longrightarrow \mathfrak{S}_\lambda(\mathbf{z}; 0)$. Let us therefore identify $\mathfrak{S}^\circ_\lambda(\mathbf{z}; w\mathbf{c}_0)$ and $\mathfrak{S}^\bullet_\lambda(\mathbf{z}; w\mathbf{c}_0)$ with their images in $\mathfrak{S}_\lambda(\mathbf{z}; 0)$. Since

$$Y^\bullet_\lambda(\mathbf{z}; w\mathbf{c}_0) = \sum_{y \leqslant w} Y^\circ_\lambda(\mathbf{z}; w\mathbf{c}_0)$$

(which is equivalent to the last result in Lecture 10) we expect that

$$\mathfrak{S}^{\bullet}_{\lambda}(\mathbf{z}; w\mathbf{c}_0) = \bigcup_{y \leqslant w} \mathfrak{S}^{\circ}_{\lambda}(\mathbf{z}; y\mathbf{c}_0) \qquad \text{(disjoint)}.$$

The problem is to prove this. The path discussion in Section 2 is a promising starting point, but I do not think this is obvious.

References

[1] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. *Comm. Amer. Math. Soc.*, 3:400–630, 2023.

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