Lecture 10

1 Bruhat order

Most of the facts that I need about the Bruhat order are covered in Chapter 25 of [4]. Therefore I recommend that you read this chapter. (The book is available on-line through the Stanford Libraries.) Watch out for the following typo: in (25.7) the wrong font is used and D should be ∂. For this section W can be the symmetric group, or a more general Coxeter group, though the geometric

I will give geometric proofs of the fact that S_n is a Coxeter group, that is, that it has a presentation:

$$S_n \cong \left\langle s_i | s_i^2 = 1, \text{braid relations} \right\rangle$$

and Matsumoto's theorem (Lecture 9). Referring to the book, these proofs are Theorem 25.1 (page 214) and Theorem 25.2 (page 217). These types of geometric arguments might be unsatisfactory since the results can be proved by purely algebraic methods in greater generality. However the technique is very powerful and useful. See [5] for applications of such geometric ideas.

I will also give a similar geometric proof of the *exchange principle* which is Proposition 20.3 or Proposition 20.4.

Proposition 1.1. *Let $w = s_{i_1} \cdots s_{i_k}$ be a product of k simple reflections such that $\ell(w) < k$. Then it is possible to omit two of the factors and get another reduced expression:*

$$w = s_{i_1} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k},$$

where the "hat" means a factor is omitted, with $1 \leqslant a < b \leqslant k$.

Proof. This is Proposition 20.4 in [4], and in class I will give a geometric proof similar to the geometric proofs of the Coxeter property and Matsumoto's theorem mentioned above. The exchange property is valid for any Coxeter group, and a purely algebraic proof may be found in [2], Section IV.1.5. Another proof can be found in [7] Section 1.7 (pages 13–15). □

Proposition 1.2 (Exchange principle)**.** *Suppose that $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression and s_j a simple reflection such that $\ell(s_j w) < \ell(w)$. (Reduced means that $k = \ell(w)$.) Then we may find another reduced expression*

$$w = s_j s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k} \tag{1}$$

for some $1 \leqslant a \leqslant k$, where the "hat" means a factor is omitted.

Proof. Let us observe how this follows from Proposition 1.1. We have

$$s_j w = s_j s_{i_1} \cdots s_{i_k} = s_{i_0} s_{i_1} \cdots s_{i_k}, \qquad i_0 := j.$$

Since $\ell(s_j w) < k$ this expression is not reduced. Therefore we may omit two factors on the right and obtain a reduced expression for $s_j w$:

$$s_j w = s_{i_0} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k}.$$

Now we claim that $a = 0$, since if not, we have

$$w = s_{i_1} \cdots \widehat{s_{i_a}} \cdots \widehat{s_{i_b}} \cdots s_{i_k}$$

contradicting our assumption that $\ell(w) = k$. Thus

$$s_j w = s_{i_1} \cdots \widehat{s_{i_b}} \cdots s_{i_k},$$

proving (1). □

Proposition 1.3. *Suppose that s is a simple reflection and $\ell(sw) < \ell(w)$. Then w has a reduced expression $w = s_{i_1} \cdots s_{i_k}$ such that $s_{i_1} = s$.*

Proof. Let $w = s_{j_1} \cdots s_{j_k}$ be a reduced expression. Then by the exchange principle, $w = s s_{j_1} \cdots \hat{s}_{j_a} \cdots s_{j_k}$ for some a, and this is the required reduced expression. □

Next we come to the *Bruhat order* on $W = S_n$ (or a more general Coxeter group). This is defined on page 222 of [4]. See [1] for more information about this very important concept.

Let $u, v \in W$, and let $v = s_{i_1} \cdots s_{i_k}$ be a reduced expression. We write $u \leqslant v$ if there is a subsequence $(j_1, \cdots, j_l)$ of $(i_1, \cdots, i_k)$ such that $u = s_{j_1} \cdots s_{j_l}$.

Proposition 1.4. *(i) This definition does not depend on the choice of reduced expression $v = s_{i_1} \cdots s_{i_k}$.*

(ii) If there exists any sequence $(j_1, \cdots, j_l)$ such that $u = s_{j_1} \cdots s_{j_l}$ then there exists such a sequence such that this is a reduced expression.

Proof. For (i) see [4], Proposition 25.4 for a deduction of this from Matsumoto's theorem. For (ii), if the expression $u = s_{j_1} \cdots s_{j_l}$ is found and is not reduced (so $\ell(u) < l$) then by Proposition 1.1 we may discard entries in pairs to shorten the expression until it is reduced. □

Lemma 1.5. *If s is a simple reflection and $w \in W$ then either $sw > w$ or $sw < w$. Indeed $sw < w$ if and only if $\ell(sw) < \ell(w)$, in which case $\ell(sw) = \ell(w) - 1$; and $sw > w$ if and only if $\ell(sw) = \ell(w) + 1$.*

Proof. It follows easily from the definition that $\ell(w)$ is the length of the shortest expression of w as a product of simple reflections that $\ell(sw) = \ell(w) \pm 1$.

Write $w = s_{i_1} \cdots s_{i_k}$. If $\ell(sw) > \ell(w)$ then $sw = s s_{i_1} \cdots s_{i_k}$ is a reduced expression so that $w < sw$ in the Bruhat order. On the other hand if $\ell(sw) < \ell(w)$ by the exchange principle, we may write $sw = s_{i_1} \cdots \hat{s}_{i_a} \cdots s_{i_k}$ so $sw < w$. □

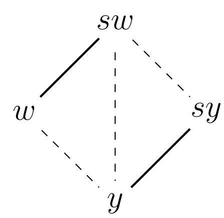
Proposition 1.6 (Deodhar's Property Z [6]). *Let $y, w \in W$ and let s be a simple reflection. Assume that $w < sw$ and $y < sy$. Then the following are equivalent:*

(i) $y \leqslant w$*;*

(ii) $y \leqslant sw$*;*

(iii) $sy \leqslant sw$*.*

Here is a lattice diagram illustrating this fact:



The solid lines represent the assumed inequalities $w < sw$ and $y < sy$. Then the dotted lines are the three equivalent statements.

Proof. (i) $\Rightarrow$ (iii): Assume $y \leqslant w$. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression for w and let $y = s_{j_1} \cdots s_{j_l}$ be a reduced expression for y such that $(j_1, \cdots, j_l)$ is a subword of $(i_1, \cdots, i_k)$. Since $\ell(sw) = \ell(w)+1$ the expression $ss_1 \cdots s_{i_k}$ is a reduced expression for sw, and $ss_{j_1} \cdots s_{j_l}$ is a subexpression representing sy, so $sy \leqslant sw$.

(iii) $\Rightarrow$ (ii): Assume $sy \leqslant sw$. Then $y < sy \leqslant sw$, as required.

(ii) $\Rightarrow$ (i). Assume $y < sw$. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression for w. Since $\ell(sw) = \ell(w) + 1 = k + 1$, the expression $sw = ss_{i_1} \cdots s_{i_k}$ is reduced, and y can be obtained from this by discarding factors. So if we take $s_{i_0} = s$, then we have a reduced expression $y = s_{j_1} \cdots s_{j_l}$ where $(j_1, \cdots, j_l)$ is a subsequence of $(i_0, i_1, \cdots, i_k)$. Now j_1 cannot be i_0 because this would imply that $sy = s_{j_1} y < y$. Therefore $(j_1, \cdots, j_l)$ is a subsequence of $(i_1, \cdots, i_k)$ which implies that $y \leqslant w$. $\square$

2 The relationship between ∂_w° and ∂_w

Let $\partial_i^\circ = (\mathbf{z}^{\alpha_i} - 1)^{-1}(1 - s_i)$ and $\partial_i = (1 - \mathbf{z}^{-\alpha_i})^{-1}(1 - \mathbf{z}^{\alpha_i} s_i)$ as before. We proved in Lecture 9 that both species of Demazure operators satisfy the braid relation, and so we may define

$$\partial_w^\circ = \partial_{i_1}^\circ \cdots \partial_{i_k}^\circ, \qquad \partial_w^\circ = \partial_{i_1} \cdots \partial_{i_k}$$

where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression, and by Matsumoto's theorem these are well-defined.

Theorem 2.1. *We have*

$$\partial_w = \sum_{y \leqslant w} \partial_y^\circ \tag{2}$$

Proof. (From [3].) We prove this by induction on $\ell(w)$. If $w = 1$, then $\partial_1 = \partial_1^\circ$ is the identity operator and (2) is certainly true. So assume (2). Let s be a simple reflection such that $\ell(sw) > \ell(w)$. This is equivalent to $sw > w$ in the Bruhat order. We must prove

$$\partial_{sw} = \sum_{y \leqslant sw} \partial_y^\circ. \tag{3}$$

Using our induction hypothesis

$$\partial_{sw} = \partial_s \partial_w = \partial_s \sum_{y \leqslant w} \partial_y^\circ.$$

Now suppose that $sy < y$. Then $\partial_y^\circ = \partial_{s \cdot sy}^\circ = \partial_s^\circ \partial_{sy}^\circ$ and since $\partial_s \partial_s^\circ = 0$ (as is easily checked) we have $\partial_s \partial_y^\circ = 0$. We may thus discard such terms from the sum and obtain

$$\partial_{sw} = \partial_s \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ. \tag{4}$$

We can divide W up into pairs $\{y, sy\}$ such that $y < sy$. These pairs are just the left cosets of W by the 2 element group $\langle s \rangle$. So we may write

$$\sum_{y \leqslant sw} \partial_y^\circ = \sum_{\substack{y \leqslant sw \\ y < sy}} \partial_y^\circ + \sum_{\substack{y \leqslant sw \\ sy < y}} \partial_y^\circ = \sum_{\substack{y \leqslant sw \\ y < sy}} \partial_y^\circ + \sum_{\substack{sy \leqslant sw \\ y < sy}} \partial_{sy}^\circ,$$

where we have made a variable change $y \to sy$ in the second term. By Deodhar's Property Z, if $y < sy$ then

$$y \leqslant w \quad \Leftrightarrow \quad y \leqslant sw \quad \Leftrightarrow \quad sy \leqslant sw$$

and if this is true then $\partial_{sy}^\circ = \partial_s^\circ \partial_y^\circ$. Also $\partial_s = 1 + \partial_s^\circ$.

$$\sum_{y \leqslant sw} \partial_y^\circ = \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ + \sum_{\substack{y \leqslant w \\ y < sy}} \partial_s^\circ \partial_y^\circ = (1 + \partial_s^\circ) \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ = \partial_s \sum_{\substack{y \leqslant w \\ y < sy}} \partial_y^\circ. \tag{5}$$

Combining this with (4) gives (3), and we are done. □

References

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[3] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Colored five-vertex models and Demazure atoms. *J. Combin. Theory Ser. A*, 178:105354, 48, 2021, arXiv:1902.01795.

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