Exercises

These exercises are *optional* but if you do them I will try to give you some feedback. See [2] Chapter 25 for more information about Demazure operators. Note that the open models are discussed in [1], while the closed models are (as far as I know) unpublished.

Exercise 1. In Lecture 8, we stated the Yang-Baxter equation for the open colored model, but we did not prove it. Check it for the case of two colors. (A computer program is an acceptable way to do this. One solution is posted.)

Exercise 2. Remember from Lecture 9 that $\mathcal{M}(T)$ is a field of meromorphic functions on T, and that $\mathcal{M}(T)$ and W both act on $\mathcal{M}(T)$. For $f \in \mathcal{M}(T)$ the action is by multiplication by f, and for $w \in W$ and $g \in \mathcal{M}(T)$ the action is $w : g \longmapsto {}^w g$ where ${}^w f(\mathbf{z}) = f(w^{-1}\mathbf{z})$. Check that if $f \in \mathcal{M}(T)$ and $w \in W$ then $wfw^{-1} = {}^w f$ as operators.

Let δ° be the operator defined in Lecture 8 as:

$$\delta_i^\circ f(\mathbf{z}) = \frac{z_{i+1} f(\mathbf{z}) - z_i f(s_i \mathbf{z})}{z_i - z_{i+1}}.$$

Exercise 3. We proved in Proposition 3.2 of Lecture 8 that if $d_i > d_{i+1}$ then

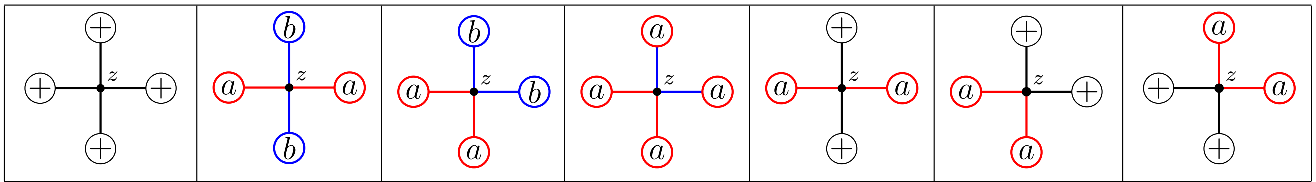
$$Z_\lambda(\mathbf{z}; s_i \mathbf{d}) = \delta_i^\circ Z_\lambda(\mathbf{z}; \mathbf{d}). \tag{1}$$

Prove that if instead $d_i < d_{i+1}$ then $Z_\lambda(s_i \mathbf{z}; \mathbf{d}) = Z_\lambda(\mathbf{z}; \mathbf{d})$. You can prove this two different ways:

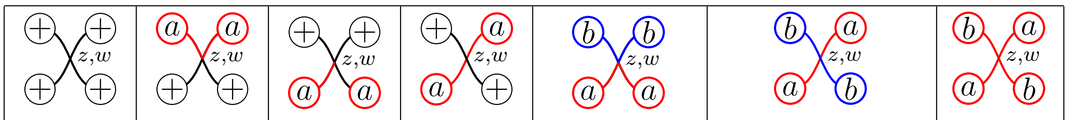
(i) Prove it directly from the Yang-Baxter equation;

(ii) Alternatively, work in the ring $\mathcal{R}$ defined in Lecture 9 prove that $s_i \delta_i^\circ = \delta_i^\circ$, then find a way to use (1).

Here are the Boltzmann weights for the closed models and their R-matrices:



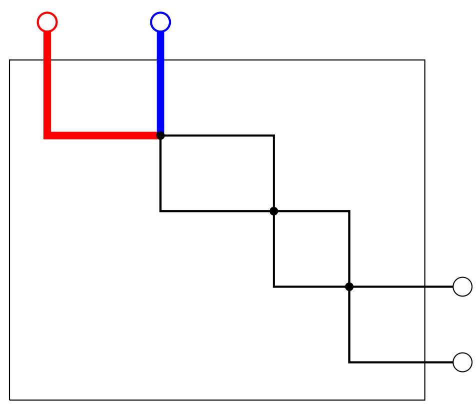
1	$z \quad a > b$ $0 \quad a < b$	$z \quad a > b$ $0 \quad a < b$	z	z	z	1



w	w	z	$z - w$	$z \quad a > b$ $w \quad a < b$	$z - w \quad a > b$ $0 \quad a < b$	z

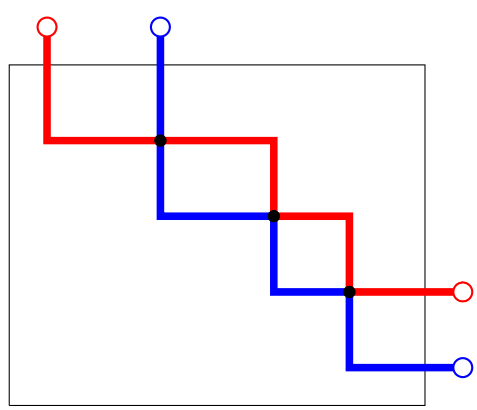
Exercise 4. Check the Yang-Baxter equation for the closed model.

Exercise 5. Here is a point that was explained in class but didn't appear yet in the lecture notes. For the open model, let us assume that two paths meet more than once:



(2)

We have indicated the colors at the start of the paths, and rest of the paths but not how they are colored. It is assumed the color red > color blue. Show that as a consequence of the Boltzmann weights, the colored lines cross the first time they meet but never again, thus:



Exercise 6. Repeat the last exercise with the closed model. Start again with this scenario in (2). Show that in the closed model there are *two* possible ways of coloring these paths. Discuss the implications for the partition function.

Exercise 7. (i) Prove that $s_i\partial_i = \partial_i$. (Just calculate in the ring $\mathcal{R}$.)

(ii) Prove that if $\ell(s_i w) < \ell(w)$ for $w \in W$ then w has a reduced expression $s_{i_1} \cdots s_{i_k}$ with $s_{i_1} = s_i$. Deduce that $\partial_w f$ is invariant under s_i.

(iii) Therefore if w_0 is the longest element of W, then $\partial_{w_0} f$ is symmetric, i.e. W-invariant.

Let δ_i be the operator

$$\delta_i f(\mathbf{z}) = \frac{z_i(f(\mathbf{z}) - f(s_i\mathbf{z}))}{z_i - z_{i+1}},$$

or equivalently

$$\delta_i = (1 - \mathbf{z}^{-\alpha_i})^{-1}(1 - s_i)$$

in the ring $\mathcal{R}$.

Exercise 8. (i) Let $w \in W = S_n$ and let $\mathbf{d} = w\mathbf{c}_0$. Prove that $\ell(s_i w) > \ell(w)$ if and only if $d_i > d_{i+1}$.

(ii) Prove that (for the open models) if $\mathbf{d} = w\mathbf{c}_0$, then

$$Z_\lambda(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^\rho \partial_w^\circ \mathbf{z}^\lambda.$$

Exercise 9. (i) Imitate the arguments in Lecture 8 and prove that the partition functions $Z_\lambda^{\text{closed}}(\mathbf{z}; \mathbf{d})$ of the closed models satisfy

$$Z_\lambda(\mathbf{z}; s_i\mathbf{d}) = \delta_i Z(\mathbf{z}; \mathbf{d})$$

if $d_i > d_{i+1}$.

(ii) Prove that $\partial_i = \mathbf{z}^{-\rho}\delta_i \mathbf{z}^\rho$.

(ii) Prove that

$$Z_\lambda^{\text{closed}}(\mathbf{z}; \mathbf{c}_0) = \mathbf{z}^\rho \partial_w^\circ \mathbf{z}^\lambda.$$

References

[1] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Colored five-vertex models and Demazure atoms. *J. Combin. Theory Ser. A*, 178:105354, 48, 2021, arXiv:1902.01795.

[2] D. Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.