Lecture 9

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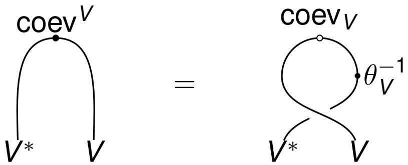
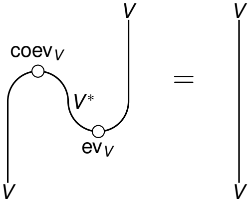
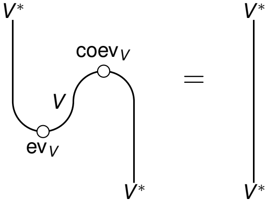


Dual objects

Let V be an object in a rigid monoidal category. We recall the assumptions we made of the dual, which we call the left dual V^*. It comes with morphisms $\mathrm{ev}_V : V^* \otimes V \to I$ and $\mathrm{coev}_V : I \to V \otimes V^*$ subject to

$$(1_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes 1_V) = 1_V, \qquad (\mathrm{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \mathrm{coev}_V) = 1_{V^*}.$$



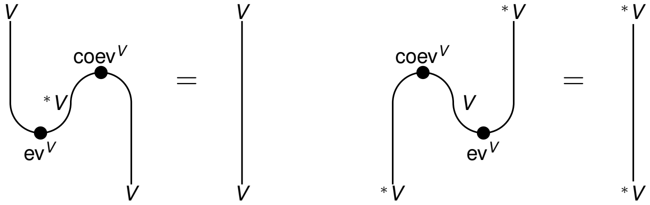


The right dual

Dually, we can ask for a right dual ${}^*V$ with morphisms $\mathrm{ev}^V : V \otimes {}^*V \to I$ and $\mathrm{coev}^V : I \to {}^*V \otimes V$ subject to

$$(\mathrm{ev}^V \otimes 1_V) \circ (1_V \otimes \mathrm{coev}^V) = 1_V,$$

$$(1_{{}^*V} \otimes \mathrm{ev}^V) \circ (\mathrm{coev}^V \otimes 1_{{}^*V}) = 1_{V^*}.$$

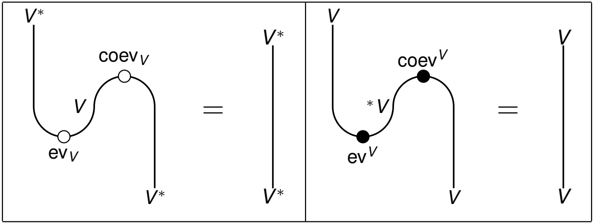


Left and right duals

The notions of left and right dual are not really different. V^* is a right dual of V if and only if V is a left dual of V^*. If every object in the category has both a right and a left dual, then by definition

$$^*(V^*) = V, \qquad (^*V)^* = V,$$

since the only difference between the defining properties



is the labelling of V, V^* and $^*V$.

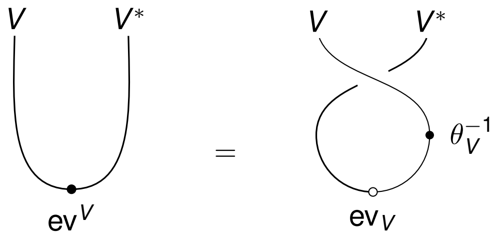
Duals in ribbon categories

Proposition

In a ribbon category, every left dual is also a right dual.

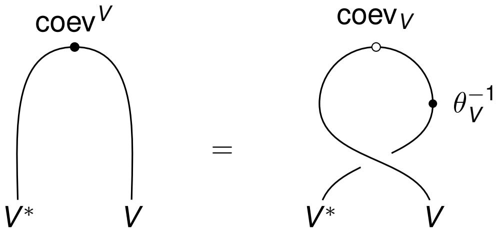
We describe the morphisms $\mathrm{ev}^V : V \otimes V^* \to I$ and $\mathrm{coev}^V : I \to V^* \otimes V$.

$$\mathrm{ev}^V = \mathrm{ev}_V \circ (1_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*}$$



Duals in ribbon categories (continued)

$$\mathrm{coev}^V = c_{V,V^*} \circ (1_{V^*} \otimes \theta_V^{-1}) \circ \mathrm{coev}_V$$



We will leave it to the reader to check that these definitions of ev^V and coev^V make the left dual V^* satisfy the definition of the right dual.

Knot invariants

Let us pick a module V in a ribbon category with unit object K. We now interpret any framed knot or link as a morphism $K \to K$. The framing is important because the framed knot invariant we describe depends on the framing. Turning it into a actual knot invariant requires modifying the definition to take into account the effect of Type I Reidemeister moves.

Let us project the knot onto a plane, resulting in a knot diagram, with crossings. Then there is a natural framing, namely we may take the normal vectors to be perpendicular to the plane.

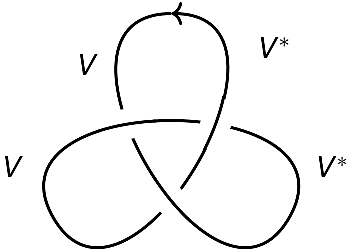
Labeling the edges

We now describe a way of assigning a vector space to an edge. (Turaev and Reshetikin worked in greater generality, but this is enough for the Jones polynomial.)

On every segment that points down, we label every edge by V. If the segment points up we label it by V^*. Then we interpret every local maximum of the curve as a coevaluation, either coev_V or coev^V depending on whether the arrow points left or right.

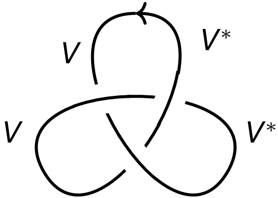
Labelling the trefoil knot

In other words we label segments of the curve where the orientation tangent vector points down by V, and segments where it points up by the dual V^*:



Reading the diagram from top to bottom gives a morphism $K \to K$, a scalar which should be a framed knot invariant.

The trefoil knot unravelled



$$\begin{array}{ccccc} K & \xrightarrow{\mathrm{coev}_V} & V\otimes V^* & \xrightarrow{1_V\otimes \mathrm{coev}_V\otimes 1_{V^*}} & V\otimes V\otimes V^*\otimes V^* \\ &&&& \downarrow c_{V,V}\otimes c_{V^*,V^*} \\ K & \xleftarrow[\mathrm{coev}^V\otimes \mathrm{coev}^V]{} & V\otimes V^*\otimes V\otimes V^* & \xleftarrow[1_V\otimes c_{V,V^*}\otimes 1_{V^*}]{} & V\otimes V\otimes V^*\otimes V^* \end{array}$$

We use the ribbon element implicitly since we use coev^V.

The standard module

Now let us consider the case where V is the standard module of $U_q(\mathfrak{sl}_2)$. A simplification here is that $V = V^*$ is self-dual. We will not completely study this example today. For example we have not computed the twist. But we have computed the R-matrix, which we denote T today:

$$T = \tau R = \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & & \\ & & & q \end{pmatrix}.$$

As we noted, T satisfies the quadratic equation:

$$T^2 = (q-q^{-1})T + 1,$$

which appeared in the definition of the Hecke algebra.

Normalization

Recall that we obtained T in Lecture 6 not by deducing it from the universal R-matrix, but by requiring it to be an H-module homomorphism $V \otimes V \to V \otimes V$, where $H = U_q(\mathfrak{sl}_2)$, and then imposing the Yang-Baxter equation. This procedure only determines T up to a constant.

Thus let $c = c_{V,V}$ be the correct ribbon category normalization of the braiding $V \otimes V \to V \otimes V$. In fact $c = q^{-1/2}T$. We need to justify this normalization properly, but at the moment we simply note that it gives $c_{V,V^*}c_{V^*,V}$ determinant 1. (We are refraining from identifying $V = V^*$ for the purpose of this last statement.)

The skein relation

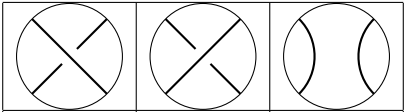
Now

$$T^2 = (q - q^{-1})T + 1, \qquad T - T^{-1} = q - q^{-1}$$

can be written:

$$q^{1/2}c - q^{-1/2}c^{-1} = q - q^{-1}.$$

Let us see what the implications of this are for the quantum trace. We consider three knots L_+, L_-, L_0 that are the same except for the circled region:



L^+	L^-	L_0

The Kauffman bracket as a quantum trace

We have explained how to attach to a knot or link L a morphism $K \to K$, which is after all just a scalar. We will denote this scalar $\mathrm{tr}(V)$.

Returning to the three knots L_+, L_- and L_0, we are using c to implement the morphism in L_+, c^{-1} to implement the morphism in L_- and $1_{V\otimes V}$ in L_0. Taking traces, the quadratic relation gives us

$$q^{1/2}\,\mathrm{tr}_{L_+} - q^{1/2}\,\mathrm{tr}_{L_+} = (q - q^{-1})\,\mathrm{tr}_{L_0}\,.$$

We recognize this as the skein relation for the Kauffman bracket (from Lecture 8):

$$a\langle L_+\rangle - a^{-1}\langle L_-\rangle = (a^2 - a^{-2})\langle L_0\rangle.$$

The parameter a has been replaced by $q^{1/2}$.

Noncommutative geometry

One can generalize some of algebraic geometry to include noncommutative variables. A common example is the strong analogy between the symmetric algebra of a vector space $S(V)$, which equals the polynomial ring on the dual space V^*, and the exterior algebra $\bigwedge V$, in which variables anticommute: $x \wedge y = -y \wedge x$.

A generalization involves variables (say x and y) that commute this way:

$$yx = qxy.$$

If $q = 1$, the algebra generated by x and y is a polynomial ring, the affine algebra (coordinate ring) of the complex plane. more generally, the algebra generated by x and y such that $yx = qxy$ may be regarded as the quantized function algebra associated with the quantum affine plane.

Noncommutative geometry and q-everything

In (commutative) algebraic geometry, the Nullstellensatz tells us that there is a bijection between the points of an algebraic variety (over an algebraically closed field) and the maximal ideals of its affine algebra. This fails in noncommutative geometry, and the notion of a point loses meaning to some extent. However the ring of functions survives.

Gaussian binomial coefficients emerge quickly in any commputation involving x and y that satisfy $yx = qxy$. These are polynomials in q that generalize the usual binomial coefficients $\binom{m}{n}$. They are the tip of the iceberg in the world of q-series which includes the work of Ramanujan and many others on partitions, so-called basic hypergeometric functions, modular forms, characters of affine Lie algebras, etc.

Gaussian binomial coefficients

The equation $yx = qxy$ leads to identities for Gaussian binomial coefficients, that are important both for quantized function algebras and for quantized enveloping algebras.
Gaussian binomial coefficients permeate the subject of quantum groups. Let q be a parameter. In one commonly used normalization (Gauss')

$$\begin{pmatrix} m \\ n \end{pmatrix}_{(q)} = \frac{(q^m - 1)\cdots(q^{m-n+1} - 1)}{(q^n - 1)\cdots(q - 1)} = \frac{m_{(q)}!}{n_{(q)}!(m-n)_q!}$$

where we define

$$m_{(q)} = \frac{q^m - 1}{q - 1}, \qquad m_{(q)}! = \prod_{k=1}^{m} k_{(q)}.$$

We define $\binom{m}{n}_{(q)} = 0$ unless $0 \leqslant n \leqslant m$.

Gaussian binomial coefficients (continued)

This normalization is useful for combinatorial purposes; for example if q is a prime power it counts the number of points in a Grassmannian over the finite field $\mathbb{F}_q$. That is:

Proposition

The number of vector subspaces of $\mathbb{F}_q^m$ of dimension n is $\binom{m}{n}_{(q)}$.

Indeed, $\mathrm{GL}(m, \mathbb{F}_q)$ acts transitively with stabilizer

$$P = \left\{ \begin{pmatrix} g_1 & * \\ & g_2 \end{pmatrix} \middle| g_1 \in \mathrm{GL}(n), g_2 \in \mathrm{GL}(m-n) \right\}$$

the index is $\binom{m}{n}_{(q)}$. An approach to quantum groups using flag varieties over finite fields (like this Grassmannian) was given by Beilinson, Lusztig and MacPherson.

Gaussian binomial coefficients (continued)

We will also encounter the normalization

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \qquad [m]_{(q)} = \frac{q^m - q^{-m}}{q - q^{-1}},$$

$$[m]_q! = \prod_{k=1}^{m} [k]_q.$$

When $q \to 1$ we have $m_{(q)} \longrightarrow m$ and $[m]_q \longrightarrow 1$ since

$$m_{(q)} = \frac{(q^m - 1)}{q - 1} = q^{m-1} + \ldots + 1 \qquad (m \text{ terms}).$$

So $\binom{m}{n}_q \to \binom{m}{n}$ and the Gaussian binomial coefficients really are a generalization of the ordinary binomial coefficients.

Some important formulas

The familiar formula for the usual binomial coefficients:

$$\binom{m+1}{n+1} = \binom{m}{n} + \binom{m}{n+1}$$

has a generalization to Gaussian binomial coefficients:

$$\boxed{\binom{m+1}{n+1}_{(q)} = q^{m-n}\binom{m}{n}_{(q)} + \binom{m}{n+1}_{(q)}.} \qquad (1)$$

Equivalently

$$\begin{bmatrix} m+1 \\ n+1 \end{bmatrix}_q = q^{n+1}\begin{bmatrix} m \\ n+1 \end{bmatrix}_q + q^{n-m}\begin{bmatrix} m \\ n \end{bmatrix}_q$$

Proof of (1)

Let us prove (1). First let us check that

$$\boxed{q^{m-n}n_{(q)}+(m-n)_{(q)}=(m)_{(q)}.} \tag{2}$$

indeed

$$q^{m-n}\frac{(q^n-1)}{q-1}+\frac{q^{m-n}-1}{q-1}=\frac{q^m-1}{q-1}.$$

Now

$$q^{m-n}\left(\begin{array}{c} m \\ n \end{array}\right)_{(q)}+\left(\begin{array}{c} m \\ n+1 \end{array}\right)_{(q)}=$$

$$q^{m-n}\frac{m_{(q)}!}{n_{(q)}!(m-n)_{(q)}!}+\frac{m_{(q)}!}{(n+1)_{(q)}!(m-n-1)_{(q)}!}.$$

Put this over a common denominator, use (2) and (1) follows.

A variant

There is a variant of the formula we just proved:

$$\binom{m+1}{n+1}_{(q)} = q^{m-n}\binom{m}{n}_{(q)} + \binom{m}{n+1}_{(q)}$$

The variant is

$$\boxed{\binom{m+1}{n+1}_{(q)} = q^{n+1}\binom{m}{n+1}_{(q)} + \binom{m}{n}_{(q)}}$$

Indeed note that $\binom{m}{n}_{(q)} = \binom{m}{m-n}_{(q)}$ so

$$\binom{m+1}{n+1}_{(q)} = \binom{m+1}{m-n}_{(q)} = q^{n+1}\binom{m}{m-n-1}_{(q)} + \binom{m}{m-n}_{(q)},$$

proving the second formula.

The Gaussian binomial theorem

There is a Gaussian binomial theorem. Let x and y be noncommuting variables such that

$$yx = qxy.$$

Then

$$\boxed{(x+y)^m = \sum_{n=0}^{m} \left(\begin{array}{c} m \\ n \end{array} \right)_{(q)} x^n y^{m-n}.}$$

Proof of the Gaussian Binomial Theorem

By induction

$$(x+y)^{m+1} = (x+y)\sum_{n=0}^{m} \left(\begin{array}{c} m \\ n \end{array}\right)_{(q)} x^n y^{m-n}.$$

$$\sum_{n=0}^{m} \left(\begin{array}{c} m \\ n \end{array}\right)_{(q)} x^{n+1} y^{m-n} + \sum_{n=0}^{m} q^n \left(\begin{array}{c} m \\ n \end{array}\right)_{(q)} x^n y^{m-n+1}.$$

The coefficient of $x^{n+1}y^{m-n}$ is

$$\left(\begin{array}{c} m \\ n \end{array}\right)_{(q)} + q^{n+1} \left(\begin{array}{c} m \\ n+1 \end{array}\right)_{(q)} = \left(\begin{array}{c} m+1 \\ n+1 \end{array}\right)_{(q)}$$

Substituting this and changing n to $n-1$ completes the proof.

The comultiplication of $U_q(\mathfrak{sl}_2)$

As an example, consider $U_q(\mathfrak{sl}_2)$. Recall that

$$\Delta E = 1 \otimes E + E \otimes K$$

where $KEK^{-1} = q^2 E$. So if $x = E \otimes K$ and $y = 1 \otimes E$ then

$$yx = q^{-2} xy.$$

Therefore (since Δ is a homomorphism)

$$\Delta E^m = (\Delta E)^m = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ n \end{array} \right)_{(q^{-2})} E^n \otimes K^n E^{m-n}.$$

A similar formula applies for ΔF^m.

A kind of Frobenius

If q is a root of unity, this has an interesting implication, analogous to properties of the Frobenius map. Remember that if p is a prime then $\binom{p}{k} = 0$ in $\mathbb{F}_p$ for $0 < k < p$, because the binomial coefficient is $\frac{p!}{k!(p-k)!}$ and the numerator is divisible by p, while the denominator is not. Using the binomial theorem,

$$(x + y)^p = x^p + y^p.$$

A kind of Frobenius (continued)

Very similarly, let N be an odd positive integer. If q is a primitive N-th root of unity, then $(N)_q = 0$ but $(k)_q \neq 0$ if $k < N$. Moreover the same is true for $(p)_{(q^{-2})}$ since q^2 is also a primitive N-th root of unity. This means that $\binom{N}{k}_{(q^{-2})} = 0$ if $0 < k < N$. Therefore

$$\Delta E^N = E^N \otimes 1 + E^N \otimes K^N,$$

and similarly

$$\Delta F^N = F^N \otimes 1 + K^{-N} \otimes F^N.$$

From this it may be deduced that if I is the ideal in $H = U_q(\mathfrak{sl}_2)$ generated by E^N, F^N and $K^N - 1$ then $\Delta(I) \subseteq H \otimes I \oplus I \oplus H$. This implies that Δ induces a comultipliccation on H/I, which then becomes a finite-dimensional Hopf algebra,

Exercises

Exercise 1. Prove that coev^V and ev^V as defined in the first section by

$$\mathrm{ev}^V = \mathrm{ev}_V \circ (1_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*},$$

$$\mathrm{coev}^V = c_{V,V^*} \circ (1_{V^*} \otimes \theta_V^{-1}) \circ \mathrm{coev}_V$$

make V^* into a right dual.

Exercise 2. (a) Prove that if H is a Hopf algebra and I is an ideal such that $\Delta(I) \subseteq H \otimes I + I \otimes H$, $\varepsilon(I) = 0$ and $S(I) \subset I$ then Δ induces a comultiplication on the quotient ring H/I and deduce that H/I is a Hopf algebra.
(b) Check these conditions for I as above (last page).

Exercises (continued)

Exercise 3. Suppose q is a nonzero complex number that is not a root of unity. Let x and y be noncommuting variables such that $yx = qxy$. Define

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n_{(q)}!}.$$

Prove that

$$e_q(x+y) = e_q(x)e_q(y).$$