Lecture 8

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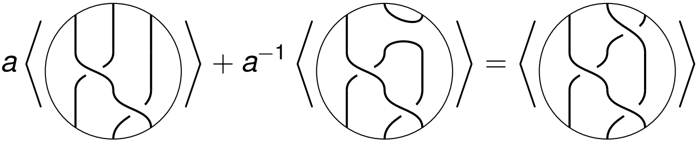
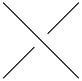
November 3, 2022



Ambient isotopy

Two knots, links or tangles are equivalent by ambient isotopy if the ambient space $\mathbb{R}^3$ or (for tangles) $\mathbb{R}^2 \times [0, 1]$ can be continuously deformed taking the first knot to the second.

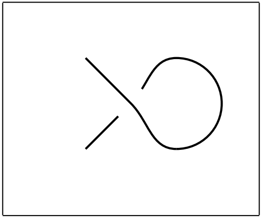
Knot invariants, or pre-invariants are described by recipes that apply to two-dimensional link diagrams that show crossings such as:

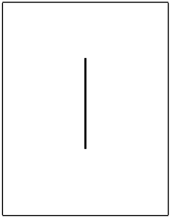


Reidemeister proved that two such diagrams describe the same knot (up to ambient isotopy) if and only if they are equivalent by Reidemeister moves.

Reidemeister moves of Type I

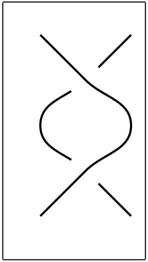
If the knots are represented by their two-dimensional projections, a necessary and sufficient condition is that these projections be related by a sequence of Reidemeister moves. There are three kinds. A Reidemeister move of Type I undoes a twist:

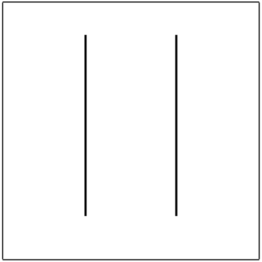




Reidemeister moves of Type II

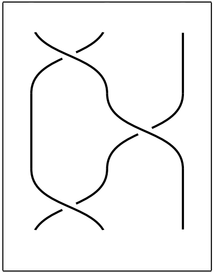
A Reidemeister move of type II changes:

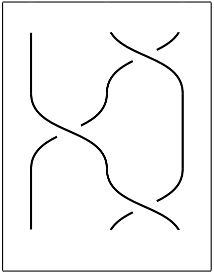




Reidemeister moves of Type III

A Reidemeister move of type III changes:





Framing of link diagrams

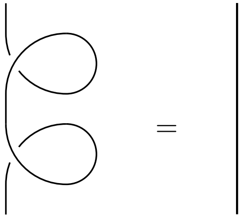
There is a natural framing of a knot diagram: we choose the normal vector field of the 3-dimensional knot as pointing perpendicular to the plane of the 2-dimensional diagram. This is actually not quite the right framing for links: we may want to twist the framing on a component of a link depending on its winding number with other components.

We will not worry about this point. Instead, we will proceed by defining polynomials of link diagrams such as the Kauffman bracket that are unchanged by Reidemeister moves of Types O, II and III, then note that these are invariants of framed links.

Ultimately we wish to modify these invariants to be independent of the framing. This means that we want invariance under Type I moves.

Type O moves

For framed links, knots or tangles, which we can visualize as ribbons, Type I Reidemeister moves no longer correspond to ambient isotopy, because they introduce a twist. Instead we have the following equivalence:



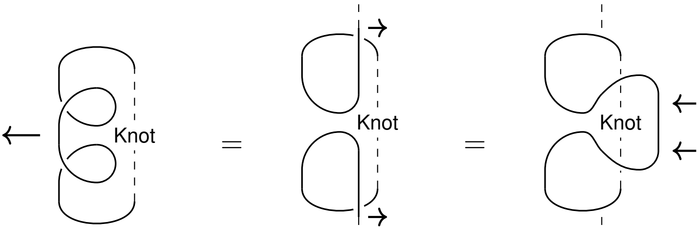
In a rigid braided category, the corresponding equivalence would bethis would be $\mathfrak{vv}^{-1} = 1$. This type of move substitutes for Type I Reidemeister moves in the category of framed tangles. We call it Type O.

Reidemeister's theorem for framed links

Proposition

A necessary and sufficient condition for two framed links (ribbons) to be equivalent by ambient isotopy is that they be equivalent by Type O, II and III moves.

Type O moves may be reduced to Type II and III if we adjoin a point at infinity to the ambient plane to make it a sphere.



The Kauffman bracket

Let L be a link diagram. The Kauffman bracket $\langle L \rangle$ is a polynomial in a variable a. We will denote

$$d = -a^2 - a^{-2}.$$

If L consists of n unlinked circles, $\langle L \rangle = d^{n-1}$. Thus if O is the unlinked circle then $\langle O \rangle = 1$.

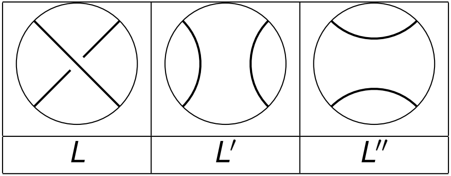
More generally, adding an unlinked circle O to a link L multiplies the Kauffman bracket by d:

$$\langle L \cup O \rangle = d \langle L \rangle.$$

To say the circle O embedded in $\mathbb{R}^3$ is unlinked from link L means that O bounds a disk disjoint from L. For the link diagram, it means the interior of the circle O doesn't intersect L.

The Kauffman bracket

Let L be a link diagram with n crossings. Now we assume recursively that $\langle L' \rangle$ is defined for L' with fewer than n crossings. We select a crossing in the diagram L and let L' and L'' be the diagrams replacing the crossing by parallel lines as follows:



The figures agree outside the circled portion. We define:

$$\langle L \rangle = a\langle L' \rangle + a^{-1}\langle L'' \rangle.$$

The Kauffman bracket is an invariant of framed links

The identity

$$\langle L\rangle = a\langle L'\rangle + a^{-1}\langle L''\rangle$$

is called the skein relation.

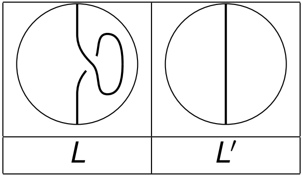
We must argue that this definition does not depend on the choice of crossing. But it is actually clear that the order in which we resolve crossings into parallels is unimportant: for if we resolve all or any subset of k crossings, the 2^k end configurations are the same regardless of what order we do it in, so the Kauffman bracket is well-defined.

Theorem

The Kauffman bracket is unchanged by Reidemeister moves of Types O, II or III. Therefore it is an invariant of the framed links.

Adding or removing a curl

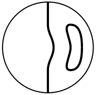
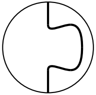
First we compute what happens when we remove a curl from a link. Suppose that L and L' agree outside the circled portion:



Then

$$\langle L\rangle = -a^3\langle L'\rangle.$$

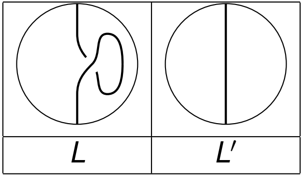
To see this, note that by the skein relation $\langle L\rangle$ equals

$a\Big\langle$  $\Big\rangle + a^{-1}\Big\langle$  $\Big\rangle = ad\langle L'\rangle + a^{-1}\langle L'\rangle.$

Since $d = -a^2 - a^{-2}$, the statement follows.

Adding or removing a curl (continued)

Similarly for a negative curl:

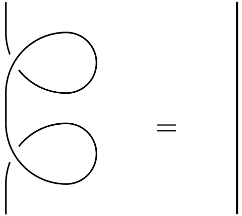


A similar calculation shows

$$\langle L \rangle = -a^{-3} \langle L' \rangle .$$

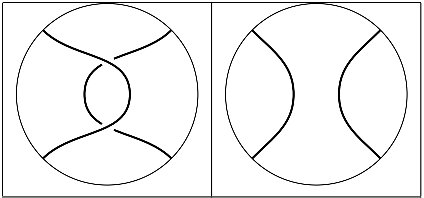
Type O moves

Now it is clear that the Kauffman bracket is invariant under Type O moves, since adding curls of opposite parities multiplies and divides the bracket by $-a^3$:



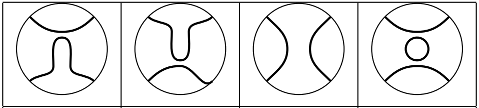
Reidemeister moves of Type II

Let us show that if link diagrams L and L' agree except on the interior of the circle, then $\langle L \rangle = \langle L' \rangle$.



L $\qquad$ L'

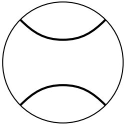
$$\langle L \rangle = a^{-2}\langle L1 \rangle + a^{2}\langle L2 \rangle + \langle L3 \rangle + \langle L4 \rangle \ .$$



$L1$ $\qquad$ $L2$ $\qquad$ $L3$ $\qquad$ $L4$

Reidemeister moves of Type II (continued)

Now

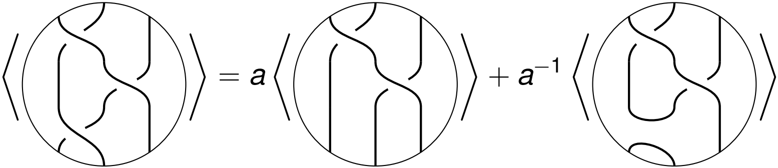
$$a^{-2}\langle L1\rangle + a^{2}\langle L2\rangle + \langle L4\rangle = (a^{-2} + a^{2} + d)\left\langle \vphantom{\Bigg|} \right.$$  $$\left. \vphantom{\Bigg|} \right\rangle = 0.$$

Therefore

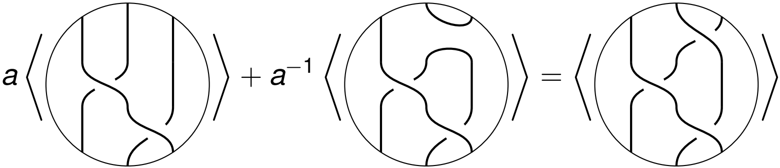
$$\langle L\rangle = \langle L3\rangle = \langle L'\rangle$$

proving that the Kauffman bracket is unchanged by Reidemeister moves of Type II.

Reidemeister moves of Type III



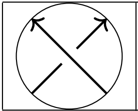
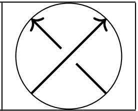
Using Reidemeister moves of Type II on the second term this equals



The writhe

The writhe is a very simple invariant of oriented framed links. An orientation of a (smooth) knot or link assigns a unit tangent vector field. A knot has 2 orientations, so if a link has n components it obviously has 2^n possible orientations. We assume an orientation is selected.

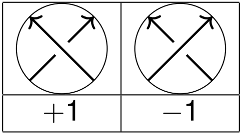
Let us consider an oriented link diagram. Every crossing in an oriented link is assigned a sign as follows.

	
$+1$	-1

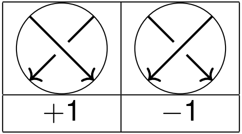
The writhe is the sum of the signs over all crossings.

The wrihe of a knot does not depend on the orientation

The writhe of a knot does not depend on orientation. Indeed, if we change orientation, both arrows are switched and the sign of a crossing is not changed.



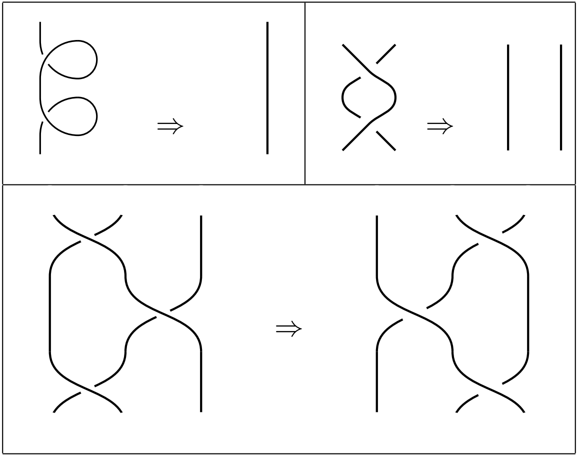
becomes:



(For links, we have to be careful.)

The writhe is an invariant of framed, oriented links

It is clearly invariant under moves of Type O,II and III. We do not draw the orientations.

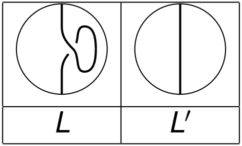


The normalized Kauffman bracket

Let $w(L)$ denote the writhe of an oriented link. By combining it with the Kauffman bracket we can make an invariant of unframed links. Let

$$f(L) = (-a)^{-3w(L)} \langle L \rangle.$$

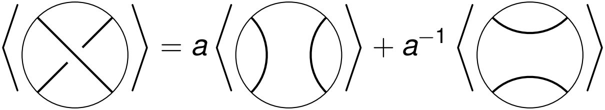
This is unchanged under Reidemeister moves of Types O, II and III. What about moves of Type I? If we add a twist:



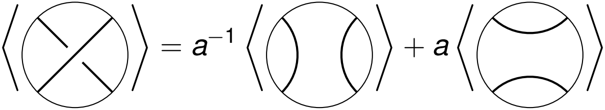
$$\langle L \rangle = -a^3 \langle L' \rangle, \qquad w(L) = w(L') + 1, \qquad f(L) = f(L').$$

The skein relations revisited

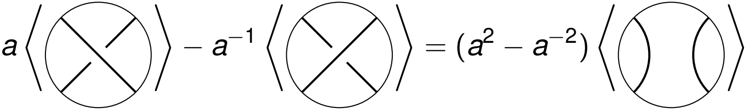
We may rewrite the skein relations.



Rotating:

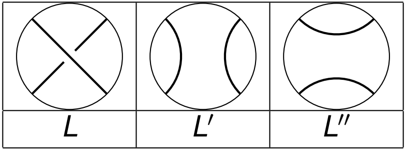


Solving:

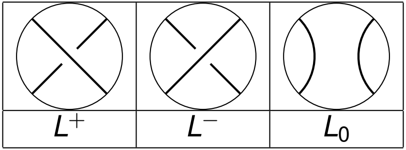


Two kinds of skein relations

So there are two different types of skein relations. We may ask for linear relations between:

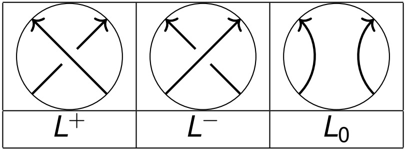


or (possibly contingent on an assumed orientation)



Skein relations and orientation

Skein relations of the L^+, L^-, L^0 kind are possible for invariants of oriented knots and links. Then we of course require the orientations to be consistently matched.

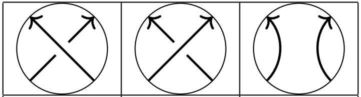


The dependence is important for links since we could change the orientation of one line in a crossing but not the other. This would interchange L_+ and L_-. For knots, the distinction is not so important since both orientations would have to be changed, and L_+ is never turned into L_- by a change in orientation.

The skein relation for the normalized Kauffman bracket

We have seen that the Kauffman bracket satisfies the skein relation

$$a\langle L_+\rangle - a^{-1}\langle L_-\rangle = (a^2 - a^{-2})\langle L_0\rangle$$



L^+	L^-	L_0

Now recall that $f_L(a) = (-a)^{-3w(L)}\langle L\rangle$ is the normalized Kauffman bracket, which is an isotopy invariant of oriented links. We have $w(L_+) = w(L_0) + 1$, $w(L_-) = w(L_0) - 1$ and so

$$-a^4 f_{L_+}(a) + a^{-4} f_{L_-}(a) = (a^2 - a^{-2}) f_{L_0}(a).$$

The Jones polynomial

The normalized Kauffman bracket is essentially the Jones polynomial. Note that the Kauffman polynomial of a circle is $d = -a^2 - a^{-2}$. We replace a by $t^{-1/4}$:

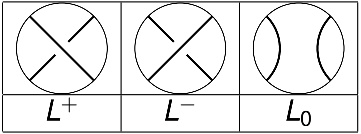
$$V_L(t) = f_L(t^{-1/4}).$$

This is a polynomial in t for knots or links with an even number of components. For links with an odd number of components it is a polynomial in $\sqrt{t}$. The skein relation becomes

$$t^{-1} V_{L^+} - tV_{L_-} = (t^{1/2} - t^{-1/2}) V_{L_0}.$$

The Jones and Conway-Jones polynomials

In 1923 Alexander discovered a knot invariant known as the Alexander polynomial $\Delta(L)$. Conway defined a variant denoted $\nabla(L)$ and showed that it satisfied a skein relation. If as usual



then

$$\nabla_{L^+}(t) - \nabla_{L_-}(t) = t\nabla_{L_0}(t).$$

The Alexander polynomial (slightly modified from Alexander's normalization) is then

$$\Delta_L(t^2) = \nabla_L(t - t^{-1}).$$

The HOMFLYPT polynomial

The Jones polynomial was discovered in 1984 by Vaughn Jones through a study of the Temperley-Lieb algebra, an algebra similar to the Hecke algebra that is related to exactly solvable models in statistical mechanics.

Meanwhile Conway had found the Conway-Alexander polynomial which utilized the skein relation in its definition. In retrospect, the skein relation was known to Alexander, but Conway showed its importance.

The HOMFLYPT polynomial was found independently and simultaneously by 4 different groups: Freyd and Yetter, Ocneanu, Millet and Lickorish and Hoste. All three sent their manuscripts to the same journal (BAMS) in October 1984. The editors noticed and a joint paper resulted.

The HOMFLYPT polynomial (continued)

The HOMFLY polynomial was also found around the same time by Przytycki and Traczyk and when this was recognized their initials were added to the acronym. With this addition the acronym becomes apt since “flyping,” a procedure used extensively by the Peter Guthrie Tait (the founder of knot theory) is a generalization of Type III Reidemeister moves and hence is related to the Yang-Baxter equation.

We may define the HOMFLYPT polynomial $P(\alpha, z)$ by $P_O = 1$ where and the skein relation

$$\alpha P_{L_+} - \alpha^{-1} P_{L_-} = z P_{L_0}.$$

The Alexander and Jones polynomials are then

$$\Delta_L(t) = P_L(1, t^{1/2} - t^{-1/2}), V_L(t) = P_L(t^{-1}, t^{1/2} - t^{-1/2}).$$

Hecke algebras and quantum groups

In Jones's original work, he made use of the Temperley-Lieb algebra. This is a close relative of the Hecke algebra that we have already seen related to the theory of quantum groups through Frobenius-Schur-Duality. After the HOMFLY paper Jones explained the two-variable representations through the Hecke algebra representations in his 1987 Annals paper.

Through Schur-Weyl-Jimbo duality, we understand that Hecke algebra representations are closely connected to quantum group representations, and also to solvable lattice models. Roughly we can say that the Jones polynomial is related to $U_q(\mathfrak{sl}_2)$ and the Alexander polynomial is related to $U_q(\mathfrak{gl}(1|1))$, a quantized enveloping algebra of a Lie superalgebra; or rather affinizations of these quantum groups.

Witten's approach

In his 1989 PJM paper on connections of solvable lattice models, Jones complained about the lack of a direct 3-dimensional connection; instead knots were studied by means of 2-dimensional projections. He compared the situation to Plato's Allegory of the Cave.

A direct 3-dimensional approach was found by Witten (1989) who obtained the Jones polynomial from 3-dimensional quantum field theories called Chern-Simons theories. Eventually (2016) modified Chern-Simons theories were found by Costello that connect directly to 2-dimensional solvable lattice models.