Lecture 7

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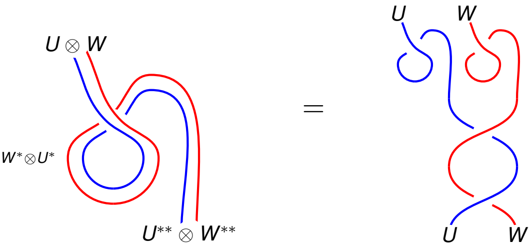
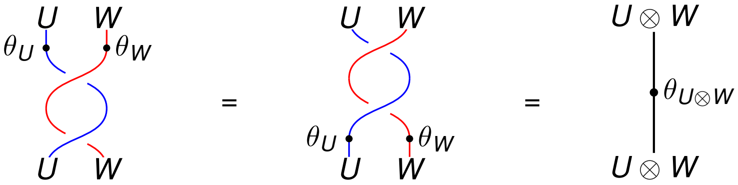
May 28, 2019



Review of ribbon categories

The notion of a ribbon category, which we have already discussed, is due to Joyal and Street, who called them tortile. We require a natural endomorphism θ_V of each object subject to the ribbon tensor axiom

$$\theta_{U\otimes W} = c_{U,W}^{-1} \circ c_{W,U}^{-1} \circ \theta_U \otimes \theta_W = \theta_U \otimes \theta_W \circ c_{U,W}^{-1} \circ c_{W,U}^{-1}$$



And such that $\theta_I = 1_I$ and $\theta_{V^*} = \theta_V^*$.

Ribbon Hopf algebras

The archetypal ribbon category is the category of framed tangles.

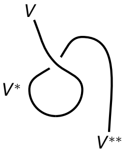
The notion of a ribbon Hopf algebra is due to Reshetikhin and Turaev. This is a QTHA with a central element $\boldsymbol{\theta}$ such that multiplication by $\boldsymbol{\theta}$ induces an endomorphism θ_V in a module V, making the category of finite-dimensional modules (already braided by quasitriangularity) into a ribbon category.

For example, if $H = U_q(\mathfrak{sl}_2)$ we may take $\theta = K^{-1}\mathbf{u}$ where $\mathbf{u} = S(R^{(2)})R^{(1)}$ was introduced in Lecture 5.

We will impose suitable axioms on $\boldsymbol{\theta}$ after further discussing the properties of $\mathbf{u}$.

The elements u and $S(\mathbf{u})$

Let H be a QTHA with universal R-matrix $R \in H \otimes H$. We recall that $\mathbf{u} = S(R^{(2)})\, R^{(1)}$. Let V be a finite-dimensional H-module. We proved that this is related to the isomorphism $\mathfrak{u} : V \to V^{**}$ as follows.



If $\iota : V \to V^{**}$ is the usual vector space isomorphism between V and its double dual, then for $x \in V$,

$$\mathfrak{u}(x) = \iota(\mathbf{u} \cdot x).$$

Note that ι is not an H-module homorphism but $\mathfrak{u}$ is.

Properties of u

Let H be a QTHA and let $\mathbf{u} = S(R^{(2)})R^{(1)}$.

Proposition

If $x \in H$ then

$$S(x_{(2)})\mathbf{u}x_{(1)} = \varepsilon(x).$$

Recall that $\tau\Delta(x) = R\Delta(x)R^{-1}$ in Sweedler notation becomes

$$R^{(1)}x_{(1)} \otimes R^{(2)}x_{(2)} = x_{(2)}R^{(1)} \otimes x_{(1)}R^{(2)}.$$

So

$$S(x_{(2)})S(R^{(2)})R^{(1)}x_{(1)} = S(R^{(2)}x_{(2)})R^{(1)}x_{(1)} =$$

$$S(x_{(1)}R^{(2)})x_{(2)}R^{(1)} = S(R^{(2)})S(x_{(1)})x_{(2)}R^{(1)}.$$

Now $S(x_{(1)})x_{(2)} = \varepsilon(x)$ which is a scalar that can be pulled out leaving **u** and we are done.

Properties of u, continued

We will denote $\mathbf{v} = S(\mathbf{u})$.

Proposition

The element $\mathbf{u}$ *is invertible with inverse* $\mathbf{u}^{-1} = R^{(2)} S^2(R^{(1)})$. *Conjugation by* $\mathbf{u}$ *implements the square of the antipode:*

$$\mathbf{u} x \mathbf{u}^{-1} = S^2(x).$$

$$\mathbf{v} x \mathbf{v}^{-1} = S^{-2}(x).$$

We refer to Majid Chapter 5 for the proof.

Note that this implies that $\mathfrak{uv}$ is central since $\mathfrak{uv}(x)(\mathfrak{uv})^{-1} = \mathfrak{u} S^{-2}(x) \mathfrak{u}^{-1} = x$. Then $\mathfrak{uv} = \mathfrak{vu}$ since $\mathfrak{uv} = \mathfrak{u}^{-1}(\mathfrak{uv})\mathfrak{u} = \mathfrak{vu}$.

Properties of u, continued

We come to another very significant property of **u**.

Theorem

We have

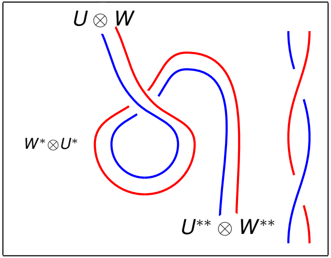
$$\Delta(\mathbf{u}) = R^{-1}R_{21}^{-1}(\mathbf{u}\otimes\mathbf{u}).$$

Here R_{21} means $R^{(2)}\otimes R^{(1)}$. Again we refer to Majid for the proof, but one step is worth pointing out. This is the "Yang-Baxter equation in a suitable form" mentioned at the end of the proof on page 33. This is the Exercise 3 in Lecture 5.

Instead of repeating the proof here, we will discuss the context and meaning of the Theorem.

Reminder of Lecture 4

The theorem is related to a topic that concerned us earlier, in Lecture 4, where the morphism $\mathfrak{u} : V \to V^{**}$ was introduced. We considered what happens when we apply $\mathfrak{u}$ to a tensor product $U \otimes V$. We gave the following figure.

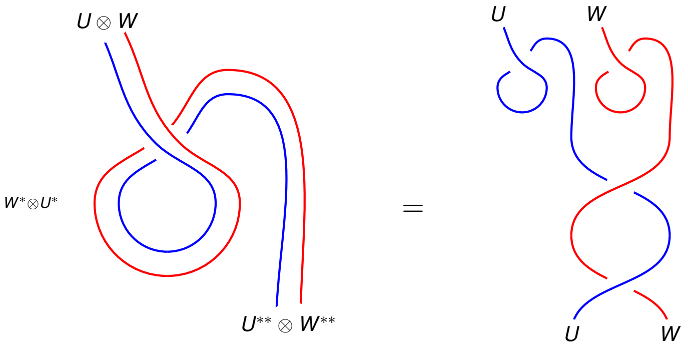


Topologically $\mathfrak{u}$ amounts to a double twist, but only if we allow ourselves a Reidemeister Type I move, or to put it another way, if we ignore the distinction between V and V^{**}.

$\mathfrak{u}_{U\otimes W}$

In an arbitrary rigid braided category, we have (please check)

$$\mathfrak{u}_{U\otimes W} = c_{U,W}^{-1} c_{W,U}^{-1} (\mathfrak{u}_U \otimes \mathfrak{u}_W) = (\mathfrak{u}_U \otimes \mathfrak{u}_W) c_{U,W}^{-1} c_{W,U}^{-1}.$$



About $R_{21}R$

If U, W are H-module homomorphisms, then the braiding $c_{U,W} : U \otimes W \to W \otimes U$ is $\tau \circ R$, meaning that we multiply an element of $U \otimes W$ by R, then apply the flip τ.

Now $R_{21}R\Delta(x) = R_{21}(\tau\Delta(x))R = \Delta(x)R_{21}R$ for $x \in H$ because $\tau\Delta(x) = R\Delta(x)R^{-1}$. This implies that multiplication by $R_{21}R$ is an H-module endomorphism of $U \otimes W$ for modules U and W. The precise endomorphism is indeed $c_{U,W}c_{W,U} = \tau R\tau R = R_{21}R$.

The meaning of the theorem

We will now explain how the result in the theorem

$$\Delta(\mathbf{u}) = (R_{12}R)^{-1}(\mathbf{u} \otimes \mathbf{u})$$

is equivalent to the formula

$$\mathfrak{u}_{U\otimes W} = c_{U,W}^{-1} c_{W,U}^{-1} (\mathfrak{u}_U \otimes \mathfrak{u}_W) = (\mathfrak{u}_U \otimes \mathfrak{u}_W) c_{U,W}^{-1} c_{W,U}^{-1}.$$

We note that $\mathbf{u} \otimes \mathbf{u}$ commutes with R and R_{21} because conjugation by $\mathbf{u}$ is the square of the antipode so

$$(\mathbf{u} \otimes \mathbf{u}) R (\mathbf{u} \otimes \mathbf{u})^{-1} = S^2(R^{(1)}) \otimes S^2(R^{(2)})$$

but using $(S \otimes S)R = R$ twice, this is R.

The meaning of the theorem (continued)

Now let $\iota_V : V \to V^{**}$ be the usual vector space isomorphism so $\mathfrak{u}(x) = \iota(\mathbf{u})$. Naturally $\iota_{U\otimes W} = \iota_U \otimes \iota_W$. So applying $\iota_{U\otimes W}\Delta(\mathbf{u})$ to $U \otimes W$ and using

$$\Delta(\mathbf{u}) = (R_{12}R)^{-1}(\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\otimes\mathbf{u})(R_{12}R)^{-1}$$

gives

$$\mathfrak{u}_{U\otimes W} = (\mathfrak{u}_U \otimes \mathfrak{u}_W)c_{U,W}^{-1}c_{W,U}^{-1},$$

where we reiterate

$$\mathfrak{u}_V(x) = \iota(\mathbf{u}x), \qquad x \in V$$

for V any module.

The ribbon element

Now following Turaev and Reshetikhin, we define the QTHA H to be ribbon if it contains an central element θ such that $\theta^2 = \mathbf{vu}$, $S(\theta) = \theta$, $\varepsilon(\theta) = 1$ and

$$\Delta(\theta) = (RR_{21})^{-1}(\theta \otimes \theta).$$

Theorem (Turaev and Reshetikhin)

The category of finite-dimensional modules for a ribbon Hopf algebra is a ribbon category.

Discussion

If V is a module since $\boldsymbol{\theta}$ is central, multiplication by $\boldsymbol{\theta}$ induces an H-module endomorphism θ_V Just as the property

$$\Delta(\mathbf{u}) = (RR_{21})^{-1}(\boldsymbol{\theta} \otimes \mathbf{u})$$

implied

$$\mathfrak{u}_{U \otimes W} = (\mathfrak{u}_U \otimes \mathfrak{u}_W) c_{U,W}^{-1} c_{W,U}^{-1},$$

the ribbon axiom

$$\Delta(\boldsymbol{\theta}) = (RR_{21})^{-1}(\boldsymbol{\theta} \otimes \boldsymbol{\theta})$$

implies

$$\theta_{U \otimes W} = (\theta_U \otimes \theta_W) c_{U,W}^{-1} c_{W,U}^{-1},$$

one of the properties we need for a ribbon category. The other properties, $\theta_{V^*} = \theta_V^*$ and $\theta_K = 1_K$ follow from $S(\boldsymbol{\theta}) = \boldsymbol{\theta}$ and $\varepsilon(\boldsymbol{\theta}) = 1$.

Schur-Weyl duality

Schur-Weyl duality is a relationship between the representations of the symmetric group S_r and the general linear group $\mathrm{GL}(n, \mathbb{C})$.
The groups S_r and $\mathrm{GL}(n, \mathbb{C})$ both act on the same vector space $\otimes^r V$ where $V = \mathbb{C}^n$, the standard module of $\mathrm{GL}(n)$. The group $\mathrm{GL}(n)$ acts diagonally:

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n, \qquad g \in \mathrm{GL}(n).$$

The symmetric group acts by permuting the components.

$$w(v_i \otimes \cdots v_r) = v_{w^{-1}i} \otimes \cdots v_{w^{-1}r}, \qquad w \in S_r.$$

The two actions obviously commute. The problem is to decompose $\otimes^r V$ into irreducibles.

Irreducibles of $\mathrm{GL}(n)$ and S_r

Let λ be a partition of r of length $\leqslant n$. Thus $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_n \geqslant 0$ and $\sum \lambda_i = r$. Then λ indexes both an irreducible representation of S_r and of $\mathrm{GL}(r, \mathbb{C})$.

The irreducible representations of S_r are indexed by partitions of r. We will denote the corresponding irreducible of S_r as $\pi_\lambda^{S_r}$.

The irreducibles of $\mathrm{GL}(n, \mathbb{C})$ are indexed by dominant weights. A weight is a rational character of the diagonal subgroup; the group of weights is in bijection with $\mathbb{Z}^n$ as follows: if $\lambda = (\lambda_1, \cdots, \lambda_n)$, then λ is called dominant if $\lambda_1 \geqslant \cdots \geqslant \lambda_r$. In particular, a partition of length $\leqslant r$ is a dominant weight. Let $\pi_\lambda^{GL(n)}$ be the corresponding irreducible.

Irreducibles of GL(n) and S_r (continued)

If $G = GL(n)$ and λ is a partition of length $\leqslant n$ (hence a dominant weight) then $\pi_\lambda^{\mathrm{GL}(n)}$ is the representation whose character is the Schur polynomial s_λ, i.e.

$$\mathrm{tr}(\pi_\lambda^{\mathrm{GL}(n)}(g)) = s_\lambda(x_1, \cdots, x_n)$$

x_i the eigenvalues of g.

Irreducibles of $\mathrm{GL}(n)$ and S_r (continued)

The partitions λ of r index the irreducible representations of S_r. One way of describing this indexing is as follows. If $\lambda = (\lambda_1, \cdots, \lambda_r)$ is a partition of r let

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r} \subseteq S_r.$$

Let λ' be the conjugate partition. Then the induced representations

$$\mathrm{ind}_{S_\lambda}^{S_r}(1), \qquad \mathrm{ind}_{S_{\lambda'}}^{S_r}(\mathrm{sgn})$$

have a unique irreducible constituent in common. Call this $\pi_\lambda^{S_r}$.

Schur-Weyl duality (concluded)

Then as a $S_r \otimes GL(n)$-module, Schur-Weyl duality is the isomorphism

$$\otimes^r V \cong \bigoplus_{\lambda \vdash r} \pi_\lambda^{S_r} \otimes \pi_\lambda^{GL(n)}.$$

This has many applications. There are no repetitions among the representations that appear on either side, so this gives a bijection between the representations of GL(n) that appear, and the representations of S_r.

In Jimbo's generalization, GL(n) is replaced by $U_q(\mathfrak{gl}_n)$ and S_r is replaced by its Hecke algebra.

Before we explain the Hecke algebra we consider the braid group.

Coxeter groups

A Coxeter group W is a group with generators $s_1, \cdots s_r$ of order 2. Let $n(i,j)$ be the order of s_i. Since the s_i have order 2 we may write

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

where there are $n(i,j)$ factors on both sides. This is called the braid relation. It is assumed that these relations

$$s_i^2 = 1, \qquad s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

are a presentation of W. For example S_r a presentation with generators $s_i = (i, i+1)$ and relations

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1.$$

The braid group of a Coxeter group

The braid group B associated with the Coxeter group W has generators t_i satisfying the braid relations but which are not assumed to be of order 2. So we have a homomorphism $W \to B$ mapping s_i to t_i.

For example The braid group B_r of the symmetric group S_r is the Artin braid group generated by braids $t_1, \cdots, t_{r-1}$. These may be identified with isotopy classes of braids, embedding the braid group in the Grothendieck group of the tangle category.

Matsumoto's theorem

We will need the following very important fact about Coxeter groups. Let W be a Coxeter group with generators s_i (called simple reflections) and let $w \in W$. A reduced expression or reduced word is a representation of w as a product of simple reflections that is as short as possible: $w = s_{i_1} \cdots s_{i_k}$. Then if $w = s_{j_1} \cdots s_{j_k}$ is another reduced word, Matsumoto's theorem asserts that second reduced word may be obtained from the first by successive applications of the braid group. For example in S_4 we have $s_1 s_2 s_1 s_3 s_2 s_1 = s_3 s_2 s_1 s_3 s_2 s_3$ and:

$$212321 \Rightarrow 121321 \Rightarrow 123121 \Rightarrow 123212 \Rightarrow$$

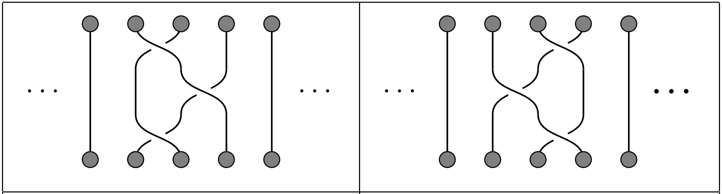
$$132312 \Rightarrow 312132 \Rightarrow 321232 \Rightarrow 321323$$

Thus $t_{i_1} \cdots t_{i_k} = t_{j_1} \cdots t_{j_k}$ in the braid group.

The braid relations

The braid relations for the Artin braid group correspond to Reidemeister moves of Type III:

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \text{ if } |i-j| > 1.$$



$t_i t_{i+1} t_i$	$t_{i+1} t_i t_{i+1}$

Braid group representations

Braid group representations are an important topic.

- Turaev: Faithful representions of the braid group
- Jones: a polynomial invariant for knots via von Neumann algebras

We obtain braid group representations from the Yang-Baxter equation. Let V be a vector space and $R \in \mathrm{End}(V \otimes V)$ satisfy

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathrm{End}(V \otimes V \otimes V)$. Then $\mathcal{R} = \tau R$ satisfies

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}.$$

This means that if we define $T_i \in \mathrm{End}(\otimes^r V)$ to be the endomorphism $R_{i,i+1}$, then the satisfy the braid relation

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

So there is a homomorphism $B_n \to \mathrm{GL}(\otimes V^k)$ mapping $t_i \to T_i$.

The SL_2 R-matrix

Let us recall the $U_q(\mathfrak{sl}_2)$ R-matrix that we obtained in Lecture 6. I have replaced q by q^{-1} so this is actually the $U_{q^{-1}}(\mathfrak{sl}_2)$ R-matrix.

$$R = \begin{pmatrix} q & & & \\ & & 1 & \\ & q-q^{-1} & 1 & \\ & & & q \end{pmatrix}.$$

It is actually τR that is the endomorphism of $V \otimes V$, where V is the standard two-dimensional module. So we are concerned with

$$\tau R = \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & & \\ & & & q \end{pmatrix}.$$

A generalization

Jimbo found a more general R-matrix for $U_{q^{-1}}(\mathfrak{sl}_n)$. Let $x_1, \cdots, x_n$ be basis vectors of an n-dimensional vector space. We will denote by E_{ij} the rank one elementary transformation that takes x_i to x_j and other basis vectors to zero. Let

$$R_q = \sum_i q\, E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}$$

If $n = 2$, this is the same R-matrix as before. This is the R-matrix for the n-dimensional standard module of the quantum group $U_{q^{-1}}(\mathfrak{sl}_n)$.

The quadratic relation

Let R be the R-matrix for $U_q(\mathfrak{sl}_2)$ or more generally the $U_q(\mathfrak{sl}_n)$ R-matrix found by Jimbo which contains the $\mathfrak{sl}_2$ R-matrix as a special case. Then $T = \tau R$ satisfies a quadratic relation:

$$T^2 = (q - q^{-1})T + 1,$$

as can easily be checked. As we will now explain, this implies that the braid group representation that we have constructed extends to a representation of the Hecke algebra of S_r, a certain deformation of the group algebra $\mathbb{C}[S_r]$.

The Hecke algebra

Let W be a Coxeter group. The Hecke algebra $\mathcal{H}_q$ (in one normalization) has generators T_i corresponding to the simple reflections. They are assumed to satisfy the braid relations and the quadratic relations

$$T_i^2 = (q - q^{-1})T_i + 1.$$

If $w \in W$, let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression and define

$$T_w = T_{i_1} \cdots T_{i_k}.$$

By Matsumoto's theorem this is well-defined. Moreover, it is not hard to show that this is a basis of $\mathcal{H}_q$, which is therefore a finite-dimensional algebra whose dimension is $|W|$.

$\mathcal{H}_q$ is a deformation of W

T_i is invertible since

$$T_i(T_i - q + q^{-1}) = 1, \qquad T_i^{-1} = T_i - q + q^{-1}.$$

So we may map the braid group into the multiplicative group of the Hecke algebra by $t_i \to T_i$.

Note that if $q \to 1$ then the quadratic relation becomes $T_i^2 = 1$, so in the limit, the relations satisfies by the T_i are the same as the s_i. Thus the Hecke algebra is a deformation of $\mathbb{C}[W]$.

$\mathcal{H}_q$ is ubiquitous.

The Hecke algebra arises in a remarkable variety of different settings.

- Iwahori and Matsumoto showed the Hecke algebras of W and of its (infinite) affine Weyl group appear as convolution rings of functions on a p-adic group.
- Howlett and Lehrer showed that the Hecke algebras appear as endomorphism rings of induced representations of finite groups.
- Lusztig interpreted the Hecke algebra as the equivariant K-theory of the flag variety.
- Jimbo showed that $\mathcal{H}_q$ appears in a deformation of Schur duality.
- The Temperley-Lieb algebras in statistical mechanics are a version of the Hecke algebra.

Schur-Weyl-Jimbo duality

Now let V be the standard module for $U_q(\mathfrak{sl}_2)$ or more generally $U_q(\mathfrak{sl}_n)$. It is actually better to enlarge the quantum group a little to obtain generally $U_q(\mathfrak{gl}_n)$, and the same R-matrix still works.

We have noted that we may obtain a braid group representation $T_i \in \mathrm{End}(\otimes^r V)$ to be the endomorphism $R_{i,i+1}$, then the satisfy the braid relation

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

Since the T_i satisfy the quadratic relations

$$T_i^2 = (q - q^{-1}) T_i + 1$$

this is actually a representation of the Hecke algebra. It commutes with the action of $U_q(\mathfrak{gl}_n)$, and in the limit $q \to 1$ these two representations essentially reduce to the representations of S_r and $\mathrm{GL}(n)$ in Schur duality.

Schur-Weyl-Jimbo duality (concluded)

If q is not a root of unity, then just as the irreducible representations of $\mathrm{GL}(n)$ are parametrized by dominant weights, so are those of $U_q(\mathfrak{gl}_n)$.

(This is not quite true: we have to specify how the group-like generators K_i act. We are ignoring this nuance.)

The irreducible representations of the Hecke algebra are, like those of S_r, indexed by partitions. And

$$\otimes^r V \cong \bigoplus_\lambda \pi_\lambda^{U_q(\mathfrak{gl}_n)} \otimes \pi_\lambda^{\mathcal{H}_q}.$$