Lecture 6

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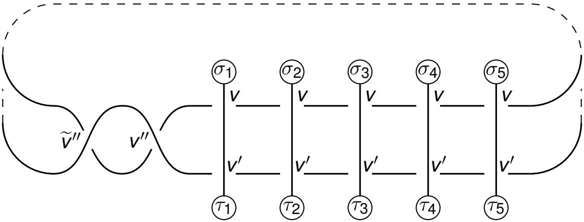


$U_q(\mathfrak{sl}_2)$

We remind the reader of $U_q(\mathfrak{sl}_2)$. It has generators E, F and K; K is "group-line" and has a multiplicative inverse K^{-1}. The relations are

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The counit and comultiplication are

$$\Delta(K) = K \otimes K,$$

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$$\varepsilon(K) = 1, \qquad \varepsilon(E) = \varepsilon(F) = 0.$$

The antipode satisfies $S(K) = K^{-1}$, $S(E) = -E$, $S(F) = -F$.

Representations of SL_2

With a caveat, the following representation theories are the same:

- Finite-dimensional representations of $SL(2, \mathbb{R})$;
- Finite-dimensional representations of $SU(2)$;
- Finite-dimensional analytic representations of $SL(2, \mathbb{C})$;
- Finite-dimensional complex representations of their Lie algebras $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{su}_2$, $\mathfrak{sl}_2(\mathbb{C})$;
- The enveloping algebras of these Lie algebras;
- The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, where q is not a root of unity.

The caveat is that these categories are the same simple objects but $U_q(\mathfrak{sl}_2)$ has an interesting comultiplication and R-matrix so they are not the same as braided categories.

Representations of SL_2 (continued)

Each of the above categories is semisimple, that is, every object is a direct sum of irreducibles.

There is one unique irreducible for every dimension. The two-dimensional standard representation V generates the category: every irreducible is a submodule of $\bigotimes^k V$ for some k.

More precisely, the k-dimensional irreducible module is the symmetric power $\mathrm{Sym}^{k-1}(V)$.

Modules of $\mathfrak{sl}_2$

The standard module of $\mathfrak{sl}_2$ has two basis vectors x and y; with respect to this basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $H = [E, F]$. For the 4-dimensional irreducible:

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Modules of $U_q(\mathfrak{sl}_2)$

The standard module of $U_q(\mathfrak{sl}_2)$ again has basis vectors x and y; with respect to this basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

For the 4-dimensional irreducible:

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & [3] \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ [3] & 0 & 0 & 0 \\ 0 & [2] & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}$$

where the “quantum integer” $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The standard module of $U_q(\mathfrak{sl}_2)$

So the quantized standard module has two basis vectors x, y

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$

where recall $EF - FE = (K - K^{-1})/(q - q^{-1})$.

The R-matrix

We are not ready to prove that $H = U_q(\mathfrak{sl}_2)$ is (in a suitable sense) quasitriangular. The universal R-matrix lives not in H but in a suitable completion, and its existence is a subtle matter.

Nevertheless it is not the universal R-matrix itself, but the morphisms it induces. We will study the R-matrix for the standard module V of $U_q(\mathfrak{sl}_2)$.

We will determine the endomorphisms of $V \otimes V$, then pick out the ones that satisfy the Yang-Baxter equation.

Endomorphism of $V \otimes V$

Let x, y be the standard basis of V. Recall

$$K(x) = qx, \qquad K(y) = q^{-1}y, \qquad E(x) = 0, \qquad E(y) = x,$$

$$F(x) = y, \qquad F(y) = 0,$$

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

We consider an endomorphism τR of $V \otimes V$ with respect to the basis $x \otimes x$, $x \otimes y$, $y \otimes x$ and $y \otimes y$. The eigenspaces of K:

eigenvalue	basis
q^2	$x \otimes x$
1	$x \otimes y, y \otimes x$
q^{-2}	$y \otimes y$

These must be invariant under τR.

E and F

$$K(x) = qx, \qquad K(y) = q^{-1}y, \qquad E(x) = 0, \qquad E(y) = x,$$

$$F(x) = y, \qquad F(y) = 0$$

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

$$E(x \otimes x) = 0 \qquad F(x \otimes x) = q^{-1} x \otimes y + y \otimes x$$

$$E(x \otimes y) = x \otimes x \qquad F(x \otimes y) = y \otimes y$$

$$E(y \otimes x) = q(x \otimes x) \qquad F(y \otimes x) = q(y \otimes y)$$

$$E(y \otimes y) = q^{-1}(x \otimes y) + y \otimes x \qquad F(y \otimes y) = 0$$

$$E = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \end{pmatrix}$$

The R-matrix

Since τR preserves the eigenspaces it has the form

$$\begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix}.$$

Since it commutes with

$$E = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \end{pmatrix}$$

it is easy computation to see that it is a a scalar multiple of

$$\begin{pmatrix} 1 & & & \\ & 1 - qb & b & \\ & b & 1 - q^{-1}b & \\ & & & 1 \end{pmatrix}.$$

The Yang-Baxter equation

We must now have the Yang-Baxter equation

$$(\tau R)_{12}(\tau R)_{23}(\tau R)_{12} = (\tau R)_{23}(\tau R)_{12}(\tau R)_{23},$$

or equivalently

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

With

$$\tau R = \begin{pmatrix} 1 & & & \\ & 1-qb & b & \\ & b & 1-q^{-1}b & \\ & & & 1 \end{pmatrix},$$

this implies $b = 0, q$ or q^{-1}. We discard the solution $b = 0$.

The R-matrices

We arrive at two R-matrices for $U_q(\mathfrak{sl}_2)$. We multiply the $b = q$ solution by q^{-1}:

$$R = \begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ & q^{-1} - q & 1 & \\ & & & q^{-1} \end{pmatrix}.$$

Similarly $b = q^{-1}$ gives the alternative R-matrix:

$$\widetilde{R} = \begin{pmatrix} q & & & \\ & 1 & q - q^{-1} & \\ & & 1 & \\ & & & q \end{pmatrix}.$$

Both satisfy

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and produce H-module endomorphism of $V \otimes V$.

Dual R-matrices

Why are there two R-matrices? Let us consider the case of a QTHA H with universal R-matrix $R = R^{(1)} \otimes R^{(2)}$. Then it is easy to see from the axioms that $\widetilde{R} = R_{21}^{-1}$ is also an R-matrix. (This is proved in Majid Chapter 5.)

Here $R_{21} = R^{(2)} \otimes R^{(1)}$. Now consider the case of $\mathfrak{sl}_2$. By abuse of notation we will use the same letter R to denote both the universal R-matrix and the endomorphism of V that it induces.

Dual R-matrices, continued

So what endomorphism of $V \otimes V$ does R_{21} induce? Note that R_{21} is R conjugated by the flip endomorphism $\tau : V \to V$, which respect to the basis $x \otimes x$, $x \otimes y$, $y \otimes x$ and $y \otimes y$ is the matrix

$$\tau = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}.$$

Consistent with this, for

$$R = \begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ & q^{-1} - q & 1 & \\ & & & q^{-1} \end{pmatrix}, \qquad \widetilde{R} = \begin{pmatrix} q & & & \\ & 1 & q - q^{-1} & \\ & & 1 & \\ & & & q \end{pmatrix},$$

we find that $\widetilde{R} = \tau R^{-1} \tau$.

A parametrized Yang-Baxter equation

For later reference, we make note of the following parametrized Yang-Baxter equation.

Define

$$R(x,y) = xR - y\widetilde{R}.$$

Then

$$R_{12}(x,y)R_{13}(x,z)R_{23}(y,z) = R_{23}(y,z)R_{13}(x,z)R_{12}(x,y).$$

If $\mathfrak{g}$ is a simple Lie algebra there is always a parametrized Yang-Baxter equation associated with $U_q(\mathfrak{g})$ or more precisely its affinization $U_q(\widehat{\mathfrak{g}})$. Is not always linear in R and R_{21}^{-1} as above. It can be quadratic: see Jimbo, Introduction to the Yang-Baxter equation, Internat. J. Modern Phys. A 4 (1989), equation (5.2).

Statistical mechanics

In statistical mechanics one is concerned with large ensembles of states. Highly energetic states are less probable.

If T is the temperature, the probability of a state $\mathfrak{s}$ is proportional to the Boltzmann weight $e^{-k\beta(\mathfrak{s})/T}$. Here β is the energy of the state. The actual probability is

$$\frac{1}{Z(\mathfrak{S})} e^{-k\beta(\mathfrak{s})/T},$$

where

$$Z(\mathfrak{S}) = \sum_{\mathfrak{s}} e^{-k\beta(\mathfrak{s})/T}$$

is the partition function.

Solvable lattice models

Certain systems may be solved exactly. The first such example was the 2-dimensional Ising model (Onsager 1944). The six-vertex model was solved by Lieb and Sutherland in the 1960's, then solved another way by Baxter whose method extended to the eight-vertex model. Baxters results also had applications to Heisenberg spin chains in quantum mechanics.

The six-vertex model was proposed by Linus Pauling (1936) in connection with two-dimensional ice, so these models are also called ice-type models.

The six-vertex model was a key example leading to quantum groups. Its generalizations have applications far beyond its origins in statistical mechanics.

Ice type models

The models that we will be concerned with take place on planar graphs. In using the term graph to describe these arrays we are deviating from usual terminology, where edges have always two vertices, for we will allow open edges with only a single endpoint.

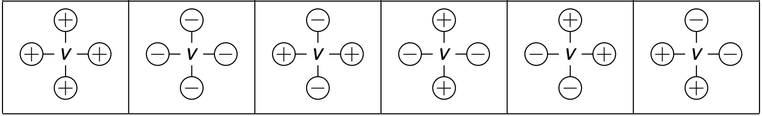
Thus we have a set of vertices which are points in the plane, together with edges that are arcs which either join two vertices, or which are attached to only a single vertex. The edges which are only attached to a single vertex are called boundary edges. Edges attached to two vertices are called interior edges. Every vertex is adjacent to four edges. Edges can only cross at a vertex.

Boltzmann weights

A spin is an element of the set $\{+, -\}$. In a state of the system $\mathfrak{S}$ every edge of the graph is assigned a spin. The spins of the boundary edges are fixed. Those of the interior edges are variable.

Every vertex v is assigned a set of Boltzmann weights. These assign a value $\beta(v)$ depending on the configuration of spins on the adjacent edges. Only six configurations are allowed.

$a_1(v)$	$a_2(v)$	$b_1(v)$	$b_2(v)$	$c_1(v)$	$c_2(v)$



The partition function

A state of the system is an assignment of spins to the edges, with the spins of the boundary edges fixed and part of the data describing the system.

The Boltzmann weight of the state:

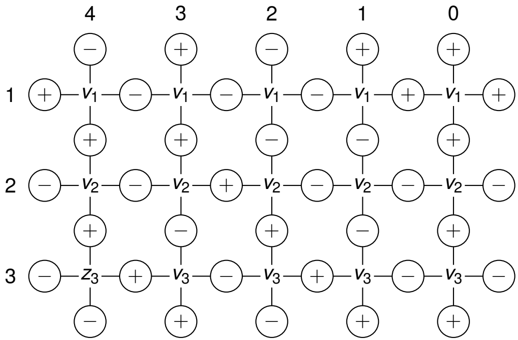
$$\beta(\mathfrak{s}) = \prod_{v} \beta(v).$$

The partition function

$$Z(\mathfrak{S}) = \sum_{\mathfrak{s}} \beta(\mathfrak{s}).$$

Toroidal boundary conditions

We consider a planar grid. We use use toroidal boundary conditions so that the left and right boundary edges are identified. We allow different Boltzmann weights from row to row, even if we are only interested in the case all v_i are the same.



Row transfer matrix

We consider the sequence of spins $\sigma = (\sigma_i)$ in a row of vertical edges to be a vector. If σ, τ are two such vectors, the product of the Boltzmann weights of the vertices in this row may be regarded as an entry in a matrix $T_v(\sigma, \tau)$. Here all vertices have the same Boltzmann weight v. But the vertices in different rows may have different weights.

If $v_1, \cdots, v_k$ are the Boltzmann weights in the rows then the partition function

$$Z(\mathfrak{S}) = \mathrm{tr}\big(T_{v_1} \cdots T_{v_r}\big)\,.$$

Commuting transfer matrices

Baxter's insight was that it is possible to diagonalize the row transfer matrix if it can be embedded in a large family of commuting row transfer matrices. If this can be done, the model is called solvable or integrable.

We restrict ourselves to weights that are field free meaning

$$\mathrm{a}_1(\mathrm{v}) = \mathrm{a}_2(\mathrm{v}), \qquad \mathrm{b}_1(\mathrm{v}) = \mathrm{b}_2(\mathrm{v}), \qquad \mathrm{c}_1(\mathrm{v}) = \mathrm{c}_2(\mathrm{v}).$$

We denote these as just $\mathrm{a}(\mathrm{v})$, $\mathrm{b}(\mathrm{v})$ and $\mathrm{c}(\mathrm{v})$. Let

$$\Delta(v) = \frac{\mathrm{a}(\mathrm{v})^2 + \mathrm{b}(\mathrm{v})^2 - \mathrm{c}(\mathrm{v})^2}{2\mathrm{a}(\mathrm{v})\mathrm{b}(\mathrm{v})}.$$

Commuting transfer matrices

The parameter Δ determines the gross characteristics of the system such as the largest eigenvalue of the row transfer matrix. Assuming the Boltzmann weights to be positive and real, Δ has physical significance. If $\Delta > 1$ or $\Delta < -1$, the system is "frozen" and there are correlations between distant spins. If $|\Delta| < 1$ the system is in a disordered state. Thus $\Delta = \pm 1$ are phase transitions.

To study the eigenvalues of T_v, we embedd T_v in a large family of commuting matrices.

Theorem (Baxter)

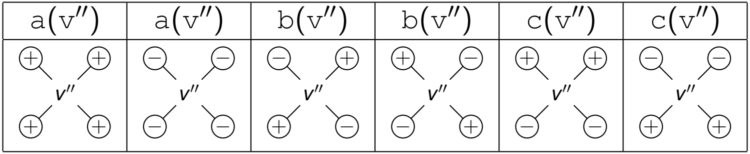
Suppose v and v' are two vertices such that $\Delta(v) = \Delta(v')$ Then T_v and $T_{v'}$ commute.

The Yang-Baxter equation

The key to the last result is a Parametrized Yang-Baxter equation which in retrospect is related to the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Here q is chosen so that

$$\Delta = \frac{1}{2}(q + q^{-1}).$$

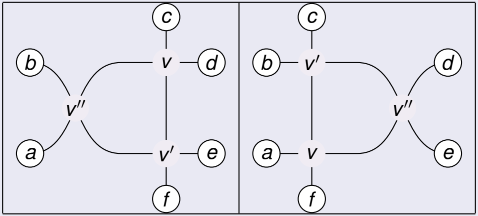
We will describe the relevant Yang-Baxter equations as Baxter understood them. We must introduce an auxiliary vertex v'' that we will draw rotated as follows.



The Yang-Baxter equation

Theorem (Baxter)

If $\Delta(v) = \Delta(v')$ then there exists a vertex v'' such that following two partition functions are equal for every choice of boundary spins $a, b, c, d, e, f \in \{+, -\}$.



Thus the boundary spins are to be fixed and we sum over the spins of the internal unnumbered edges. The value $\Delta(v'')$ also equals $\Delta(v) = \Delta(v')$.

A parametrized Yang-Baxter equation

Let us fix $q \in \mathbb{C}^\times$. If $x \in \mathbb{C}^\times$ define $v(x)$ to be the vertex with Boltzmann weights

$$a = q - x^2/q, \qquad b = 1 - x^2, \qquad c = x(q - 1/q).$$

Then if $v = v(x)$ and $v' = v(y)$, Baxter's identity may be checked by direct computation with $v'' = v(x/y)$.

This special case implies Baxter's theorem since given any v with $\Delta(v) = \frac{1}{2}(q + q^{-1})$, we may find a λ and x such that

$$a = \lambda(q - x^2/q), \qquad b = \lambda(1 - x^2), \qquad c = \lambda\, x(q - 1/q).$$

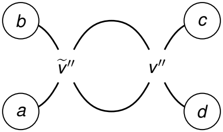
Multiplying the weights by a constant just multiplies the partition functions on both sides by the same constant, and so this parametrized YBE implies Baxter's theorem.

The inverse R-matrix

Let $v = v(x)$, $v' = v(y)$ and $v'' = v(x/y)$. Let $\widetilde{v}'' = c^{-1}v(y/x)$ where with $z = y/x$

$$c = (qz)^{-2}(z-q)(z+q)(qz-1)(qz+1).$$

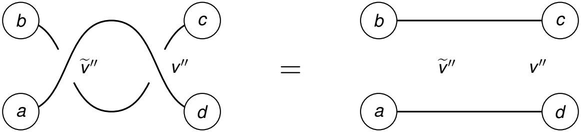
We mean the Boltzmann weights of $v(y/x)$ are multiplied by this constant to obtain $\widetilde{v}''$. We compute that the partition function of:



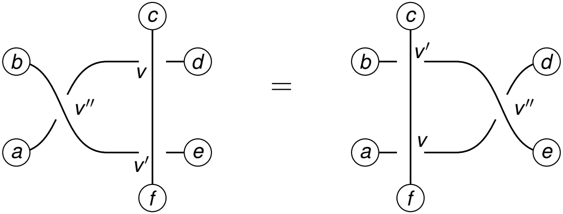
is $\delta(a,d)\,\delta(b,c)$ (Kronecker δ). The same is true if v, v'' are switched.

The inverse R-matrix

In order to show that v'' and v'' cancel, we will draw them as over-crossings and under-crossing.

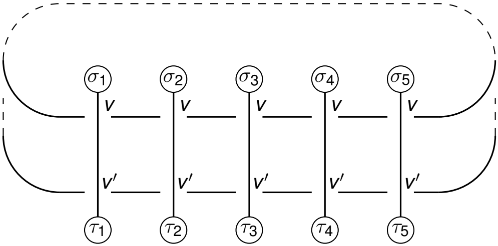


This resembles a Reidemeister move, so we give the Yang-Baxter equation the same treatment:



Commuting transfer matrices

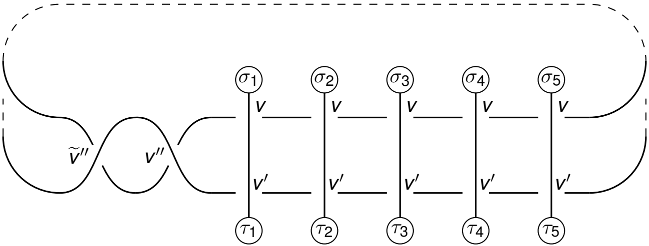
We may now explain how Baxter proved the commutativity of the row transfer matrices in the case of toroidal boundary conditions. We will fix v and v' with $\Delta(v) = \Delta(v')$ and prove that the row transfer matrices T_v and $T_{v'}$ commute.



Here we compute $\langle \tau, T_{v'} T_v \sigma \rangle$.

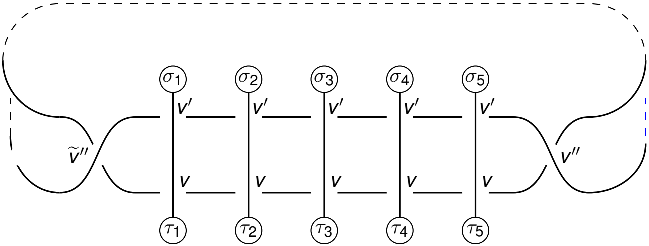
Setting up the train argument

We find v'' and $\widetilde{v}''$ as above and insert them between the rows. This does not change the partition function.



The train argument

Repeatedly using the Yang-Baxter equation, the braid moves to the right.

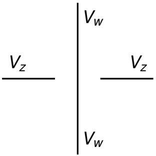


After repeatedly applying the Yang-Baxter equation, the braid v'' moves across the two rows. Note that v and v' are now switched. Due to the toroidal boundary conditions, the braids v'' and $\tilde{v}''$ are again in juxtaposition and may be discarded.

Edges as modules

The parameter Δ is fixed. Moreover we choose and fix q such that $\Delta = \frac{1}{2}(q + q^{-1})$.

We postulate a category some of whose objects are vector spaces V_z indexed by a nonzero complex number z. Every edge will be assigned a V_z such that opposite edges through the same vertex v will be assigned the same V_z.



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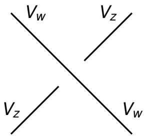
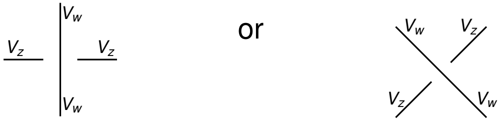


Interpretation of Boltzmann weights

The module V_z will be a two-dimensional vector space with basis $v_+ = v_+(z)$, $v_- = v_-(z)$ indexed by the possible spins $\pm$. At a vertex



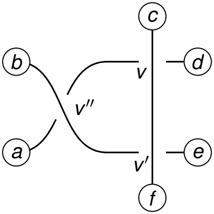
we take the Boltzmann weights with

$$a = \lambda(q - x^2/q), \qquad b = \lambda(1 - x^2), \qquad c = \lambda\, x(q - 1/q),$$

where $x = z/w$.

Example

We can assign V_z's consistently, for example in this case

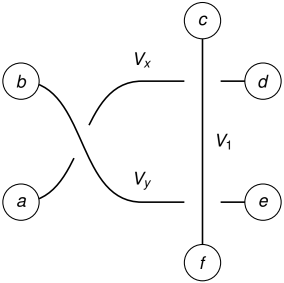


we want v, v' and v'' to have Boltzmann weights

$$a = \lambda(q - z^2/q), \qquad b = \lambda(1 - z^2), \qquad c = \lambda\, z(q - 1/q),$$

with z taking values $x, y, x/y$ respectively. So we label the edges thus:

Example (continued)



All the Boltzmann weights are then correct, and similarly for the other side of the Yang-Baxter equation.

Interpretation of Boltzmann weights (continued)

Now we will interpret the Boltzmann weights as describing an endomorphism $R(z,w) \in \mathrm{End}(V_z \otimes V_w)$. With a, b, c as above this is the transformation

$$\begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}$$

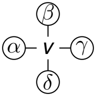
with respect to the basis

$$v_+(z) \otimes v_+(w), \qquad v_+(z) \otimes v_-(w),$$

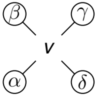
$$v_-(z) \otimes v_-(w) \qquad v_-(z) \otimes v_-(w).$$

The R-matrix

In other words, if $\alpha, \beta, \gamma, \delta$ are spins, and we define $R^{\gamma,\delta}_{\alpha,\beta} = R^{\gamma,\delta}_{\alpha,\beta}(z, w)$ to be the Boltzmann weight of the vertex at



or

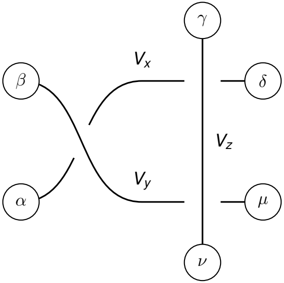


then

$$R(z, w) : v_\alpha \otimes v_\beta \mapsto \sum_{\gamma,\delta} R^{\gamma,\delta}_{\alpha,\beta}(v_\gamma \otimes v_\beta).$$

The left side of the Yang-Baxter equation

Now let us determine the matrix S such that the partition function of:

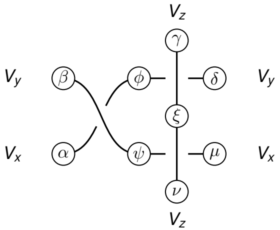


is $S^{\delta,\mu,\nu}_{\alpha,\beta,\gamma}$. Regard this as an endomorphism of $V_y \otimes V_x \otimes V_z$:

$$S(v_\alpha \otimes v_\beta \otimes v_\gamma) = \sum_{\delta,\mu,\nu} S^{\delta,\mu,\nu}_{\alpha,\beta,\gamma}(v_\delta \otimes v_\mu \otimes v_\nu).$$

The left side of the Yang-Baxter equation

We have to sum over the possible spins of the three interior edges, labeled ϕ, ψ, ξ in the following:



Thus

$$S^{\delta,\mu,\nu}_{\alpha,\beta,\gamma} = \sum_{\phi,\psi,\xi} R^{\phi,\xi}_{\alpha,\beta}(x,y) R^{\delta,\xi}_{\phi,\gamma}(x,z) R^{\mu,\nu}_{\psi,\xi}(y,z).$$

In matrix notation,

$$S = R(x,y)_{12} R(x,z)_{13} R(y,z)_{23}.$$

The parametrized Yang-Baxter equation reappears

The other side of the Yang-Baxter equation is treated similarly, and so it is equivalent to the matrix identity:

$$R(x,y)_{12}R(x,z)_{13}R(y,z)_{23} = R(y,z)_{23}R(x,z)_{13}R(x,y)_{12}.$$

This is equivalent to the parametrized Yang-Baxter equation we encountered earlier in this lecture.

Affine $\widehat{\mathfrak{sl}}_2$

To get this R-matrix from the theory of quantum groups, we need a quantum group with one two-dimensional module V_z for each $z \in \mathbb{C}^\times$. This is $U_q(\widehat{\mathfrak{sl}}_2)$ for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. It can be constructed as a central extension of the “loop” Lie algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$, the Lie algebra defined by

$$[X \otimes t^k, Y \otimes t^\ell] = [X, Y] \otimes t^{k+\ell}.$$

The central extension is needed for infinite-dimensional representations. If V is a finite-dimensional $\mathfrak{sl}_2$-module and $z \in \mathbb{C}^\times$ then there is a representation V_z of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$ in which

$$(X \otimes t^k) \cdot v = z^k X \cdot v.$$

Relation with $U_q(\mathfrak{sl}_2)$

A variant replaces this R-matrix

$$R(x,y) = \begin{pmatrix} q - z^2/q & & & \\ & 1 - z^2 & z(q-1/q) & \\ & z(q-1/q) & 1-z^2 & \\ & & & q - z^2/q \end{pmatrix}$$

(where $z = x/y$) by

$$\widehat{R}(x,y) = \begin{pmatrix} q - z^2/q & & & \\ & 1 - z^2 & z^2(q-1/q) & \\ & (q-1/q) & 1-z^2 & \\ & & & q - z^2/q \end{pmatrix}.$$

It is not obviously equivalent but still true that the parametrized Yang-Baxter holds:

$$\hat{R}(x,y)_{12}\hat{R}(x,z)_{13}\hat{R}(y,z)_{23} = \hat{R}(y,z)_{23}\hat{R}(x,z)_{13}\hat{R}(x,y)_{12}.$$

Relation with $U_q(\mathfrak{sl}_2)$ (continued)

This has the advantage that we may specialize $z \to 0$ and obtain

$$R = \begin{pmatrix} q & & & \\ & 1 & & \\ & q - 1/q & 1 & \\ & & & q \end{pmatrix}.$$

This is the R-matrix for $U_q(\mathfrak{sl}_2)$ in its standard 2-dimensional representation.

(Comparing the R-matrix from earlier in this lecture, q has become q^{-1}, so more correctly this is the R-matrix for $U_{q^{-1}}(\mathfrak{sl}_2)$.)