Lecture 5

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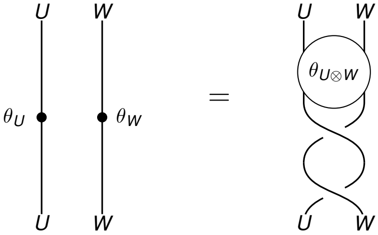


Review of Tangles

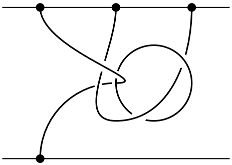
In our last lecture, we introduced the the tangle category, the category of framed tangles, and the notion of a ribbon category. Let us remind ourselves of these definitions.

If m, n are nonnegative integers an (m, n)-tangle is a collection of (piecewise smooth) arcs and circles in $\mathbb{R}^2 \times [0, 1]$ with m endpoints on $\mathbb{R}^2 \times \{0\}$ and n endpoints on $\mathbb{R}^2 \times \{1\}$. We take the endpoints to be

$$(k, 0, 0), \quad 1 \leqslant k \leqslant m, \qquad (\ell, 0, 0), \quad 1 \leqslant \ell \leqslant m.$$

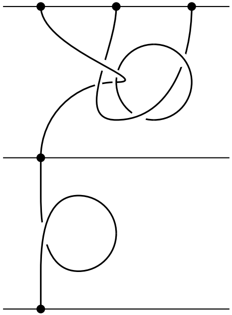
We identify tangles equivalent by ambient isotopy.

A $(3, 1)$ tangle represented by its $\mathbb{R}^2$ projection (always drawn upside-down)



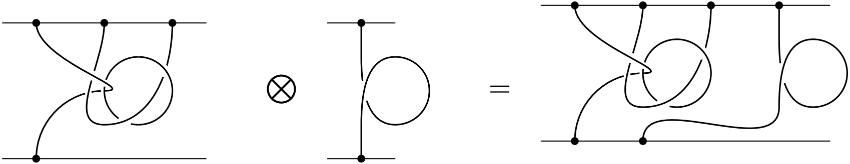
Tangles form a category

Objects in the tangle category are nonnegative integers. Given tangles T in $\mathrm{Hom}(a, b)$ and U in $\mathrm{Hom}(b, c)$, join the b lower endpoints of T to the b endpoints of U. After rescaling everything into $\mathbb{R}^2 \times [0, 1]$ obtain a tangle $U \circ T \in \mathrm{Hom}(a, c)$.



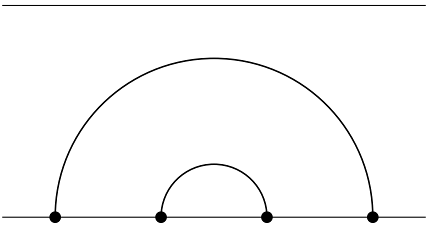
Tangles form a monoidal category

If $T \in \mathrm{Hom}(m, n)$ and $U \in \mathrm{Hom}(m', n')$, replace T and U by isotopic tangles with the endpoints of U moved to the right of those of T. Arrange that the curves in the modified T and U are in disjoint topological half-spaces. Then $T \otimes U$ will be the union of all arcs and closed curves of T and U.



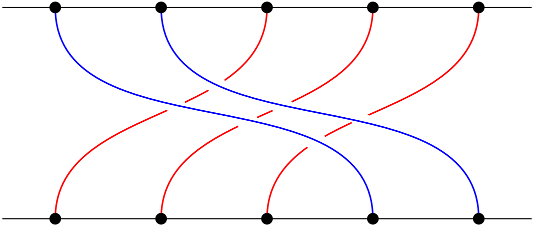
Tangles form a rigid category

If m is an object in the tangle category we define $m^* = m$ and we can make the tangle category into a rigid category. Here is the coevaluation morphism $0 \to 2 \otimes 2^*$.



Tangles form a braided category

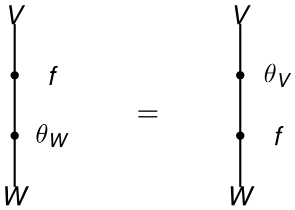
We may introduce a braiding by specifying morphisms in $\mathrm{Hom}(m \otimes n, n \otimes m)$. Here is the braiding for $m = 2, n = 3$.



Review of ribbon categories

Just now we will only restate the axioms. A ribbon category is a rigid braided category with a bit of extra structure. We require for each object V a natural morphism $\theta_V : V \to V$.

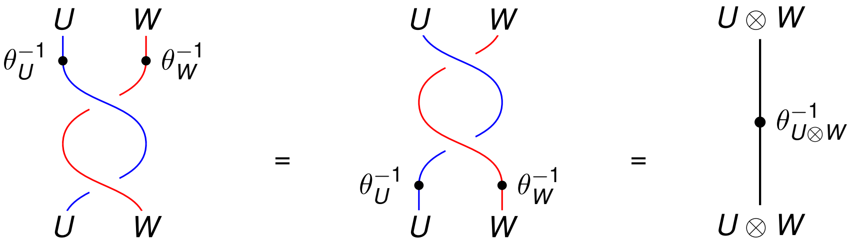
Naturality means if $f : V \to W$ then $\theta_W f = f\theta_V$:



The Tensor Ribbon Axiom

We must have (using naturality of $c_{U,W}$ and $c_{W,U}$:

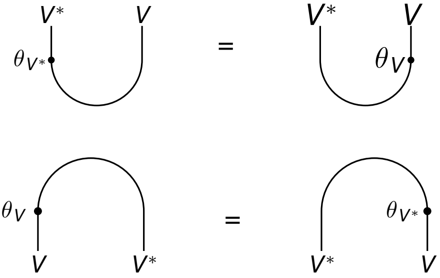
$$\theta_{U\otimes W}^{-1} = c_{W,U} \circ c_{U,W} \circ \theta_U^{-1} \otimes \theta_W^{-1} = \theta_U^{-1} \otimes \theta_W^{-1} \circ c_{W,U} \circ c_{U,W}$$



Ribbon axioms, concluded

Unit axiom. I is the unit object in the category then $\theta_I = 1_I$.

Dual axiom. If V is an object then $\theta_{V^*} = \theta_V^*$. By Exercise 3 from Lecture 4 this amounts to compatibility with ev_V and coev_V:



These last two axioms seem innocuous but they are powerful and important as we will see later.

Framed tangles

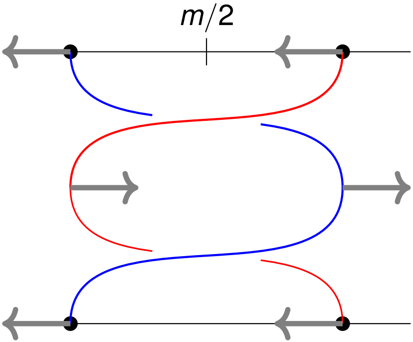
The category of framed tangles adds a normal vector field to each curve of the tangle. At the endpoints of the arcs adjoining the boundary, the direction is to be $(0, -1, 0)$ so that the field remains continuous when two morphisms are glued.

Just like tangles, framed tangles form a braided, rigid category.

Even better, they form a ribbon category.

Twists in the category of framed tangles

If m is an object in the framed tangle category, then m is an integer. Define $\theta_m : m \to m$ to be the parametrized arc $(0, m/2, t) - (\frac{m}{2}\cos(2\pi t), \frac{m}{2}\sin(2\pi t), 0)$, $t \in [0, 1]$. We also rotate the vector field. If $m = 2$:



The morphisms u_V and $\mathfrak{v}_V$

In Lecture 4 we introduced (in a rigid braided category) two invertible morphisms $\mathfrak{u}_V : V \to V^{**}$ and $\mathfrak{v}_V : V^{**} \to V$. They are not inverses of each other. Here are $\mathfrak{u}_V$ and $\mathfrak{v}_V$ and their inverses.

$$\mathfrak{u}_V : V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$

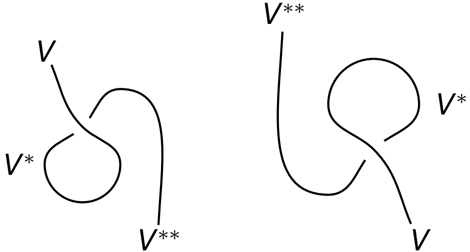
$$\mathfrak{u}_V^{-1} : V^{**} \xrightarrow{1_{V^{**}} \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^* \xrightarrow{c_{V^{**},V} \otimes 1_{V^*}} V \otimes V^{**} \otimes V^* \xrightarrow{1_V \otimes \mathrm{ev}_{V^*}} V$$

$$\mathfrak{v}_V : V^{**} \xrightarrow{1_V \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^* \xrightarrow{1_{V^{**}} \otimes c_{V,V^*}} V^{**} \otimes V^* \otimes V \xrightarrow{\mathrm{ev}_{V^*} \otimes 1_V} V$$

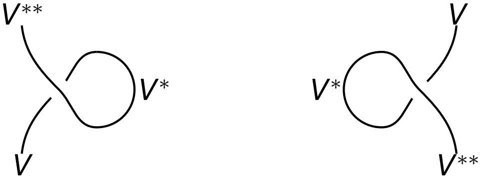
$$\mathfrak{v}_V^{-1} : V \xrightarrow{\mathrm{coev}_{V^*} \otimes 1_V} V^* \otimes V^{**} \otimes V \xrightarrow{1_{V^*} \otimes c_{V^{**},V}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$

$\mathfrak{u}_V$, $\mathfrak{v}_V$, $\mathfrak{u}_V^{-1}$, $\mathfrak{v}_V^{-1}$ in pictures

$\mathfrak{u}_V, \mathfrak{v}_V$:



$\mathfrak{u}_V^{-1}, \mathfrak{v}_V^{-1}$:



The relation between θ_V and $\mathfrak{u}_V$ and $\mathfrak{v}_V$

We argued heuristically in Lecture 3 that u and v both twist the strand by 2π; we came to this by studying $V = U \otimes W$. However $\mathfrak{u}_V$ and $\mathfrak{v}_V$ do not relate V to itself, so the best we can do is to consider $\mathfrak{v}_V\mathfrak{u}_V : V \to V$, which should be a self-twist in 4π.

Now assume that the category is ribbon. The case of the framed tangle category shows that $\theta_V : V \to V$ is a twist in 2π, we want θ_V to be a square root of $\mathfrak{v}_V\mathfrak{u}_V$. We do not have to prove this, since it follows from the axioms!

Theorem

We have $\theta_V^2 = \mathfrak{v}_V\mathfrak{u}_V$.

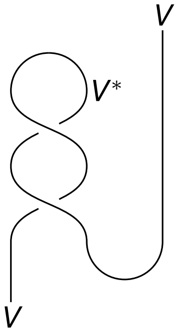
The proof will demonstrate how to work with the ribbon axioms.

About θ_V^2

We will begin by showing that

$$\theta_V^2 = (I_V \otimes \mathrm{ev}_V)(c_{V^*,V}c_{V,V^*} \otimes I_V)(\mathrm{coev}_V \otimes I_V),$$

that is:



Then our task will be to show that this morphism equals $\mathfrak{v}_V \mathfrak{u}_V$.

Equivalent form of the ribbon axiom

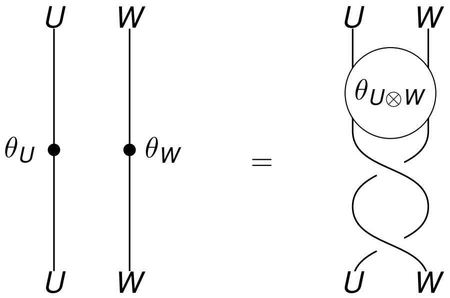
Note that the tensor ribbon axiom

$$\theta_{U\otimes W}^{-1} = \theta_U^{-1} \otimes \theta_W^{-1} \circ c_{W,U} \circ c_{U,W}$$

is equivalent to

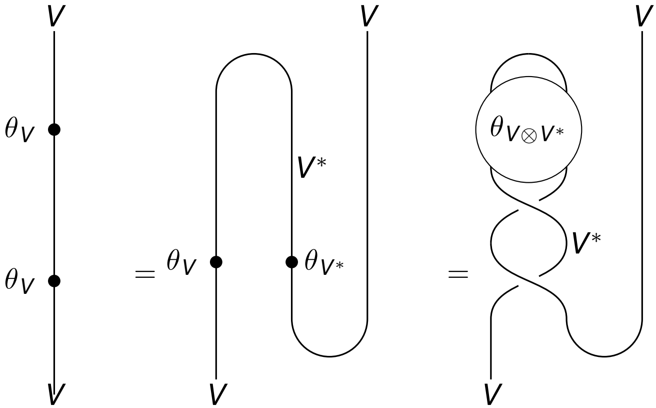
$$\theta_U \otimes \theta_W = c_{W,U} \circ c_{U,W} \circ \theta_{U\otimes W}$$

That is:



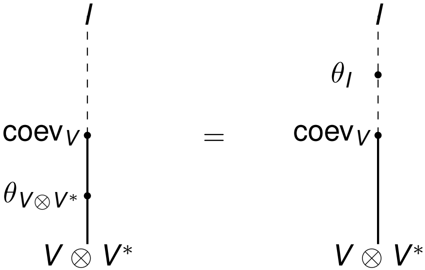
The first evaluation of θ_V^2

We apply this using the adjointness of θ and the straightening property of the dual, θ_V equals



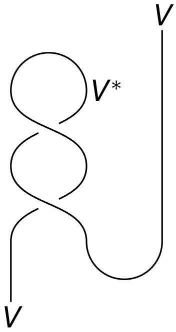
Getting rid of $\theta_{V\otimes V^*}$

We can discard the $\theta_{V\otimes V^*}$ for the following Use naturality. Representing the unit object in the category by I, we use the naturality to move the θ past the coev_V and then remember the axiom that $\theta_I = 1_I$:



So far so good ...

We have proved that θ_V^2 equals:

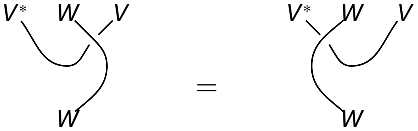


We will now show that $\mathfrak{v}_V \mathfrak{u}_V$ also equals this.

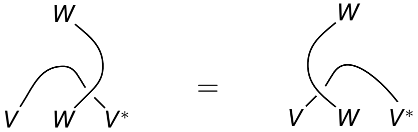
Reminder of Lecture 3

In Lecture 3 we proved that

$$(\mathrm{ev}_V \otimes 1_W)(1_{V^*} \otimes c_{W,V}) = (1_W \otimes \mathrm{ev}_V)(c^{-1}_{W,V^*} \otimes 1_V)$$

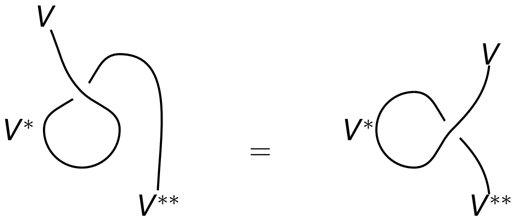


We will call this the coevaluation crossing identity. The evaluation crossing identity is similar:



Evaluation of $\mathfrak{v}_V \mathfrak{u}_V$

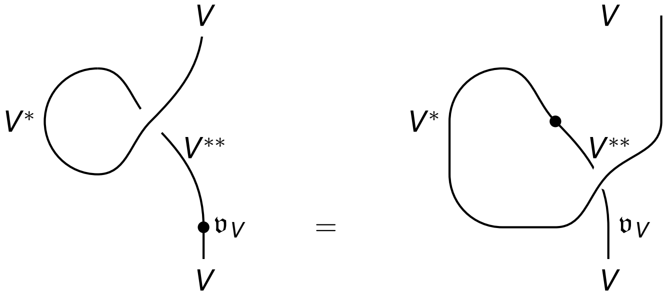
An immediate consequence of the coevaluation crossing identity is that we may rewrite $\mathfrak{u}_V$:



Normally we prefer the first representation since the crossing c_{V,V^*} is preferred to $c^{-1}_{V^*,V}$, but for our next step the second representation of $\mathfrak{u}_V$ works better.

Manipulation of $\mathfrak{v}_V\mathfrak{u}_V$

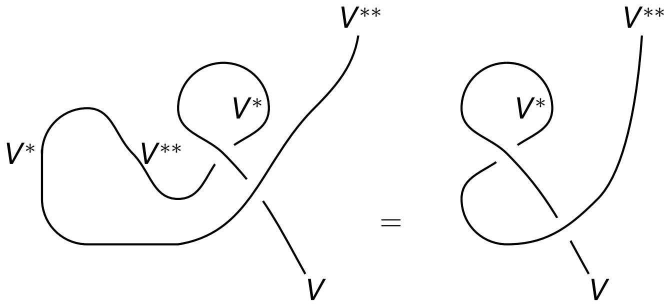
We may now write $\mathfrak{v}_V\mathfrak{u}_V$ as follows:



We have used the naturality of the braiding.

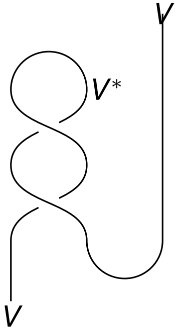
Manipulation of $\mathfrak{v}_V \mathfrak{u}_V$ (continued)

Now we substitute the definition of $\mathfrak{v}_V$, then use the straightening property of the dual:



Manipulation of $\mathfrak{v}_V \mathfrak{u}_V$ (concluded)

Finally, using the evaluation crossing identity shows that $\mathfrak{v}_V \mathfrak{u}_V$ equals:



Since this is our expression for θ_V^2 we have proved the theorem.

Coboundary Hopf Algebras

We introduced quasitriangular Hopf algebras (QTHA) in Lecture 3 and showed that the category of modules is a braided monoidal category. Let us review the definition before developing their properties further.

Let H be a Hopf algebra and R be an invertible element of $H \otimes H$ such that for all $x \in H$

$$\tau\Delta(x) = R\Delta(x)R^{-1}.$$

Here and always τ denotes the flip map $\tau(x \otimes y) = y \otimes x$.

If $R = 1$ this means $\tau \circ \Delta = \Delta$ so H is cocommutative. More generally, the existence of such R makes H into a (Drinfeld) coboundary Hopf algebra.

Notation

Assuming the coboundary property

$$\tau\Delta(x) = R\Delta(x)R^{-1}.$$

we checked in Lecture 3 that if U, W are H-modules, then the map $\tau R : U \otimes W \to W \otimes U$ is an H-module homomorphism.

To state the remaining axioms we need elements R_{12}, R_{23} and R_{13} in $H \otimes H \otimes H$. In words, R_{ij} means R distributed on the i and j tensor components of $H \otimes H \otimes H$. Let

$$R = R^{(1)} \otimes R^{(2)}.$$

(We are suppressing a summation.) Then

$$R_{12} = R^{(1)} \otimes R^{(2)} \otimes 1_H, \qquad R_{13} = R^{(1)} \otimes 1_H \otimes R^{(2)},$$

$$R_{23} = 1_H \otimes R^{(1)} \otimes R^{(2)}.$$

The braiding axioms

The remaining axioms of a QTHA are

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R = R_{13}R_{12}.$$

We also checked in Lecture 2 that if (for modules U, W) we define $c_{U,W} : U \otimes W \to W \otimes U$ to be τR then these axioms imply that $c_{U,W}$ is a braiding, that is:

$$\begin{array}{ccccc} A \otimes B \otimes C & & \xrightarrow{c_{A,B\otimes C}} & & B \otimes C \otimes A \\ & \searrow^{c_{A,B}\otimes 1_C} & & \nearrow^{1_B \otimes c_{A,C}} & \\ & & B \otimes A \otimes C & & \end{array}$$

$$\begin{array}{ccccc} A \otimes B \otimes C & & \xrightarrow{c_{A\otimes B,C}} & & C \otimes A \otimes B \\ & \searrow^{1_A\otimes c_{B,C}} & & \nearrow^{c_{A,C} \otimes 1_B} & \\ & & A \otimes C \otimes B & & \end{array}$$

Sweedler notation

The kinds of manipulations that we carried out in our study of rigid braided categories have analogs in QTHA. Let us begin by writing the braid axioms

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R = R_{13}R_{12}$$

in Sweedler notation. We can write

$$(\Delta \otimes 1)R = \Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)}{}_{(1)} \otimes R^{(1)}{}_{(2)} \otimes R^{(2)}.$$

But for the other side $R_{13}R_{23}$ we have two copies of R, and we have to keep them straight, so we will denote the second one $\widetilde{R}_{23}$. Therefore we arrive at

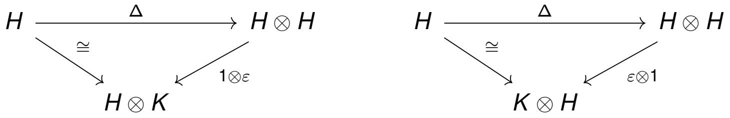
$$R^{(1)}{}_{(1)} \otimes R^{(1)}{}_{(2)} \otimes R^{(2)} = R^{(1)} \otimes \widetilde{R}^{(1)} \otimes R^{(2)}\widetilde{R}^{(2)}$$

and similarly for the other axiom

$$R^{(1)} \otimes R^{(2)}{}_{(1)} \otimes R^{(2)}{}_{(2)} = R^{(1)}\widetilde{R}^{(1)} \otimes \widetilde{R}^{(2)} \otimes R^{(2)}.$$

Counit reminder

We remind the reader that the counit $\epsilon(x)$ for $x \in H$ is a scalar and can be moved around at your convenience wherever it is needed in a calculation. The counit axiom



in Sweedler notation becomes

$$x_{(1)}\varepsilon(x_{(2)}) = \varepsilon(x_{(1)})x_{(2)} = x.$$

Counit and R-matrix

We will prove

$$(1 \otimes \varepsilon)R = (\varepsilon \otimes 1)R = 1_{H \otimes H}.$$

This is actually an identity in H, as we will now explain. In Sweedler notation

$$R^{(1)} \otimes \varepsilon(R^{(2)}) = 1_H \otimes 1_H,$$

but remembering that $\varepsilon(R^{(2)})$ is a scalar, we may write

$$R^{(1)}\varepsilon(R^{(2)}) \otimes 1_H = 1_H \otimes 1_H,$$

so really the identity to be proved is

$$R^{(1)}\varepsilon(R^{(2)}) = \varepsilon(R^{(1)})R^{(2)} = 1_H.$$

Proof

let us be pendantic an restrain from identifying $H = K \otimes H$ in the counit identity. Thus (for $x \in H$)

$$(\varepsilon \otimes 1)\Delta(x) = 1 \otimes x$$

with both sides in $K \otimes H$. Then

$$(\varepsilon \otimes 1 \otimes 1)(\Delta \otimes 1)R = 1 \otimes R^{(1)} \otimes R^{(2)} = R_{23}$$

in $H \otimes H \otimes H$. On the other hand using the braid identity

$$(\Delta \otimes 1)R = R_{13}R_{23}$$

gives

$$(\varepsilon \otimes 1 \otimes 1)(\Delta \otimes 1)R = (\varepsilon \otimes 1 \otimes 1)R_{13}R_{23}.$$

Since R_{23} is invertible

$$(\varepsilon \otimes 1 \otimes 1)R_{13} = 1_{H\otimes H\otimes H}$$

so $(\varepsilon \otimes 1)R = 1$ and $(1 \otimes \varepsilon)R = 1$ is similar.

The Yang-Baxter equation

The Yang-Baxter equation satisfied by R is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

To prove this, we apply the axiom $\tau\Delta(x) = R\Delta(x)R^{-1}$ to obtain

$$(\tau \otimes 1)(\Delta \otimes 1)R = R_{12}((\Delta \otimes 1)R)R_{12}^{-1}$$

since

$$R_{23}R_{13} = (\tau \otimes 1)R_{13}R_{23} = R_{12}R_{13}R_{23}R_{12}^{-1}$$

and rearranging gives us the Yang-Baxter equation.

Antipode and R-matrix

Next we will prove the identities

$$(S \otimes 1)R = R^{-1}, \qquad (1 \otimes S)R^{-1} = R.$$

These imply

$$(S \otimes S)R = R.$$

Antipode and R-matrix proofs

We begin with the braid identity $(\Delta \otimes 1)R = R_{13}R_{23}$ which we write in Sweedler notation as

$$R^{(1)}{}_{(1)} \otimes R^{(1)}{}_{(2)} \otimes R^{(2)} = R^{(1)} \otimes \tilde{R}^{(1)} \otimes R^{(2)}\tilde{R}^{(2)}. \qquad (1)$$

We apply the map $x \otimes y \otimes z \to S(x)y \otimes z$ to both sides. To the left side, this produces

$$S(R^{(1)}{}_{(1)})R^{(1)}{}_{(2)} \otimes R^{(2)} = (\varepsilon \otimes 1)R = 1$$

by what we have already proved. To the right side it produces

$$S(R^{(1)})\tilde{R}^{(1)} \otimes R^{(2)}\tilde{R}^{(2)} = ((S \otimes 1)R) \cdot R.$$

Thus

$$((S \otimes 1)R) \cdot R = 1,$$

so $(S \otimes 1)R = R^{-1}$.

The other antipode and R-matrix identity

We start with the identity $(1 \otimes \Delta)R = R_{13}R_{12}$ which implies $(1 \otimes \Delta)R^{-1} = R_{12}^{-1}R_{13}^{-1}$ because Δ is a ring homomorphism. Let us use the notation

$$R^{-1} = R^{-(1)} \otimes R^{-(2)}$$

so

$$R^{-(1)} \otimes R^{-(2)}{}_{(1)} \otimes R^{-(2)}{}_{(2)} = R^{-(1)}\tilde{R}^{-(1)} \otimes R^{-(2)} \otimes \tilde{R}^{-(2)}.$$

Applying the map $x \otimes y \otimes z \mapsto x \otimes S(y)z$ to both sides gives

$$1 = R^{-(1)}\tilde{R}^{-(1)} \otimes SR^{-(2)}\tilde{R}^{-(2)} = ((1 \otimes S)R^{-1})R^{-1}$$

and so $((1 \otimes S)R^{-1}) = R$.

The element u

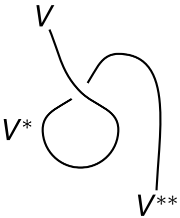
Let H be a quasitriangular Hopf algebra with R-matrix $R \in H \otimes H$. Define $\mathbf{u} = S(R^{(2)})R^{(1)} \in H$. The element $\mathbf{u}$ has remarkable properties. Its secret meaning is that it implements the morphism $\mathfrak{u}$ in the corresponding rigid braided category of finite-dimensional H-modules.

To be precise, let V be a module and let $\iota : V \to V^{**}$ be the usual identification of V with its double dual in the category of finite-dimensional vector spaces. Of course V and V^{**} both have H-module structures but ι is not necessarily an H-module homomorphism.

The meaning of u

We recall that we defined an H-module homomorphism $\mathfrak{u}_V : V \to V^{**}$ by

$$\mathfrak{u}_V : V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$



Theorem

If $v \in V$ *then* $\mathfrak{u}_V(v) = \iota(\mathbf{u} \cdot v)$.

The meaning of u (proof)

To prove this, let v_i and v_i^* be dual bases of V and V^*; then let v_i^{**} be the basis of V^{**} dual to v_i^*. The map $\iota : V \to V^*$ sends v_i to v_i^{**}.

Now if $x \in V$ we write $\text{coev}_{V^*} = v_i^* \otimes v_i^{**}$ where we are supressing the summation over i. Then

$$\begin{aligned} \mathfrak{u}_V(x) = \quad & (\text{ev}_V \otimes I_{V^{**}})(c_{V,V^*} \otimes I_{V^{**}})(1_V \otimes \text{coev}_{V^*})(x) = \\ & (\text{ev}_V \otimes I_{V^{**}})(c_{V,V^*} \otimes I_{V^{**}})(x \otimes v_i^* \otimes v_i^{**}) = \\ & (\text{ev}_V \otimes I_{V^{**}})(R^{(2)} v_i^* \otimes R^{(1)} x \otimes v_i^{**}) = \\ & \langle R^{(2)} v_i^*, R^{(1)} x \rangle v_i^{**}. \end{aligned}$$

(We use $\langle\, , \rangle$ for the dual pairing of V^* with V.) This equals

$$\iota(\langle v_i^*, S(R^{(2)}) R^{(1)} x \rangle v_i) = \iota(\mathbf{u}(x)).$$

Exercises

Exercise 1. (a) Show $S(\mathbf{u}) = S(R^{(1)})S^2(R^{(2)}) = R^{(1)}S(R^{(2)})$.
(b) Prove that if $v \in V$ then $\mathfrak{v}_V(\iota(v)) = S(\mathbf{u})(v)$.

Exercise 2. Let H be a Hopf algebra. The anti-multiplicativity property $S(ab) = S(b)S(a)$ of the antipode can be written $S \circ \mu = \mu(S \otimes S)\tau$. Prove the dual anti-comultiplicativity $\Delta \circ S = \tau(S \otimes S)\Delta$ or in Sweedler notation

$$S(x)_{(1)} \otimes S(x)_{(2)} = S(x_{(2)}) \otimes S(x_{(1)}).$$

Exercise 3. Let $U = S(R^{(2)}) \otimes R^{(1)}$. Prove that

$$(\Delta \otimes 1)U = U_{13}U_{23}$$

and deduce

$$R_{12}U_{13}U_{23} = U_{23}U_{13}R_{12}.$$