Lecture 4

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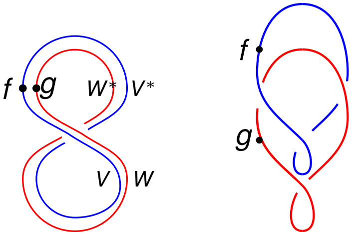
May 24, 2019



Tangles

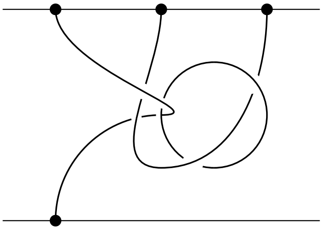
Tangles generalize braids, knots and links. A tangle is a collection of circles and arcs piecewise smoothly immersed in $\mathbb{R}^2 \times [0,1]$ with endpoints on the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$. Specifically let m and n be given nonnegative integers; we will define a tangle of type (m, n). Let us fix m points in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$: for definiteness

$$m = \{(k,0,0)|1 \leqslant k \leqslant m\}, \qquad \{n = (\ell,0,0)|1 \leqslant \ell \leqslant n\}.$$

These are to be the endpoints of the arcs. We identify two tangles if they are equivalent by an ambient isotopy that fixes the endpoints on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$.

Tangles form a category

The objects in the tangle category are the nonnegative integers $\mathbb{N}$. We think of an (m, n) tangle as a morphism $m \to n$. We will draw this upside down with the m at the top. Here is a $(3, 1)$ tangle represented by its projection onto the plane.

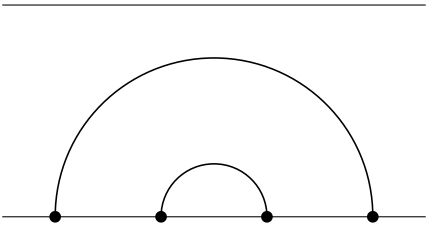


Morphisms may be composed by gluing $(k, 0, 1)$ to $(k, 0, 0)$, then rescaling to fit between the planes $z = 0$ and $z = 1$.

Tangles form a rigid monoidal category

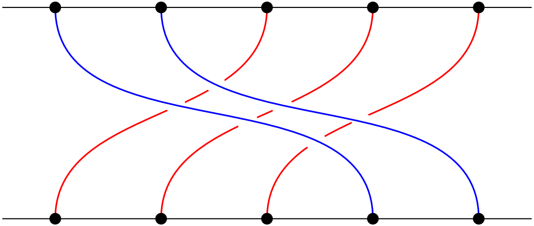
The monoidal structure identifies m_1 and m_2 with $m_1 + m_2$. Given tangles T_1 in $\mathrm{Hom}(m_1, n_1)$ and T_2 in $\mathrm{Hom}(m_2, n_2)$, we may juxtapose them to get a tangle in $\mathrm{Hom}(m_1 + m_2, n_1 + n_2)$.

We may even define $m^* = m$ and make the tangle category into a rigid category. Here is the coevaluation map for $m = 2$. It is an object in $\mathrm{Hom}(0, 4) = \mathrm{Hom}(0, 2 \otimes 2^*)$.



Tangles form a braided category

We may introduce a braiding by specifying morphisms in $\mathrm{Hom}(m \otimes n, n \otimes m)$. Here is the braiding for $m = 2, n = 3$.



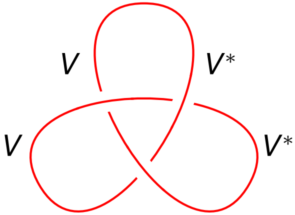
Framed Tangles

A framed tangle associates to each strand a family of normal vectors. Fattening up the strand in the direction of these normal vectors produces a ribbon.

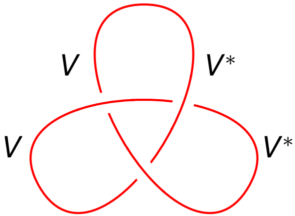
Framed tangles again form a braided monoidal category.

Knot invariants

We may try to model a knot or link in a rigid braided category. Let us pick a module V in the category. Let K be the unit object in the category. Assume that $V^{**} \cong V$ so that the evaluation morphism $\mathrm{coev}_{V^*} : V^* \otimes V^{**} \to K$ can be regarded as a morphism $V^* \otimes V \to K$. Now we label the strands of a 2-dimensional projection as follow:



Knot invariants, continued

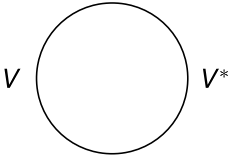


Interpreting the caps and cups as coevaluation and evaluation, this is a morphism $K \to K$. If K happens to be a field, it is a scalar. This approach to knot invariants has some problems, but ultimately can be made to succeed.

The simplest knot

The simplest knot is an unknotted circle.

$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{\text{ev}_{V^*}} K.$$

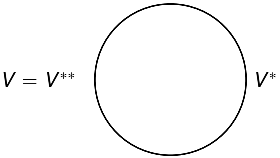


Applying the above mentioned heuristic will expose some of the problems with this plan.

Dimension

The braided category of finite-dimensional vector spaces over a field K is symmetric: the maps $c_{U,V}$ and $c_{V,U}^{-1} : U \to V$ are equal. We may identify V with its double dual V^{**} and so we a linear map

$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{\text{ev}_{V^*}} K.$$

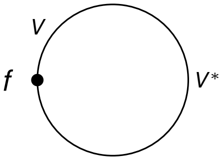


Remember that if v_i and v_i^* are dual bases of V and V^* then $\text{coev}_V(1) = \sum v_i^* \otimes v_i$. From this, this endomorphism of K is the scalar $\dim(V)$.

Trace

More generally we may include an endomorphism of V and compute its trace.

$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{f \otimes 1_{V^*}} V \otimes V^* \xrightarrow{\text{ev}_{V^*}} K.$$



The trace is multiplicative

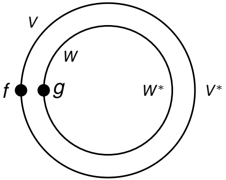
Still working in the symmetric category of vector spaces, if $f : V \to V$ and $g : W \to W$ are endomorphisms then

$$\operatorname{tr}(f \otimes g) = \operatorname{tr}(f)\operatorname{tr}(g).$$

Here is a graphical proof. Remember,

$$\operatorname{coev}_{V \otimes W} = (1_V \otimes \operatorname{coev}_W \otimes 1_{V^*})\operatorname{coev}_W,$$

$$\operatorname{ev}_{(V \otimes W)^*} = \operatorname{ev}_{W^* \otimes V^*} = \operatorname{ev}_{V^*}(1_V \otimes \operatorname{ev}_{W^*} \otimes 1_{V^*})$$

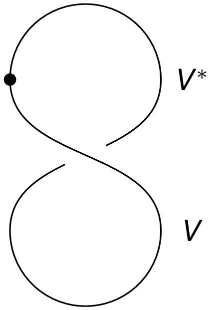


The evaluations ev_{V^*}, ev_{W^*} may be carried out separately, then multiplied together.

The trace in a braided rigid category

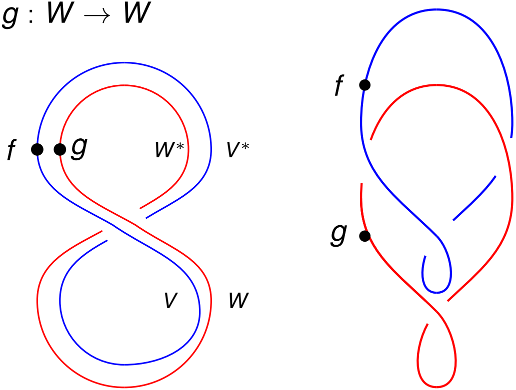
We can try to make a trace in a braided rigid category. We create $V \otimes V^*$ with coev. We have to interchange them before we evaluate:

$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{\text{ev}_V} K$$



This trace is not multiplicative

$f : V \to V, g : W \to W$

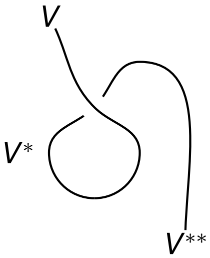


If we try to prove multiplicativity for $\mathrm{tr}(f \otimes g)$ we cannot because the two paths are linked and cannot be separated. This is a sign that we need a new ingredient to make a satisfactory theory.

Isomorphisms $V \to V^{**}$

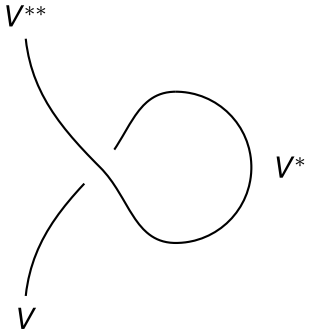
In a rigid braided category V and V^{**} are naturally isomorphic, but there are potentially an infinite number of such natural isomorphisms corresponding to increasingly twisted tangles. The following morphism will be denoted u_V:

$$V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$



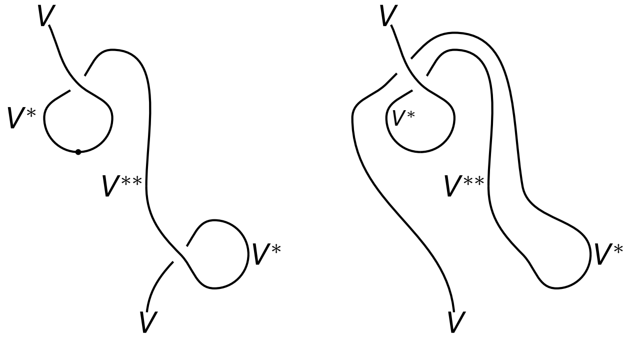
The inverse of u_V

$$V^{**} \xrightarrow{1_{V^{**}} \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^{*} \xrightarrow{c_{V^{**},V} \otimes 1_{V^{*}}} V \otimes V^{**} \otimes V^{*} \xrightarrow{1_V \otimes \mathrm{ev}_V^{*}} V$$



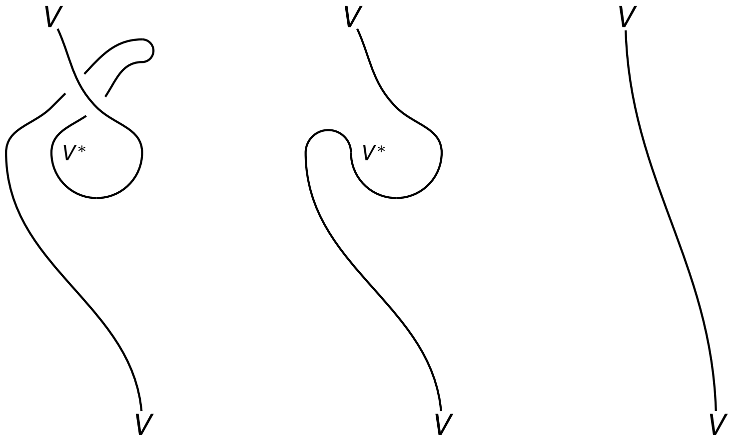
Checking the inverse

Let us check that with $u_V : V \to V^{**}$ and u_V^{-1} that indeed $u_V^{-1} u_V = 1_V$.



We use the naturality of the second (lower) crossing to move it before (above) the first crossing.

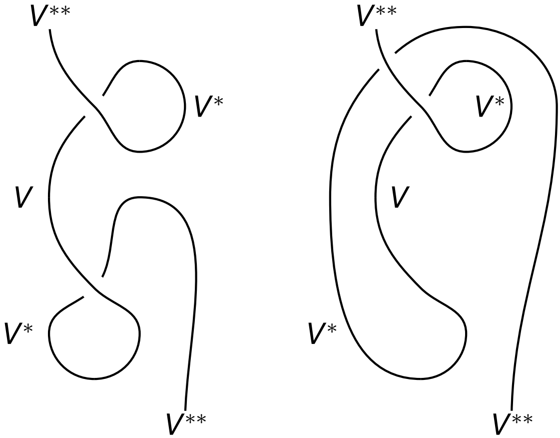
Checking the inverse (continued)



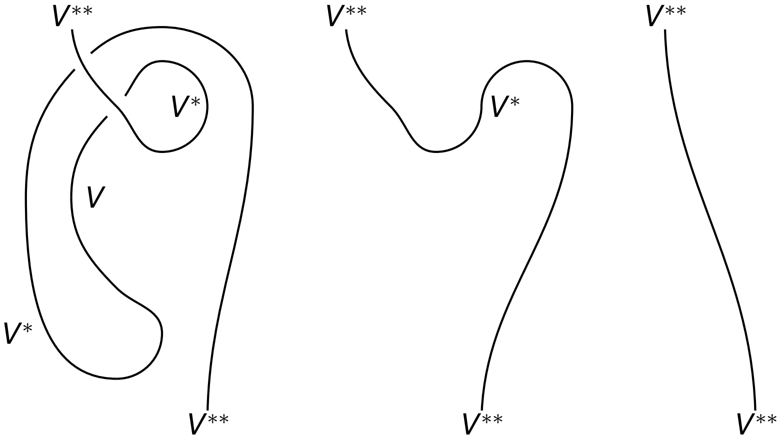
This shows that $u_V^{-1} u_V = 1_V$.

Checking the inverse (continued)

Now let us show that $u_V u_V^{-1} = 1_V$.



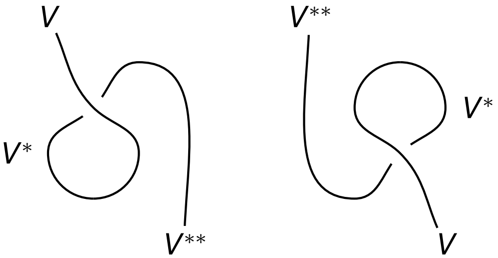
Checking the inverse (continued)



Another isomorphism

$$u_V : V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$

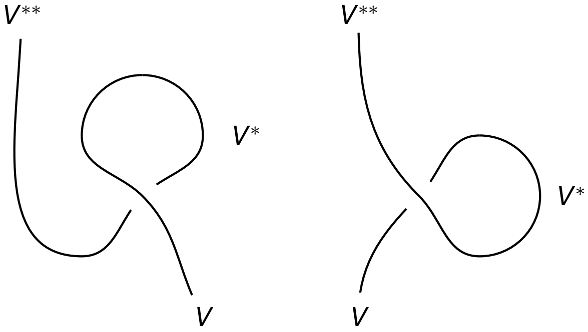
$$v_V : V^{**} \xrightarrow{1_V \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^* \xrightarrow{1_{V^{**}} \otimes c_{V,V^*}} V^{**} \otimes V^* \otimes V \xrightarrow{\mathrm{ev}_{V^*} \otimes 1_V} V$$



In addition to u_V, whose definition we repeat, we will need another isomorphism $v_V : V^{**} \to V$. This is not $u_V^{-1} : V^{**} \to V$ whose definition we have already considered.

Why are there two isomorphisms

Let us compare $v_V : V^{**} \to V$ with $u_V^{-1} : V^{**} \to V$.

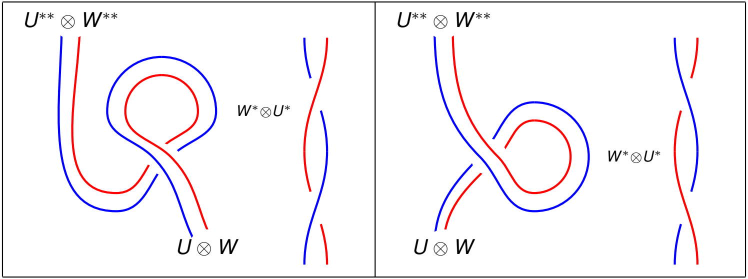


Left: v_V. Right: u_V^{-1}.

Why are there two isomorphisms

Let us compare $v_V : V^{**} \to V$ with $u_V^{-1} : V^{**} \to V$.

Let $V = U \otimes W$.

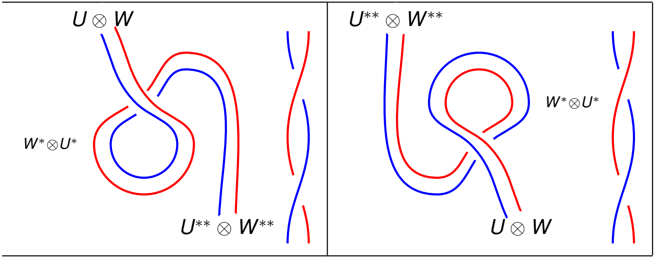


The difference between v_V and u_V^{-1} is made clear if $V = U \otimes W$: it is in the direction of twisting.

u_V and v_V

Both $u_V : V \to V^{**}$ and $v_V : V^{**} \to V$ are counter clockwise 2π twists. (Our z axis points down and the y axis points away from the viewer).

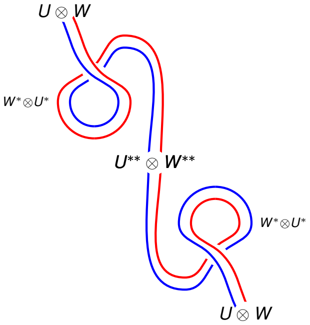
Left: u_V.
Right: v_V.

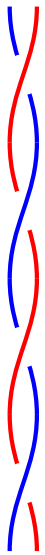


Composing them, $v_V \circ u_V : V \to V$ is a clockwise twist in 4π. We could solve many problems such as the non-multiplicativity of the trace if we had a map $V \to V$ that is a twist in 2π.

Possible twistings

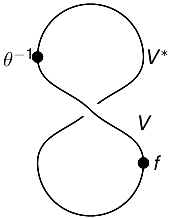
We have used the example $V = U \otimes W$ to show what kinds of twisting we can obtain with the tools we have so far. In a braided rigid category, we can construct morphisms $V \to V$ that twist a multiple of 4π times.





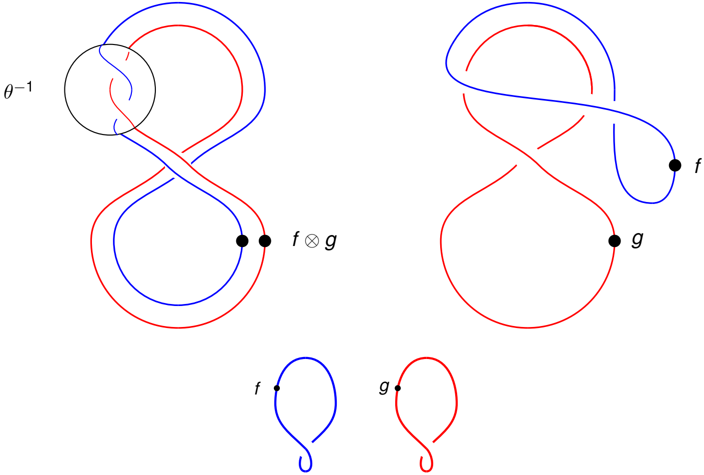
Motivating the notion of a ribbon category

What we need, however, is a natural morphism $\theta : V \to V$ that twists by 2π. We expect that $\theta^2 = v_V \circ u_V$. With such a morphism in hand, we can construct a multiplicative trace.



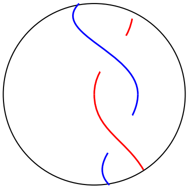
We have not yet give an proper definition of θ but heuristically show how it solves this problem.

Multiplicativity of the ribbon trace (informal)

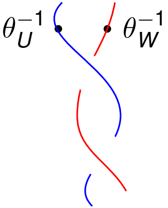


What did we forget?

We glossed over the following point. The morphism $\theta^{-1}_{U \otimes W}$ isn't actually this:



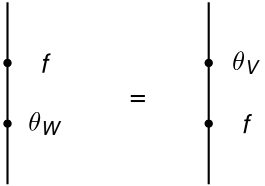
It's this, because U and W themselves are ribbons that can twist:



Twists

We will now formulate the axioms that the twist θ_V in a rigid braided category must statisfy. We want a natural isomorphism $\theta_V : V \to V$ for every object in the category satisfying certain axioms. A braided rigid category with a twist is called a ribbon category.

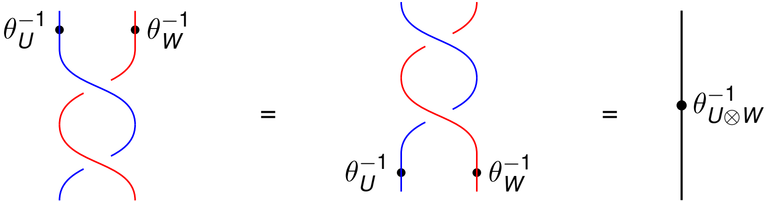
Naturality means if $f : V \to W$ then $\theta_W f = f\theta_V$:



Ribbon axioms, continued

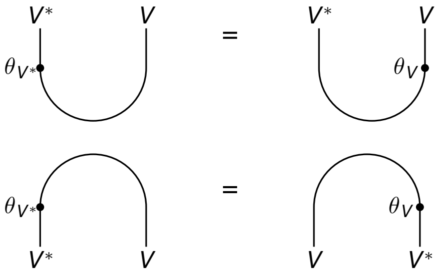
We must have (using naturality of $c_{U,W}$ and $c_{W,U}$:

$$\theta^{-1}_{U\otimes W} = c_{W,U} \circ c_{U,W} \circ \theta_U^{-1} \otimes \theta_W^{-1} = \theta_U^{-1} \otimes \theta_W^{-1} \circ c_{W,U} \circ c_{U,W}$$



Ribbon axioms, concluded

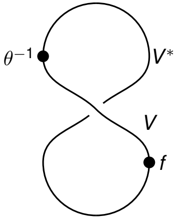
Finally it is necessary to assume that if I is the unit object in the category then $\theta_I = 1_I$, and that $\theta_{V^*} = \theta_V^*$. This last axiom means a compatibility with evaluation and coevalution:



Example: The category of framed tangles is a ribbon category.

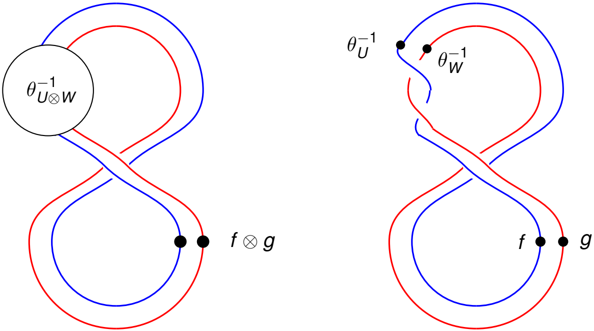
The trace in a ribbon category

The definition of a ribbon category contains all we need to define a multiplicative trace. It is an endomorphism of I, which in a category of vector spaces means a scalar.

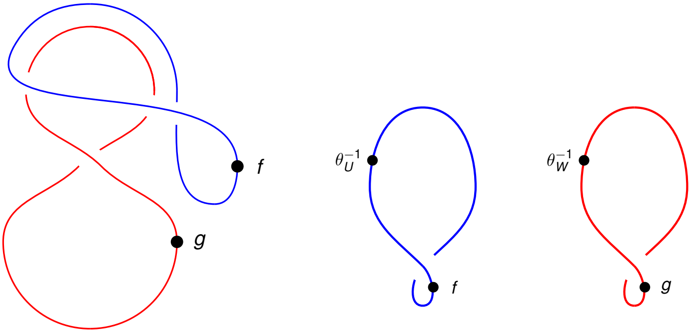


Multiplicativity of the ribbon trace (formal)

Let $f : U \to U$ and $g : W \to W$.



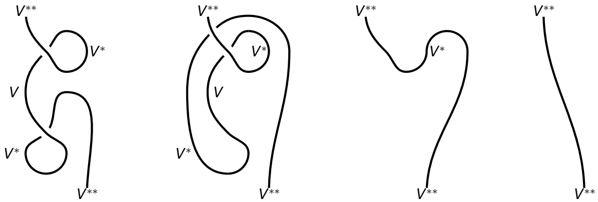
Multiplicativity of the ribbon trace (continued)



This proves $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$.

Exercises

Exercise 1. In the slides called Checking the inverse we proved that u_V and u_V^{-1} as we defined it were inverses by the following manipulations.



Explain carefully the justification of each step.

Exercise 2. What is the inverse of v_V?

Exercise 3. Prove that $\theta_{V^*} = \theta_V^*$ implies the ev_V and coev_V compatibilities under Ribbon axioms concluded.