Lecture 3

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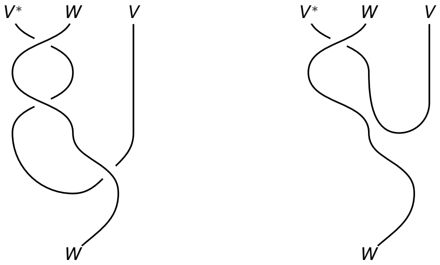


Monoidal categories

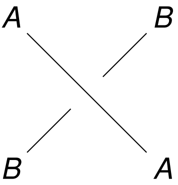
The axioms for a braided monoidal category are due to Joyal and Street. Braided Monoidal Category (Wikipedia Link)

Let $\mathcal{C}$ be a monoidal category. We recall that if A, B, C are objects in $\mathcal{C}$ then there are natural isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. We will not distinguish between these objects and just denote either as $A \otimes B \otimes C$. It is a consequence of Mac Lane's coherence theorem that this identification will never lead to any difficulties. We will never worry about this point.

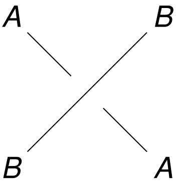
Braided categories

In a braided category there are explicit braid isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$ but now we must be careful. For example the composition $c_{B,A}c_{A,B}$ is not assumed to be the identity. So $c_{A,B}$ and $c_{B,A}^{-1}$ are distinct isomorphisms $A \to B$.

We will notate the morphism $c_{A,B}$ by an over crossing and $c_{B,A}$ by an under crossing.



$c_{A,B}$



$c_{B,A}^{-1}$

Naturality

We review the important notion of a natural transformation. We used this implicitly when we defined a monoidal category in Lecture 1, where we said that the isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

are required to be natural.

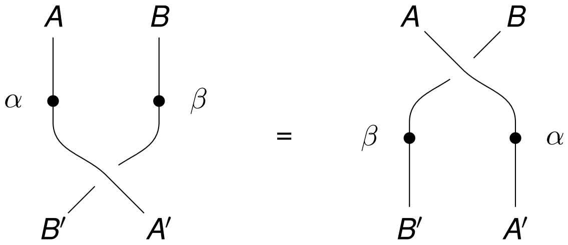
This means, explicitly, the following. Since $\otimes$ is a bifunctor, if $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ are morphisms then we have on the left and right of the following diagram.

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\cong} & A \otimes (B \otimes C) \\ \downarrow{\scriptstyle (\alpha\otimes\beta)\otimes\gamma} & & \downarrow{\scriptstyle \alpha\otimes(\beta\otimes\gamma)} \\ (A' \otimes B') \otimes C' & \xrightarrow{\cong} & A' \otimes (B' \otimes C') \end{array}$$

Naturality of the commutativity map

The first axiom of a braided category is that the morphisms $c_{A,B} : A \otimes B \to B \otimes A$ are to be natural. This means that if $\alpha : A \to A'$ and $\beta : B \to B'$ are morphisms, then

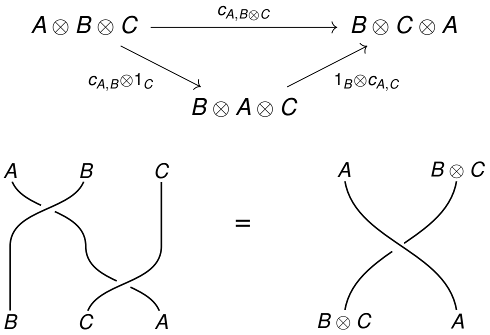
$$(\beta \otimes \alpha) \circ c_{A,B} = c_{A',B'} \circ (\alpha \otimes \beta)$$



(We are representing the morphisms α, β by dots.)

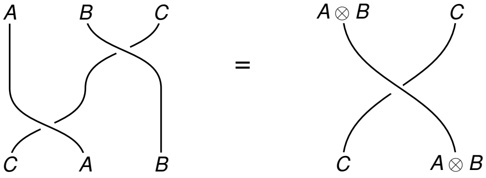
Braided category axioms

The braid morphism $c_{A,B} : A \otimes B \to B \otimes A$ is sometimes called an R-matrix. It is assumed to satisfy:



Mirror image axiom

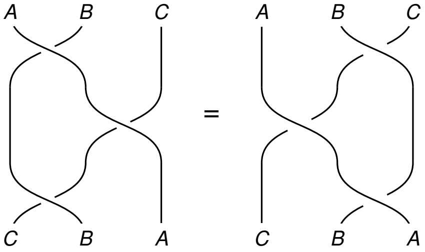
$$\begin{array}{ccccc} A\otimes B\otimes C & & \xrightarrow{\quad c_{A\times B,C}\quad} & & C\otimes A\otimes B \\ & {\scriptstyle 1_A\otimes c_{B,C}}\searrow & & \nearrow{\scriptstyle c_{A,C}\otimes 1_B} & \\ & & A\otimes C\otimes B & & \end{array}$$



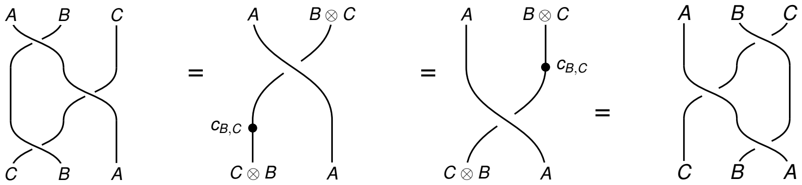
This completes the definition of a braided monoidal category.

The Yang-Baxter equation

We will show that the Yang-Baxter equation is true in a braided monoidal category. Remember that this means we have to show:



Proof of the Yang-Baxter equation



Showing that the Yang-Baxter equation is true in a braided monoidal category uses both axioms and naturality.

Slogan, revisited

Recall the slogan: modules over a quasitriangular bialgebra are a braided category.

We wish to impose properties on a bialgebra *H* that will make the module category into a braided modular category.

Thus we will arrive at the notion of a quasitriangular or braided Hopf algebra (QTHA), introduced by Drinfeld in his 1986 ICM lecture.

Quasitriangular Hopf algebras

Let R be an element of $H \otimes H$, which we assume to be invertible. If U and V are H-modules, consider the map $\tau R : U \otimes V \to V \otimes U$. ($\tau$ is always the flip map $x \otimes y \to y \otimes x$ of any tensor product $U \otimes V$.)

Proposition

Suppose that for $x \in H$ we have

$$R\Delta(x)R^{-1} = \tau\Delta(x).$$

Then for all U, V, the map $c_{U,V} : U \otimes V \to V \otimes U$ defined by $c_{u,v}(u \otimes v) = \tau R(u \otimes v)$ is an H-module homomorphism.

Proof

We are assuming $R\Delta(x)R^{-1} = \tau\Delta(x)$ which we rewrite

$$R\Delta(x) = \big(\tau(\Delta(x))\big)R.$$

We will write

$$R = R^{(1)} \otimes R^{(2)},$$

where we have omitted an implicit summation. Combining this with $\Delta(x) = x_{(1)} \otimes x_{(2)}$ (Sweedler notation)

$$R^{(1)}x_{(1)} \otimes R^{(2)}x_{(2)} = x_{(2)}R^{(1)} \otimes x_{(1)}R^{(2)}$$

in other words

$$R^{(1)}x_{(1)} \otimes R^{(2)}x_{(2)} = x_{(2)}R^{(1)} \otimes x_{(1)}R^{(2)}. \tag{1}$$

Proof (continued)

Now suppose $u \otimes v \in U \otimes V$. We need to show for $x \in H$:

$$\tau R(x(u \otimes v)) = x\tau R(u \otimes v)$$

in other words

$$\tau(R^{(1)}x_{(1)}u \otimes R^{(2)}x_{(2)}v) = (x_{(1)} \otimes x_{(2)})\tau(R^{(1)}u \otimes R^{(2)}v)$$

or

$$R^{(2)}x_{(2)}v \otimes R^{(1)}x_{(1)}u = x_{(1)}R^{(2)}v \otimes x_{(2)}R^{(1)}u$$

This follows from (1) by applying the map

$$a \otimes b \mapsto bv \otimes au$$

and we are done.

Some notation

Remember that $R \in H \otimes H$. We have written

$$R = R^{(1)} \otimes R^{(2)},$$

and this is shorthand for a sum, say

$$R = \sum_{i=1}^{N} (R_i') \otimes (R_i''),$$

where we are writting $R^{(1)}$ instead of R_i' and $R^{(2)}$ instead of R_i''. We will consider some elements of $H \otimes H \otimes H$,

$$R_{12} = R^{(1)} \otimes R^{(2)} \otimes 1_R, \qquad R_{13} = R^{(1)} \otimes 1_R \otimes R^{(2)},$$

$$R_{23} = 1_R \otimes R^{(1)} \otimes R^{(2)}.$$

Quasitriangular Hopf algebras

Definition (Drinfeld)

A quasitriangular or braided Hopf algebra is a Hopf algebra H with $R \in H \otimes H$ such that

$$R\Delta(x)R^{-1} = \tau\Delta(x)$$

and

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R = R_{13}R_{12}.$$

The element R is called the universal R-matrix. We understand the significance of the first condition: it means that if U, V are H-modules, then $c_{U,V} : U \otimes V \to V \otimes U$ defined by $(u, v) \mapsto \tau R(u \otimes v)$ is an H-module homomorphism. What about the other properties?

Braided category axioms and quasitriangularity

We will show that the axiom

$$(1 \otimes \Delta)R = R_{13}R_{12}$$

is equivalent to the axiom

$$\begin{array}{ccccc} A \otimes B \otimes C & & \xrightarrow{\quad c_{A,B\otimes C} \quad} & & B \otimes C \otimes A \\ & {\scriptstyle c_{A,B}\otimes 1_C} \searrow & & \nearrow {\scriptstyle 1_B \otimes c_{A,C}} & \\ & & B \otimes A \otimes C & & \end{array}$$

First let us argue that the top arrow $c_{A,B\otimes C}$ is

$$c_{A,B\otimes C}(a, b, c) = \theta((1 \otimes \Delta)R)(a \otimes b \otimes c),$$

where

$$\theta(a \otimes b \otimes c) = b \otimes c \otimes a.$$

Braided category axiom continued

From the previous slide, we are checking:

$$c_{A,B\otimes C}(a,b,c) = \theta((1\otimes\Delta)R)(a\otimes b\otimes c), \qquad (2)$$

where

$$\theta(a\otimes b\otimes c) = b\otimes c\otimes a.$$

Indeed let us treat $d = b\otimes c$ as a unit and remember the definition of $c_{A,D}$, with $D = B\otimes C$.

$$c_{A,B\otimes C}(a\otimes d) = \tau R(a\otimes d) = \tau(R^{(1)}a\otimes R^{(2)}d,)$$

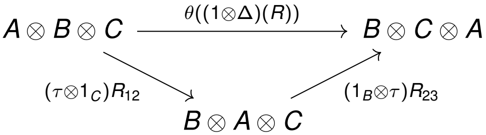
and θ is just the map τ in this setting. Now since the multiplication of Hopf elements on a tensor product of modules is through the tensor product,

$$R^{(2)}d = \Delta(R^{(2)})(b\otimes c).$$

Thus we obtain (2).

Braided category axiom continued

Now we want to prove:



We need to show that

$$\theta((1 \otimes \Delta)R) = (1_B \otimes \tau)R_{12}(\tau \otimes 1_C)R_{12}.$$

We have

$$(\tau \otimes 1_C)R_{23}(\tau \otimes 1_C) = R_{13}$$

so because $(1_B \otimes \tau)(\tau \otimes 1_C) = \theta$:

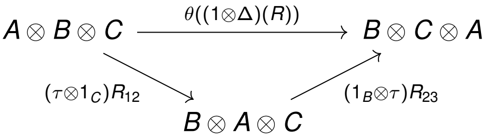
$$(1_B \otimes \tau)R_{23}(\tau \otimes 1_C)R_{12} = (1_B \otimes \tau)(\tau \otimes 1_C)R_{13}R_{23} = \theta R_{13}R_{23}$$

Braided category axiom, concluded

Now using the assumption from the definition of a QTHA:

$$(1 \otimes \Delta)R = R_{13}R_{23},$$

the commutativity of



follows and we have proved one of the braided category axioms.
That the mirror image axiom and naturality are satisfied are left to the exercises.

Example of a QTHA

Let G be the finite cyclic group of order n:

$$\langle g | g^n = 1 \rangle$$

Let $H = \mathbb{C}[G]$ be the group algebra and let $q = e^{2\pi i/n}$. It is a Hopf algebra with comultiplication

$$\Delta(g^a) = g^a \otimes g^a.$$

Define

$$R = \frac{1}{n} \sum_{a,b} q^{-ab} g^a \otimes g^b \in H \otimes H.$$

(Sum is over a, b mod n). This is invertible with

R is invertible

$$R = \frac{1}{n} \sum_{a,b} q^{-ab} g^a \otimes g^b \in H \otimes H.$$

$$R^{-1} = \frac{1}{n} \sum_{a,b} q^{ab} g^a \otimes g^b.$$

To see this, the product of these two elements is

$$\frac{1}{n^2} \sum_{a,b,c,d} q^{-ab+cd} g^{a+c} \otimes g^{b+d} = \frac{1}{n^2} \sum_{t,u} \left(\sum_{a,b} q^{-ab+(t-a)(u-b)} \right) g^t \otimes g^u.$$

The inner sum is

$$q^{tu} \sum_{a,b} q^{-tb-ua} = \begin{cases} n^2 & \text{if } t = u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that R^{-1} is indeed an inverse to R.

Cyclic group QTHA

Theorem

H is quasitriangular with universal R-matrix R.

Proof: The axiom $R\Delta(h)R^{-1} = \tau\Delta(h)$ for $h \in H$ is trival since H is both commutative and cocommutative. We have

$$R_{13}R_{12} = \frac{1}{n^2}\left(\sum_{a,b} q^{-ab} g^a \otimes 1 \otimes g^b\right)\left(\sum_{c,d} q^{-cd} g^c \otimes g^d \otimes 1\right)$$

$$= \frac{1}{n^2}\sum_{a,b,c,d} q^{-ab-cd} g^{a+c} \otimes g^d \otimes g^b$$

Proof, concluded

This equals

$$\frac{1}{n^2}\sum_{t,b,d}\left(\sum_a q^{-ab-(t-a)d}\right) g^t \otimes g^d \otimes g^b.$$

The inner sum is zero unless $b = d$, so this equals

$$\frac{1}{n}\sum_{t,b}\left(\sum_a q^{-tb}\right) g^t \otimes g^b \otimes g^b = (1 \otimes \Delta)R.$$

This proves $R_{13}R_{12} = (1 \otimes \Delta)R$ and $R_{13}R_{23} = (\Delta \otimes 1)R$ is similar.

Quantized enveloping algebras

We have mentioned in Lecture 1 that there are two classes of Hopf algebras that may be derived from a Lie group: the enveloping algebra, and the affine algebra or coordinate ring. Both admit quantum deformations.

A particular subtle point is the existence of the universal R-matrix. If $\mathfrak{g}$ is a complex semisimple Lie algebra and $H = U_q(\mathfrak{g})$ is the quantized enveloping algebra, the R-matrix does not live in H itself but in a completion. (So strictly speaking, H does not satisfy the definition of a QTHA.)

Morally, however $U_q(\mathfrak{g})$ is a QTHA. Moreover if q is a root of unity $U_q(\mathfrak{g})$ has a finite-dimensional quotient that is strictly quasitriangular.

$U_q(\mathfrak{sl}_2)$

Recall that the Lie algebra $\mathfrak{sl}_2$ is 3-dimensional with basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with bracket operations

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

To construct the quantized enveloping algebra we replace H by an invertible element K that you can think of as

$$K = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}$$

so that

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The comultiplication

So we let $U_q(\mathfrak{sl}_2)$ be the algebra with generators K, K^{-1}, E and F subject to the above relations. It is not possible to naively set $q = 1$ and recover $U(\mathfrak{sl}_2)$ but a more careful procedure will produce this result. This is done in Kassel's book.

There is an algebra homomorphism $\Delta : H \to H \otimes H$ defined by $\Delta(K) = K \otimes K$,

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

Indeed, it is not hard to check that the expressions on the right satisfy the same relations as E, F, K. There is a counit $\varepsilon : K \to \mathbb{C}$ defined by $\varepsilon(K) = 1$, $\varepsilon(E) = \varepsilon(F) = 0$ and antipode $S : K \to K$ such that $S(K) = K^{-1}$, $S(E) = -E$ and $S(F) = -F$. So H is a Hopf algebra.

At a root of unity

We refer to Majid and Kassel for the following facts.

Let $q = e^{2\pi i/n}$ where we assume that n is odd. Then E^n, F^n and K^n are central, and quotienting by them produces the finite-dimensional Hopf algebra $u_q(\mathfrak{sl}_2)$. It is quasitriangular with universal R-matrix

$$R = \frac{1}{n}\left(\sum_{a,b=0}^{n-1} q^{-2ab} K^a \otimes K^b\right) \sum_{r=0}^{n-1} \frac{(q-q^{-1})^r}{[r]_{q^{-2}}!} E^r \otimes F^r$$

where

$$r]_{q^{-2}}! = \prod_{t=1}^{r} [t]_{q^{-2}}, \qquad [r]_{q^{-2}} = \frac{1-q^{-2r}}{1-q^{-2}}.$$

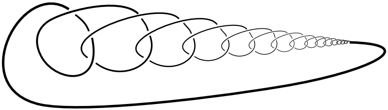
Knots and links

A knot is a smooth simple closed curve in S^3. A link is a finite union of disjoint smooth simple closed curves.

- Knot Theory (Wikipedia)
- Knotinfo web page
- The Knot Atlas

Wild knots

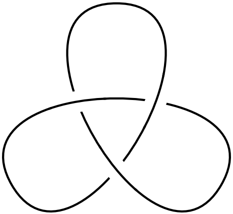
We only consider knots and links that are equivalent by ambient isotopy to a smooth curve or (equivalently) a finite union of segments, to avoid wild knots like this one:



▸ Wild knots (Wikipedia Link)

Reidemeister moves

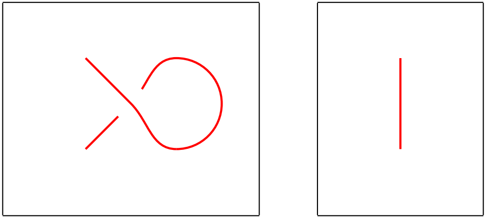
Knots are typically studied by projecting them onto the plane. A knot in 3-space is projected onto $\mathbb{R}^2$ with the crossings marked to show which strand is over and which is under.



An issue then is to determine when two knots are equivalent by an ambient isotopy (i.e. isotopy of the ambient S^3).

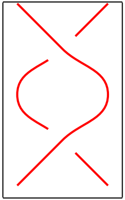
Reidemeister moves of Type I

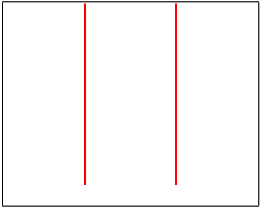
If the knots are represented by their two-dimensional projections, a necessary and sufficient condition is that these projections be related by a sequence of Reidemeister moves. There are three kinds. A Reidemeister move of Type I undoes a twist:



Reidemeister moves of Type II

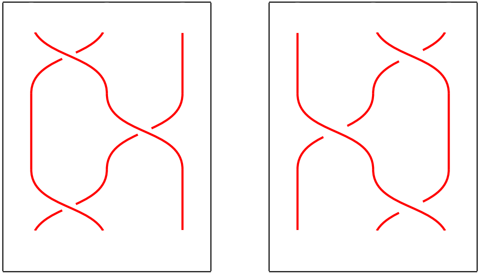
A Reidemeister move of type II changes:





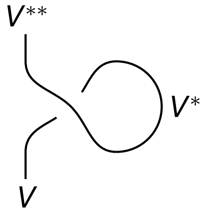
Reidemeister moves of Type III

A Reidemeister move of type III changes:



Difficulty with Type I

We note that Reidemeister moves of Types II and III have analogs in braided rigid categories. However a Reidemeister move of Type I does not. In a braided rigid category, the twist that we drew is a morphism $V^{**} \to V$:



Since this morphism doesn't map to the same object, we do not expect to be able to straighten it.

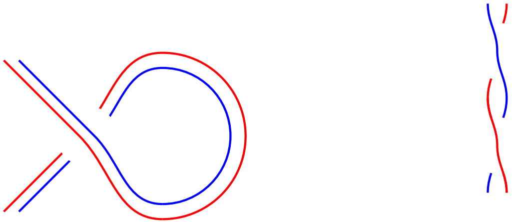
Framed knots and links

Instead of working with knots, we wish to work with framed links ("ribbons") which can also be projected to the plane, and which are then equivalent when they are related by Reidemeister moves of Types II and III, but not of Type I.

A framed knot is not just a strand, but a strand with a given normal vector field. If we fatten it up in the direction of the normal vector field, it becomes a ribbon, which can twist in space.

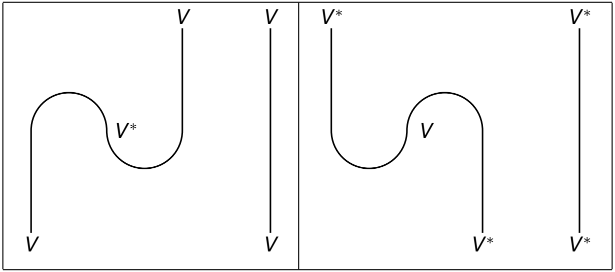
Twisting

If we double the strand in order to visualize how it will behave in 3 dimensions, the red strand passes first over, then under the blue strand. When we pull it straight, it gets a double twist.



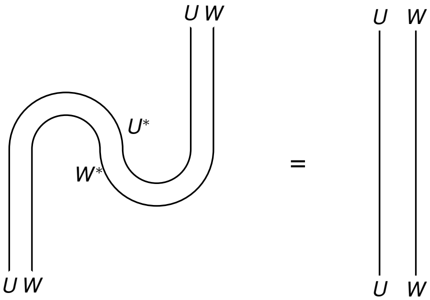
Evaluation and coevaluation of tensor products

We proved in Lecture 2 that the evaluation and coevaluation characterize the dual uniquely.



We can use this to identify ev_V when $V = U \otimes W$ is a tensor product.

Evaluation and coevaluation of tensor products



Uniqueness of the dual implies

$$\mathrm{ev}_{U\otimes W} = \mathrm{ev}_W(1_{W^*} \otimes \mathrm{ev}_U \otimes 1_W),$$

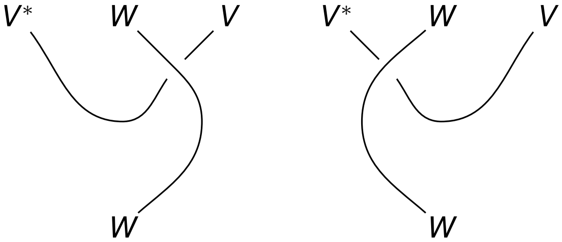
$$\mathrm{coev}_{U\otimes W} = (1_U \otimes \mathrm{coev}\, W \otimes 1_{U^*})\, \mathrm{coev}_U$$

(Other axiom can be diagrammed similarly.)

A relation between $c_{V,W}$ and $c_{V^*,W}$

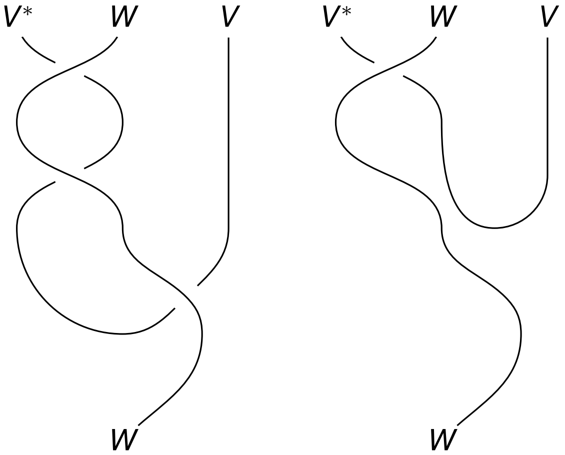
We will prove that

$$(\mathrm{ev}_V \otimes 1_W)(1_{V^*} \otimes c_{W,V}) = (1_W \otimes \mathrm{ev}_V)(c^{-1}_{W,V^*} \otimes 1_V)$$



The figures are isotopic, so this is expected, but how to prove it?

Proof

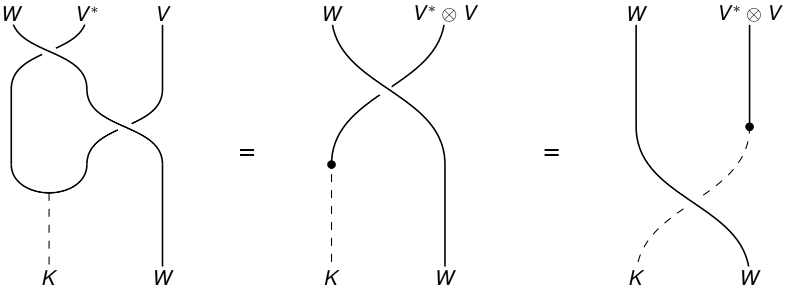


What just happened?

We used the braided category axiom and naturality

$$W \otimes V^* \otimes V \xrightarrow{c_{W,V^*\otimes V}} V^* \otimes V \otimes W \xrightarrow{\mathrm{ev}_V} K \otimes W = W$$

$$W \otimes V^* \otimes V \xrightarrow{c_{W,V^*}\otimes 1_V} V^* \otimes W \otimes V \xrightarrow{1_V^*\otimes c_{W,V}} V^* \otimes V \otimes W$$



Exercises

Exercise 1. In the proof that the category of modules for a QTHA is a braided modular category, we left naturality and the mirror image axiom to the reader. Check these.

Exercise 2. Prove that

$$c_{V^*,W} = (\mathrm{ev}_V \otimes 1_W \otimes 1_{V^*})(1_{V^*} \otimes c_{V,W}^{-1})(1_{V^*} \otimes 1_W \otimes \mathrm{coev}_V).$$