Lecture 2

Daniel Bump

May 23, 2019
Review of bialgebras and Hopf algebras

Recall that a **bialgebra** over a field $K$ is a Vector space $V$ with linear maps

$$
\mu : H \otimes H \to H, \quad \Delta : H \to H \otimes H,
$$

$$
\eta : K \to H, \quad \varepsilon : H \to K,
$$

subject to the axioms. Associativity and coassociativity:
Bialgebra axioms (continued)

Unit:

\[ H \otimes H \xrightarrow{1_{H \otimes \eta}} H \otimes K \xrightarrow{\mu} H \]  
\[ H \otimes H \xrightarrow{\eta \otimes K_H} K \otimes H \xrightarrow{\mu} H \]

Counit:

\[ H \otimes H \xrightarrow{1_{H \otimes \epsilon}} H \otimes K \xrightarrow{\Delta} H \]  
\[ H \otimes H \xrightarrow{\epsilon \otimes I_H} K \otimes H \xrightarrow{\Delta} H \]
Bialgebra axioms (continued)

The augmentation and coaugmentation axioms:

\[
\begin{align*}
H \otimes H & \xrightarrow{\varepsilon \times \varepsilon} K \times K \\
\downarrow \mu & \quad \downarrow \iota \\
H & \xrightarrow{\varepsilon} K \\
\end{align*}
\quad
\begin{align*}
K & \xrightarrow{\eta} H \\
\downarrow \iota & \quad \downarrow \Delta \\
K \times K & \xrightarrow{\eta \times \eta} H \times H
\end{align*}
\]

These say that the counit is an algebra homomorphism, and that the unit is a coalgebra homomorphism, respectively.
Bialgebra axioms (concluded)

Let $\tau(a \otimes b) = b \otimes a$. Hopf:

\[ H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{1_H \otimes \tau \otimes 1_H} H \otimes H \otimes H \otimes H \]

\[ \downarrow \mu \]

\[ H \xrightarrow{\Delta} H \otimes H \]

- The associativity and unit axioms: $H$ is an algebra.
- The coassociativity and counit: $H$ is an coalgebra.
- Augmentation axiom: counit is an algebra homomorphism
- Dually, the unit is a coalgebra homomorphism
- The Hopf axiom: $\Delta$ is an algebra homomorphism
- Equivalently $\mu$ is a coalgebra homomorphism
**Definition of a Hopf algebra**

A **Hopf algebra** is a bialgebra with a linear map \( S : H \to H \) called the **antipode** satisfying

\[
\begin{align*}
H & \xrightarrow{\Delta} H \otimes H & H & \xrightarrow{\Delta} H \otimes H \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
K & \xrightarrow{\eta} H & K & \xrightarrow{\eta} H
\end{align*}
\]

\[
\begin{align*}
H \otimes H & \xrightarrow{1_H \otimes S} H \otimes H & H \otimes H & \xrightarrow{S \otimes 1_H} H \otimes H \\
\downarrow \mu & & \downarrow \mu \\
H & & H
\end{align*}
\]

In the analogy between Hopf algebras and groups, this substitutes for \( g \cdot g^{-1} = g^{-1} \cdot g = 1 \).
Sweedler Notation (I)

We will use ordinary ring notation for the multiplication and unit \( \mu \) and \( \eta \). Thus if \( a, b \in H \) let \( a \cdot b = \mu(a \otimes b) \) and let \( 1_H = \eta(1_K) \).

Indeed we may identify \( K \) with a subring of the center of \( H \) using \( \eta \).

The Sweedler notation streamlines the formula

\[
\Delta(a) = \sum_{i=1}^{N} a_i' \otimes a_i''.
\]

Instead, write

\[
\Delta(a) = a_{(1)} \otimes a_{(2)}.
\]

We are omitting the summation from the notation.
Sweedler Notation (II)

Sweedler notation from the previous slide:

\[ \Delta(a) = a_{(1)} \otimes a_{(2)}. \]  

(1)

The co-associativity

\[(\Delta \otimes 1_H)\Delta(a) = (1 \otimes \Delta)\Delta(a)\]

means

\[ a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}. \]

We will write

\[ a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \]

for either of these decompositions. Note that \(a_{(2)}\) has a different meaning here than it did in (1).
Axioms in Sweedler notation (I)

The counit axiom:

\[
\begin{align*}
H \otimes H & \xrightarrow{\Delta} H \\
H \otimes I & \xrightarrow{\sim} H \\
I \otimes H & \xrightarrow{\Delta} H
\end{align*}
\]

In Sweedler notation,

\[
a = a_{(1)} \varepsilon (a_{(2)}) = \varepsilon (a_{(1)}) a_{(2)}.
\]
Axioms in Sweedler notation (II)

The antipode axiom:

![Diagram]

\[ \varepsilon(a) = a_{(1)} S(a_{(2)}) = S(a_{(1)}) a_{(2)}. \]

We should write

\[ \eta \varepsilon(a) = \cdots \]

but we identify \( \varepsilon(a) \in K \) with its image under \( \eta \).
Hopf axiom

If $A$ and $B$ are algebras, so is $A \otimes B$ with multiplication

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

If $\mu_A$, $\mu_B$ and $\mu_{A \otimes B}$ are the multiplications in $A$, $B$ and $A \otimes B$ this means

$$\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B).$$

Hence we write the Hopf axiom

$$(H \otimes H) \xrightarrow{\Delta \otimes \Delta} (H \otimes H) \otimes (H \otimes H)$$

In other words, $\Delta : H \rightarrow H \otimes H$ is an algebra homomorphism.
Hopf axiom in Sweedler notation

We may also translate the Hopf axiom into Sweedler notation.

\[
H \otimes H \xrightarrow{\Delta \otimes \Delta} (H \otimes H) \otimes (H \otimes H)
\]

\[
\begin{array}{c}
\downarrow \mu_H \\
H \\
\uparrow \Delta \\
H \otimes H \\
\end{array}
\]

\[
(\mu_H \otimes H)
\]

Applying this to \( x \otimes y \in H \otimes H \) gives

\[
(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}.
\]
Proposition

In a Hopf algebra $S(ab) = S(b)S(a)$.

The proof of this will be good practice for Sweedler notation so we will explain it carefully. This is an analog of the group property $(xy)^{-1} = y^{-1}x^{-1}$ so the problem will be to translate the proof of that fact into the Hopf algebra setting. Let us ponder the middle expression in

$$(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}.$$  

The Hopf analog is

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)})$$

where

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x)$$
The Hopf analog of \((xy)^{-1} yxy^{-1} x^{-1}\) is

\[
S(x_1y_1)x_2y_2 S(y_3)S(x_3)
\]

(2)

where

\[
x_1 \otimes x_2 \otimes x_3 = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x).
\]

So imitating what we did for the group identity, we will unravel this expression two ways, proving both

\[
S(x_1y_1)x_2y_2 S(y_3)S(x_3) = S(xy)
\]

and

\[
S(x_1y_1)x_2y_2 S(y_3)S(x_3) = S(y)S(x).
\]

Using the antipode axiom \(y_1 S(y_2) = \varepsilon(y)\)

\[
S(x_1y_1)x_2y_2 S(y_3)S(x_3) = S(x_1y_1)x_2\varepsilon(y_2)S(x_3).
\]

What just happened?
What just happened?

Here we are parsing

\[ y_1 \otimes y_2 \otimes y_3 = (1 \otimes \Delta) \Delta(y) = y_1 \otimes y_2(1) \otimes y_2(2). \]

Now applying the map \( a \otimes b \otimes c \mapsto a \otimes bS(c) \) to both sides of this identity gives

\[ y_1 \otimes y_2 S(y_3) = y_1 \otimes S(y_2(1))y_2(2) = y_1 \otimes \varepsilon(y_2). \]

That is how we get

\[ S(x_1y_1)x_2y_2S(y_3)S(x_3) = S(x_1y_1)x_2\varepsilon(y_2)S(x_3). \]
Going back to (2), bear in mind that $\varepsilon(a)$ is a scalar and can be moved around at will in formulas. Also $\varepsilon(a)\varepsilon(b) = \varepsilon(ab)$ since the counit is a ring homomorphism by the augmentation axiom.

$$
S(x_1)S(y_1)x_2y_2S(y_3)S(x_3) = \\
S(x_1)\varepsilon(y_2)S(x_2)S(x_3) = \\
S(x_1)\varepsilon(y_2)S(x_2)\varepsilon(y_2) = \\
S(x_1)\varepsilon(x_2)\varepsilon(y_2) = \\
S(x_1)\varepsilon(x_2)\varepsilon(y_2).
$$

Now by the Hopf axiom

$$
x_1y_1 \otimes x_2y_2 = (xy)_1 \otimes (xy)_2
$$

so

$$
S(x_1)S(y_1)x_2y_2S(y_3)S(x_3) = S((xy)_1\varepsilon((xy)_2)) = S(xy).
$$
The other side

Now let us unravel (1) a different way.

\[
\begin{align*}
S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) &= \\
S(x_{(1)}(y_{(1)})(1))x_{(1)(2)}y_{(2)(2)}S(y_{(2)})S(x_{(2)}) &= \text{Hopf axiom} \\
S((x_{(1)}y_{(1)})(1))(x_{(1)}y_{(1)})(2)S(y_{(2)})S(x_{(2)}) &= \\
\varepsilon(x_{(1)}y_{(1)})S(y_{(2)})S(x_{(2)}) &= \\
S(\varepsilon(y_{(1)}))y_{(2)}S(\varepsilon(x_{(1)}))x_{(2)} &= \\
\end{align*}
\]

and so

\[
S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(y)S(x).
\]
Slogans

- Modules over a bialgebra are a monoidal category
- Finite-Dimensional Modules over a Hopf algebra are a rigid monoidal category
- Modules over a quasitriangular bialgebra are a braided monoidal category
- Finite-Dimensional Modules over a ribbon Hopf algebra are a ribbon category

For the first item, if \( V \) and \( W \) are modules over \( H \), then \( V \otimes W \) is a module over \( H \otimes H \). Since \( \Delta \) is an \( H \)-module homomorphism \( H \rightarrow H \otimes H \), we may use it to transport this module structure to \( H \). In Sweedler notation

\[
a(v \otimes w) = a_{(1)} v \otimes a_{(2)} w.
\]

The coassociativity means that \( U \otimes (V \otimes W) \) and \( (U \otimes V) \otimes W \) have the same module structure.
We have just seen that the category of modules over a bialgebra is a monoidal category, and that the key to this is the comultiplication.

There is a dual construction. Since the axioms of a bialgebra are self-dual, this statement has a dual one, that the category of comodules over a bialgebra is a monoidal category.

This fact arises in the theory of affine algebraic groups. If $G$ is an affine group scheme over a field $K$, the affine algebra $\mathcal{O}(G)$ is a commutative Hopf algebra, and if $V$ is a module for $G$, then the dual space of $V$ is a comodule for $\mathcal{O}(G)$. For a proof, see Waterhouse, Introduction to Affine Group Schemes Section 3.2.
Our second slogan is that modules over a Hopf algebra form a rigid category. This is a category in which objects have duals. The archetype is the category finite-dimensional vector spaces over a field $K$.

Let $V$ be an object in a monoidal category with unit object $I$. We will abstract the properties of a (left) dual. This comes with morphisms $\text{ev}_V : V^* \otimes V \to I$ and $\text{coev}_V : I \to V \otimes V^*$ called evaluation and coevaluation.

The evaluation and coevaluation maps are subject to the following axioms.

$$(1_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes 1_V) = 1_V, \quad (\text{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \text{coev}_V) = 1_{V^*}. $$
Example: finite-dimensional vector spaces

In the category of finite-dimensional vector spaces, $I = K$

$V^* = \text{Hom}(V, K)$ and $\text{ev}_V(v^* \otimes v) = v^*(v)$, evaluating the linear functional $v^* \in V^*$ at the vector $v$.

Continuing with the example of a finite-dimensional vector space, we have to define the coevaluation, which is now to be a linear map $K \rightarrow V \otimes V^*$. We pick dual bases $v_i$ of $V$ and $v_i^*$ of $V^*$ and define $\text{coev}_V(a) = a \sum v_i \otimes v_i^*$. You may check that this is independent of $V$, and that the axioms are satisfied.
We can represent $ev_V : V^* \otimes V \to K$ and $coev_V : K \to V^* \otimes V$

The diagram is to be read from top to bottom.

At the top of the diagram is a sequence of spaces which are to be tensored together, hence $V^* \otimes V$ in the left figure; so the two figures represent morphisms $V^* \otimes V \to K$ and $K \to V \otimes V^*$
Diagrams for rigidity axioms (I)

The first axiom

$$(1_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes 1_V) = 1_V$$

means that the following two diagram represents the identity map $V \rightarrow V$.

Here $\text{coev}_V \otimes 1_V : V = K \otimes V \rightarrow V \otimes V^* \otimes V$ and $1_V \otimes \text{ev}_V : V \otimes V^* \otimes V \rightarrow V \otimes K = V$. 
Identifying $K \otimes V = V \otimes K$ in the above diagram, the $K$ is superfluous. Moreover we may rely on the reader to slice the diagram horizontally at intervals to obtain the sequence of spaces $V, V \otimes V^* \otimes V$; we have only to label the segments of the diagram.

Then the axiom asserts that the above two morphisms are equal. (The second one is the identity map $V \rightarrow V$.)
Here is the dual axiom

\[(\text{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \text{coev}_V) = 1_{V^*}.\]

We see that the two axioms may be understood as asserting that such bends may be straightened.
Uniqueness of the left dual

Suppose we have another left dual \( \widetilde{V}^* \). Let \( \text{ev}_V \) and \( \text{coev}_V \) be the corresponding evaluation and coevaluation maps as we can construct morphisms \( \phi : V^* \to \widetilde{V}^* \) and \( \psi : \widetilde{V}^* \to V^* \) by

\[
\phi = (\text{ev}_V \otimes I_{\widetilde{V}^*})(1_{V^*} \otimes \text{coev}_V)
\]

\[
\psi = (\text{ev}_V \otimes I_{V^*})(1_{\widetilde{V}^*} \otimes \text{coev}_V)
\]
Uniqueness of the dual (continued)

We may diagram the composition \( \psi \circ \phi \):

\[
\psi \circ \phi = (\text{ev}_V \otimes l_{\tilde{V}*})(1_{V*} \otimes \text{coev}_V)\overline{(\text{ev}_V \otimes l_{V*})(1_{\tilde{V}*} \otimes \text{coev}_V)}
\]

\[
= (\text{ev}_V \otimes l_{V*})(l_{V*} \otimes l_V \otimes \text{ev}_V \otimes l_{V*})(l_{V*} \otimes \text{coev}_V \otimes l_V \otimes l_{V*})(l_{V*} \otimes \text{coev}_V)
\]
Uniqueness of the dual (continued)

After switching the order of $\widetilde{\text{coev}}_V$ and $\widetilde{\text{ev}}_V$ as above (just using the fact that $\otimes$ is a bifunctor) we may use the axioms twice to conclude that $\psi \circ \phi = I_{V^*}:

\[
\begin{array}{c}
V^* \\
\downarrow \\
V \\
\downarrow \\
V^*
\end{array}
\quad =
\begin{array}{c}
V^* \\
\downarrow \\
V^*
\end{array}
\]

Similarly $\phi \circ \psi = I_{\widetilde{V}^*}$, so $\phi$ and $\psi$ are inverse isomorphisms. This proves the uniqueness of the dual, and also illustrates the use of diagrammatic methods in proofs.
We define an object $V$ in a monoidal category to be rigid if it has a left dual $V^*$. The category is called rigid if every object is.

Suppose that $V$ and $W$ are objects in a rigid category and $f : V \to W$ is a morphism. Define a morphism $f^* : W^* \to V^*$ by

$$f^* = (\text{ev}_W \otimes 1_{V^*})(1_{W^*} \otimes f \otimes 1_{V^*})(l_{W^*} \otimes \coev_V).$$

Representing $f$ as a dot (left) then $f^*$ is as on the right.
Exercises

Exercise 1. Let $f : V \to W$ and $g : W \to U$ be morphisms in a rigid category. Prove that $(gf)^* = f^* g^*$.

Let $V$ be a vector space and $V^*$ its dual. We will write $\langle v^*, v \rangle$ instead of $v^*(v)$ for the dual pairing.

Exercise 2. Let $H$ be a Hopf algebra, $V$ a finite-dimensional module, and let $V^*$ be its dual vector space. If $a \in H$ and $v^* \in V^*$ define $av^*$ by

$$\langle av^*, v \rangle = \langle v^*, S(a)v \rangle.$$ 

Prove that this makes $V^*$ into a module over $H$.

Exercise 3. Prove that the category of finite-dimensional modules over a Hopf algebra is a rigid category.