Lecture 2

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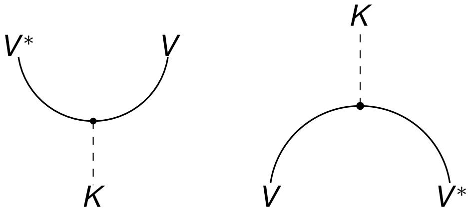
May 23, 2019



Review of bialgebras and Hopf algebras

Recall that a bialgebra over a field K is a Vector space V with linear maps

$$\mu : H \otimes H \to H, \qquad \Delta : H \to H \otimes H,$$

$$\eta : K \to H, \qquad \varepsilon : H \to K,$$

subject to the axioms. Associativity and coassociativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\mu \otimes 1} & H \otimes H \\ \downarrow{\scriptstyle 1_H \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ H \otimes H & \xrightarrow{\mu} & H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1_H \otimes \Delta} \\ H \otimes H & \xrightarrow{\Delta \otimes 1_H} & H \otimes H \otimes H \end{array}$$

Bialgebra axioms (continued)

Unit:

$$\begin{array}{ccc} H\otimes H & & \\ {\scriptstyle 1_H\otimes\eta}\uparrow & \searrow^{\mu} & \\ H\otimes K & \xrightarrow{\cong} & H \end{array} \qquad \begin{array}{ccc} H\otimes H & & \\ {\scriptstyle \eta\otimes K_H}\uparrow & \searrow^{\mu} & \\ K\otimes H & \xrightarrow{\cong} & H \end{array}$$

Counit:

$$\begin{array}{ccc} H\otimes H & & \\ {\scriptstyle 1_H\otimes\varepsilon}\downarrow & \nwarrow^{\Delta} & \\ H\otimes K & \xrightarrow{\cong} & H \end{array} \qquad \begin{array}{ccc} H\otimes H & & \\ {\scriptstyle \varepsilon\otimes I_H}\downarrow & \nwarrow^{\Delta} & \\ K\otimes H & \xrightarrow{\cong} & H \end{array}$$

Bialgebra axioms (continued)

The augmentation and coaugmentation axioms:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon \times \varepsilon} & K \times K \\ \downarrow \mu & & \downarrow \cong \\ H & \xrightarrow{\varepsilon} & K \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\eta} & H \\ \downarrow \cong & & \downarrow \Delta \\ K \times K & \xrightarrow{\eta \times \eta} & H \times H \end{array}$$

These say that the counit is an algebra homomorphism, and that the unit is a coalgebra homomorphism, respectively.

Bialgebra axioms (concluded)

Let $\tau(a \otimes b) = b \otimes a$. Hopf:

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\Delta\otimes\Delta} & H \otimes H \otimes H \otimes H & \xrightarrow{1_H\otimes\tau\otimes 1_H} & H \otimes H \otimes H \otimes H \\ \downarrow \mu & & & & \downarrow \mu\otimes\mu \\ H & & \xrightarrow{\quad\quad\Delta\quad\quad} & & H \otimes H \end{array}$$

- The associativity and unit axioms: H is an algebra.
- The coassociativity and counit: H is an coalgebra.
- Augmentation axiom: counit is an algebra homomorphism
- Dually, the unit is a coalgebra homomorphism
- The Hopf axiom: Δ is an algebra homomorphism
- Equivalently μ is a coalgebra homomorphism

Definition of a Hopf algebra

A Hopf algebra is a bialgebra with a linear map $S : H \to H$ called the antipode satisfying

$$\begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{1_H \otimes S} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\eta\quad} & & H \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{S \otimes 1_H} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\eta\quad} & & H \end{array}$$

In the analogy between Hopf algebras and groups, this substitutes for $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

Sweedler Notation (I)

We will use ordinary ring notation for the multiplication and unit μ and η. Thus if $a, b \in H$ let $a \cdot b = \mu(a \otimes b)$ and let $1_H = \eta(1_K)$. Indeed we may identify K with a subring of the center of H using η.

The Sweedler notation streamlines the formula

$$\Delta(a) = \sum_{i=1}^{N} a_i' \otimes a_i''.$$

Instead, write

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

We are omitting the summation from the notation.

Sweedler Notation (II)

Sweedler notation from the previous slide:

$$\Delta(a) = a_{(1)} \otimes a_{(2)}. \tag{1}$$

The co-associativity

$$(\Delta \otimes 1_H)\Delta(a) = (1 \otimes \Delta)\Delta(a)$$

means

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}.$$

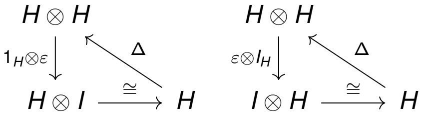
We will write

$$a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

for either of these decompositions. Note that $a_{(2)}$ has a different meaning here than it did in (1).

Axioms in Sweedler notation (I)

The counit axiom:



In Sweedler notation,

$$a = a_{(1)}\varepsilon(a_{(2)}) = \varepsilon(a_{(1)})a_{(2)}.$$

Axioms in Sweedler notation (II)

The antipode axiom:

$$\begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{1_H \otimes S} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\quad \eta \quad\quad} & & H \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{S \otimes 1_H} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\quad \eta \quad\quad} & & H \end{array}$$

$$\varepsilon(a) = a_{(1)} S(a_{(2)}) = S(a_{(1)}) a_{(2)}.$$

We should write

$$\eta\varepsilon(a) = \cdots$$

but we identify $\varepsilon(a) \in K$ with its image under η.

Hopf axiom

If A and B are algebras, so is $A \otimes B$ with multiplication

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

If μ_A, μ_B and $\mu_{A\otimes B}$ are the multiplications in A, B and $A \otimes B$ this means

$$\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B).$$

Hence we write the Hopf axiom

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta\otimes\Delta} & (H \otimes H) \otimes (H \otimes H) \\ \downarrow{\scriptstyle \mu_H} & & \downarrow{\scriptstyle \mu_{H\otimes H}} \\ H & \xrightarrow{\quad\Delta\quad} & H \otimes H \end{array}$$

In other words, $\Delta : H \to H \otimes H$ is an algebra homomorphism.

Hopf axiom in Sweedler notation

We may also translate the Hopf axiom into Sweedler notation.

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta \otimes \Delta} & (H \otimes H) \otimes (H \otimes H) \\ \downarrow \mu_H & & \downarrow \mu_{H \otimes H} \\ H & \xrightarrow{\quad\Delta\quad} & H \otimes H \end{array}$$

Applying this to $x \otimes y \in H \otimes H$ gives

$$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}.$$

The antipode

Proposition

In a Hopf algebra $S(ab) = S(b)S(a)$.

The proof of this will be good practice for Sweedler notation so we will explain it carefully. This is an analog of the group property $(xy)^{-1} = y^{-1}x^{-1}$ so the problem will be to translate the proof of that fact into the Hopf algebra setting. Let us ponder the middle expression in

$$(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}.$$

The Hopf analog is

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)})$$

where

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x)$$

The antipode (continued)

The Hopf analog of $(xy)^{-1}xyy^{-1}x^{-1}$ is

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) \tag{2}$$

where

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x).$$

So imitating what we did for the group identity, we will unravel this expression two ways, proving both

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(xy)$$

and

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(y)S(x).$$

Using the antipode axiom $y_{(1)}S(y_{(2)}) = \varepsilon(y)$

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(x_{(1)}y_{(1)})x_{(2)}\varepsilon(y_{(2)})S(x_{(3)}).$$

What just happened?

What just happened?

Here we are parsing

$$y_{(1)} \otimes y_{(2)} \otimes y_{(3)} = (1 \otimes \Delta)\Delta(y) = y_{(1)} \otimes y_{(2)(1)} \otimes y_{(2)(2)}.$$

Now applying the map $a \otimes b \otimes c \mapsto a \otimes bS(c)$ to both sides of this identity gives

$$y_{(1)} \otimes y_{(2)}S(y_{(3)}) = y_{(1)} \otimes S(y_{(2)(1)})y_{(2)(2)} = y_{(1)} \otimes \varepsilon(y_{(2)}).$$

That is how we get

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(x_{(1)}y_{(1)})x_{(2)}\varepsilon(y_{(2)})S(x_{(3)}).$$

Going back to (2), bear in mind that $\varepsilon(a)$ is a scalar and can be moved around at will in formulas. Also $\varepsilon(a)\varepsilon(b) = \varepsilon(ab)$ since the counit is a ring homomorphism by the augmentation axiom.

$$\begin{aligned} & S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) && = \\ & S(x_{(1)}y_{(1)})x_{(2)}\varepsilon(y_{(2)})S(x_{(3)}) && = \\ & S(x_{(1)}y_{(1)})x_{(2)}S(x_{(3)})\varepsilon(y_{(2)}) && = \\ & S(x_{(1)}y_{(1)})\varepsilon(x_{(2)})\varepsilon(y_{(2)}) && = \\ & S(x_{(1)}y_{(1)}\varepsilon(x_{(2)}y_{(2)})). && \end{aligned}$$

Now by the Hopf axiom

$$x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} = (xy)_{(1)} \otimes (xy)_{(2)}$$

so

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S((xy)_{(1)}\varepsilon((xy)_{(2)})) = S(xy).$$

The other side

Now let us unravel (1) a different way.

$$\begin{array}{lll}
S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) & = & \\
S(x_{(1)(1)}y_{(1)(1)})x_{(1)(2)}y_{(2)(2)}S(y_{(2)})S(x_{(2)}) & = & \text{Hopf axiom} \\
S((x_{(1)}y_{(1)})_{(1)})(x_{(1)}y_{(1)})_{(2)}S(y_{(2)})S(x_{(2)}) & = & \\
\varepsilon(x_{(1)}y_{(1)})S(y_{(2)})S(x_{(2)}) & = & \\
S(\varepsilon(y_{(1)})y_{(2)})S(\varepsilon(x_{(1)})x_{(2)}) & &
\end{array}$$

and so

$$S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(y_{(3)})S(x_{(3)}) = S(y)S(x).$$

Slogans

- Modules over a bialgebra are a monoidal category
- Finite-Dimensional Modules over a Hopf algebra are a rigid monoidal category
- Modules over a quasitriangular bialgebra are a braided monoidal category
- Finite-Dimensional Modules over a ribbon Hopf algebra are a ribbon category

For the first item, if V and W are modules over H, then $V \otimes W$ is a module over $H \otimes H$. Since Δ is an H-module homomorphism $H \to H \otimes H$, we may use it to transport this module structure to H. In Sweedler notation

$$a(v \otimes w) = a_{(1)}v \otimes a_{(2)}w.$$

The coassociativity means that $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ have the same module structure.

Bialgebras and monoidal categories

We have just seen that the category of modules over a bialgebra is a monoidal category, and that the key to this is the comultiplication.

There is a dual construction. Since the axioms of a bialgebra are self-dual, this statement has a dual one, that the category of comodules over a bialgebra is a monoidal category.

This fact arises in the theory of affine algebraic groups. If G is an affine group scheme over a field K, the affine algebra $\mathcal{O}(G)$ is a commutative Hopf algebra, and if V is a module for G, then the dual space of V is a comodule for $\mathcal{O}(G)$. For a proof, see Waterhouse, Introduction to Affine Group Schemes Section 3.2.

Dual objects

Our second slogan is that modules over a Hopf algebra form a rigid category. This is a category in which objects have duals. The archetype is the category finite-dimensional vector spaces over a field K.

Let V be an object in a monoidal category with unit object I. We will abstract the properties of a (left) dual. This comes with morphisms $\mathrm{ev}_V : V^* \otimes V \to I$ and $\mathrm{coev}_V : I \to V \otimes V^*$ called evaluation and coevaluation.

The evaluation and coevaluation maps are subject to the following axioms.

$$(1_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes 1_V) = 1_V, \qquad (\mathrm{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \mathrm{coev}_V) = 1_{V^*}.$$

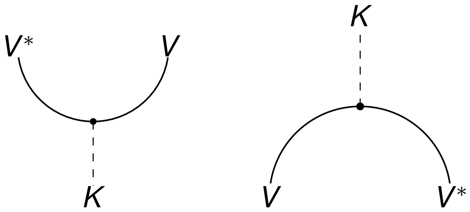
Example: finite-dimensional vector spaces

In the category of finite-dimensional vector spaces, $I = K$ $V^* = \mathrm{Hom}(V, K)$ and $\mathrm{ev}_V(v^* \otimes v) = v^*(v)$, evaluating the linear functional $v^* \in V^*$ at the vector v.

Continuing with the example of a finite-dimensional vector space, we have to define the coevaluation, which is now to be a linear map $K \to V \otimes V^*$. We pick dual bases v_i of V and v_i^* of V^* and define $\mathrm{coev}_V(a) = a \sum v_i \otimes v_i^*$. You may check that this is independent of V, and that the axioms are satisfied.

Diagrams for evaluation and coevaluation

We can represent $ev_V : V^* \otimes V \to K$ and $coev_V : K \to V^* \otimes V$
The diagram is to be read from top to bottom.



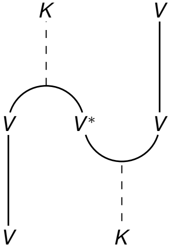
At the top of the diagram is a sequence of spaces which are to be tensored together, hence $V^* \otimes V$ in the left figure; so the two figures represent morphisms $V^* \otimes V \to K$ and $K \to V \otimes V^*$

Diagrams for rigidity axioms (I)

The first axiom

$$(1_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes 1_V) = 1_V$$

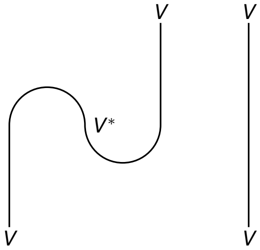
means that the following two diagram represents the identity map $V \to V$.



Here $\mathrm{coev}_V \otimes 1_V : V = K \otimes V \to V \otimes V^* \otimes V$ and $1_V \otimes \mathrm{ev}_V : V \otimes V^* \otimes V \to V \otimes K = V$.

Diagrams for rigidity axioms (II)

Identifying $K \otimes V = V \otimes K$ in the above diagram, the K is superfluous. Moreover we may rely on the reader to slice the diagram horizontally at intervals to obtain the sequence of spaces V, $V \otimes V^* \otimes V$; we have only to label the segments of the diagram.

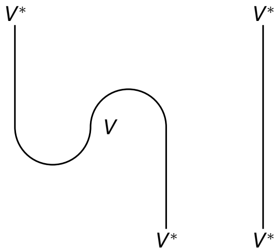


Then the axiom asserts that the above two morphisms are equal. (The second one is the identity map $V \to V$.)

Diagrams for rigidity axioms (III)

Here is the dual axiom

$$(\mathrm{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \mathrm{coev}_V) = 1_{V^*}.$$



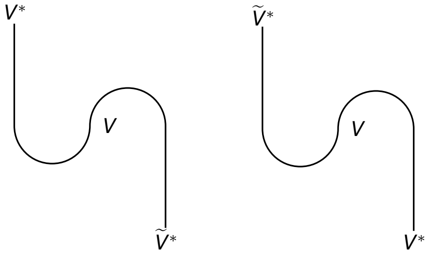
We see that the two axioms may be understood as asserting that such bends may be straightened.

Uniqueness of the left dual

Suppose we have another left dual $\widetilde{V}^*$. Let $\widetilde{\mathrm{ev}}_V$ and $\widetilde{\mathrm{coev}}_V$ be the corresponding evaluation and coevaluation maps as We can construct morphisms $\phi : V^* \to \widetilde{V}^*$ and $\psi : \widetilde{V}^* \to V^*$ by

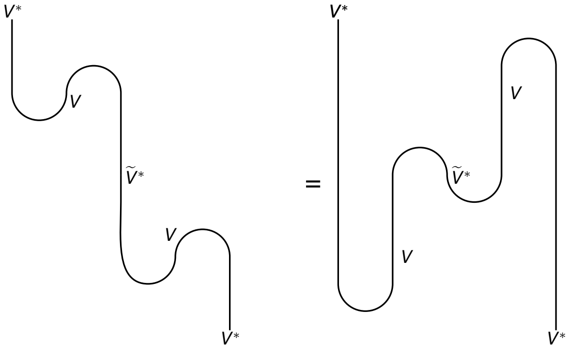
$$\phi = (\mathrm{ev}_V \otimes I_{\widetilde{V}^*})(1_{V^*} \otimes \widetilde{\mathrm{coev}}_V)$$

$$\psi = (\widetilde{\mathrm{ev}}_V \otimes I_{V^*})(1_{\widetilde{V}^*} \otimes \mathrm{coev}_V)$$



Uniqueness of the dual (continued)

We may diagram the composition $\psi \circ \phi$:

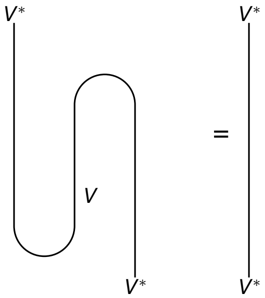


$$\psi \circ \phi = (\mathrm{ev}_V \otimes I_{\widetilde{V}^*})(1_{V^*} \otimes \widetilde{\mathrm{coev}}_V)(\widetilde{\mathrm{ev}}_V \otimes I_{V^*})(1_{\widetilde{V}^*} \otimes \mathrm{coev}_V)$$

$$= (\mathrm{ev}_V \otimes I_{V^*})(I_{V^*} \otimes I_V \otimes \widetilde{\mathrm{ev}}_V \otimes I_{V^*})(I_{V^*} \otimes \widetilde{\mathrm{coev}}_V \otimes I_V \otimes I_{V^*})(I_{V^*} \otimes \mathrm{coev}_V)$$

Uniqueness of the dual (continued)

After switching the order of $\widetilde{\text{coev}}_V$ and $\widetilde{\text{ev}}_V$ as above (just using the fact that $\otimes$ is a bifunctor) we may use the axioms twice to conclude that $\psi \circ \phi = I_{V^*}$:



Similarly $\phi \circ \psi = I_{\widetilde{V}^*}$, so ϕ and ψ are inverse isomorphisms. This proves the uniqueness of the dual, and also illustrates the use of diagrammatic methods in proofs.

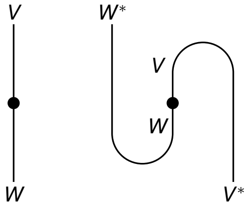
Dual morphisms and rigid categories

We define an object V in a monoidal category to be rigid if it has a left dual V^*. The category is called rigid if every object is.

Suppose that V and W are objects in a rigid category and $f : V \to W$ is a morphism. Define a morphism $f^* : W^* \to V^*$ by

$$f^* = (\mathrm{ev}_W \otimes 1_{V^*})(1_{W^*} \otimes f \otimes 1_{V^*})(I_{W^*} \otimes \mathrm{coev}_V).$$

Representing f as a dot (left) then f^* is as on the right.



Exercises

Exercise 1. Let $f : V \to W$ and $g : W \to U$ be morphisms in a rigid category. Prove that $(gf)^* = f^* g^*$.

Let V be a vector space and V^* its dual. We will write $\langle v^*, v \rangle$ instead of $v^*(v)$ for the dual pairing.

Exercise 2. Let H be a Hopf algebra, V a finite-dimensional module, and let V^* be its dual vector space. If $a \in H$ and $v^* \in V^*$ define av^* by

$$\langle av^*, v \rangle = \langle v^*, S(a)v \rangle.$$

Prove that this makes V^* into a module over H.

Exercise 3. Prove that the category of finite-dimensional modules over a Hopf algebra is a rigid category.