Lecture 18

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$$\mathcal{B}_{(1)} = \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad \mathcal{B}_{(1,1)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \xrightarrow{1} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$$

Weight lattice

Let G be a complex semisimple or reductive Lie group with Lie algebra $\mathfrak{g}$. Let α_i be the simple positive roots, and Λ the weight lattice. The index set is usually $\{1, \cdots, r\}$ so that α_i are indexed by $i \in I$.

If $G = \mathrm{GL}(r)$ we will identify the weight lattice $\Lambda = \mathbb{Z}^r$ with standard basis $\mathbf{e}_i$. Then $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

Categories of modules

Let G be a complex semisimple or reductive Lie group with Lie algebra $\mathfrak{g}$. When q is an indeterminate, the category of finite-dimensional integrable $U_q(\mathfrak{g})$-modules is similar to the theory of G modules. In both cases, representations are completely reducible, and there is a unique irreducible for every dominant weight; the dimensions of the irreducible modules of $U_q(\mathfrak{g})$ and G are equal. The main difference is that the category of $U_q(\mathfrak{g})$-modules has an interesting braiding.

In previous lectures we considered what happens to modules of $U_q(\mathfrak{g})$ when we specialize q to a root of unity. Today we will consider what happens when we specialize $q \to 0$.

Kashiwara crystals

The Hopf algebra $U_q(\mathfrak{g})$ does not make sense in the limit $q \to 0$, but if V is a $U_q(\mathfrak{g})$-module, it is possible to slightly modify the action of the E_i and F_i so that it is possible to make sense of them in the limiting case.

Then a very interesting combinatorial structure emerges. Let Λ be the weight lattice.

We define a crystal to be a set $\mathcal{B}$ with a map $\mathrm{wt} : \mathcal{B} \to \Lambda$. There are then assumed to exist maps $e_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ and $f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ (for i in the index set) subject to certain axioms.

Kashiwara crystals (continued)

It is assumed that if $e_i(x) = y$ is nonzero, then $f_i(y) = x$. In this case

$$\mathrm{wt}(e_i(x)) = \mathrm{wt}(x) + \alpha_i, \qquad \mathrm{wt}(f_i(y)) = \mathrm{wt}(y) - \alpha_i.$$

Moreover

$$\varphi_i(x) - \varepsilon_i(y) = \langle \alpha_i^\vee, \mathrm{wt}(x) \rangle.$$

The crystal is called seminormal if

$$\varepsilon_i(x) = \max\{k | e_i^k(x) \neq 0\}, \qquad \varphi_i(x) = \max\{k | f_i^k(x) \neq 0\}.$$

Tableaux

Let $G = \mathrm{GL}(n)$ so that the weight lattice $\Lambda = \mathbb{Z}^n$.

If $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a partition, so $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, a semistandard Young tableaux (SSYT) of shape λ is an array of boxes filled by integers, in which the rows are weakly decreasing, and the columns are strictly decreasing. If T is a SSYT we define the weight of T to be $(\mu_1, \mu_2, \cdots, \mu_n)$ where μ_i is the number of entries in T equal to i. Thus

$$\mathrm{wt}\left(\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\right) = (2, 2, 3)$$

Representing crystals

If $\mathcal{B}$ is a crystal, we make $\mathcal{B}$ into a directed graph with edges labeled by the index set I. We draw an edge labeled by i from x to y if $f_i(x) = y$. With this convention here are a couple of $\mathrm{GL}(3)$ crystals:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

We require if $f_i(x) = y$ then $\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha_i$. For example if

$$x = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \qquad y = \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

then $\mathrm{wt}(x) = (1,1,0)$, $\mathrm{wt}(y) = (1,0,1)$, $f_2(x) = y$, $\mathrm{wt}(x) - \mathrm{wt}(y) = \alpha_2$.

The character of a crystal

Let T be a maximal torus of G and $\mathbf{z} \in T$. If $\mathcal{B}$ is a crystal, its character is

$$\chi_{\mathcal{B}}(\mathbf{z}) = \sum_{x \in \mathcal{B}} \mathbf{z}^{\mathrm{wt}(x)}.$$

Thus the characters of the crystals:

$$\mathcal{B}_{(1)} = \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad\qquad \mathcal{B}_{(1,1)} = \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \xrightarrow{1} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$$

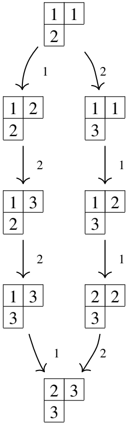
are

$$z_1 + z_2 + z_3, \qquad z_1 z_2 + z_1 z_3 + z_2 z_3.$$

These are the characters of the irreducible representations of $\mathrm{GL}(3)$ with highest weights $(1, 0, 0)$ and $(1, 1, 0)$.

Another example

$$\mathcal{B}_{(2,1)} =$$



$$\chi_{\mathcal{B}_{(2,1)}} = x_1^2x_2 + x_2^2x_1 + x_2^2x_3 + x_3^2x_2 + x_3^2x_1 + x_1^2x_3 + 2x_1x_2x_3$$

Tensor product

If $\mathcal{B}$, $\mathcal{C}$ are crystals (with the same root system), there is a crystal $\mathcal{B} \otimes \mathcal{C}$. Our notation for the tensor product differs from Kashiwara's but agrees with that of Bump and Schilling, Crystal Bases: representations and combinatorics. As a set $\mathcal{B} \otimes \mathcal{C}$ is the Cartesian product of $\mathcal{B}$ and $\mathcal{C}$, but we will denote the element (x, y) as $x \otimes y$. We will denote $\mathrm{wt}(x \otimes y) = \mathrm{wt}(x) + \mathrm{wt}(y)$.

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \varphi_i(y) \leqslant \varepsilon_i(x), \\ x \otimes f_i(y) & \text{if } \varphi_i(y) > \varepsilon_i(x), \end{cases}$$

and

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \varphi_i(y) < \varepsilon_i(x), \\ x \otimes e_i(y) & \text{if } \varphi_i(y) \geqslant \varepsilon_i(x). \end{cases}$$

It is understood that $x \otimes 0 = 0 \otimes x = 0$.

Tensor product (continued)

Also

$$\varphi_i(x \otimes y) = \max\big(\varphi_i(x), \varphi_i(y) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle\big)$$

and

$$\varepsilon_i(x \otimes y) = \max\big(\varepsilon_i(y), \varepsilon_i(x) - \langle \mathrm{wt}(y), \alpha_i^\vee \rangle\big).$$

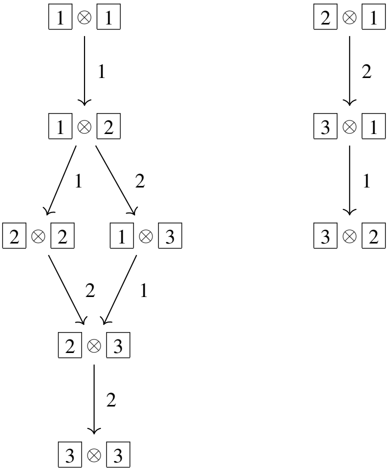
It may be checked that

$$\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \cong (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}.$$

The category of crystals is monoidal with the tensor operation. It also has a $\oplus$ operation, namely disjoint union and of course

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C).$$

Example: $\mathcal{B}_{(1)} \otimes \mathcal{B}_{(1)}$



Example (continued)

We see that the tensor product of the 3-dimensional crystal $\mathcal{B}_{(1)}$ with itself decomposes into a 6 dimensional crystal with character

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

together with a copy of the crystal $\mathcal{B}_{(1,1)}$.

This exactly mirrors the decomposition of the standard representation $\pi_{(1)}$ of $\mathrm{GL}(3)$ tensored with itself:

$$\pi_{(1)} \otimes \pi_{(1)} = \pi_{(2)} \oplus \pi_{(1,1)}$$

where if λ is a partition, π_λ denotes the irreducible representation of $\mathrm{GL}(3)$ with highest weight λ. $\pi_{(2)}$ and $\pi_{(1,1)}$ are the symmetric and exterior square representations.

Normal crystals

Let λ be a dominant weight. By Weyl character theory, G has a unique irreducible representation π_λ with highest weight λ, with character χ_λ.

Theorem (Kashiwara)

There exists a family of crystals to be called normal with the following properties.
(1) There is a unique connected normal crystal $\mathcal{B}_\lambda$ for each dominant weight λ.
(2) A crystal is normal if and only if each connected component is normal; and each connected component is a $\mathcal{B}_\lambda$.
(3) A tensor product of normal crystals is normal.
(4) The character of $\mathcal{B}_\lambda$ equals χ_λ.

Implications

The theorem implies that the decomposition of crystals under tensor product exactly mirrors the decomposition of representations into irreducibles. Thus if

$$\pi_\lambda \otimes \pi_\mu = \sum_\nu c^\nu_{\lambda,\mu} \pi_\nu$$

then $c^\nu_{\lambda,\mu}$ is also the multiplicity of $\mathcal{B}_\nu$ when $\mathcal{B}_\lambda \otimes \mathcal{B}_\mu$ is decomposed into connected components.

This follows since the characters of irreducible representations are linearly independent.

Methods of proof

Although Kashiwara arrived at crystals from the theory of quantum groups, Littelmann arrived them independently from considerations in algebraic geometry, related to the Hodge-Seshadri-Lakshmibai theory of standard monomials.

Thus there were two independent proofs of the theorem, since Littelmann's path-method proof was very different. It required proof that Littelmann's crystals were the same as Kashiwara's.

A purely combinatorial proof of the theorem was obtained by Bump and Schilling. I believe it is also possible to prove it using ideas from geometric Langlands theory (MV polytopes).

Review: $U_q(\mathfrak{g})$

For simplicity assume that the Cartan type is simply-laced. Let q be an indeterminate, so $\mathbb{C}(q)$ is the field of fractions of the polynomial $\mathbb{C}[q]$. $U_q(\mathfrak{g})$ had generators E_i, F_i and K_i $(i = 1, \cdots, r)$ with K_i inqertible and relations:

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

and the quantum Serre relations: if α_i and α_j are orthogonal then E_i and E_j commute,

$$E_i^2 E_j - [2]_q E_i E_j E_i + E_j E_i^2 = 0$$

where $[2]_q = q + q^{-1}$, and similar relations for the F_i. Let $U_q(\mathfrak{g})$ be the algebra generated by the E_i, F_i and K_i.

The comultiplication

We will deviate from Kashiwara's notation for the comultiplication, which is non-standard to begin with. This is related to the fact that our notation for the tensor product also differs from Kashiwara's. This does not change the crystals themselves.

We will take

$$\Delta(K_i) = K_i \otimes K_i, \qquad \Delta(E_i) = K_i^{-1} \otimes E_i + E_i \otimes 1,$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.$$

Representations

There is a suitable category of representations $\mathcal{O}_{\text{int}}$ that is completely reducible, and generated by irreducible highest weight modules V_λ which look like the corresponding irreducibles $L(\lambda)$ of G. That is, they have the same character.

Sometimes we wish to consider crystals of more general representations, such as Verma modules that live in the BGG Category $\mathcal{O}$.

Indeed Kashiwara's theories make essential use of a crystal $\mathcal{B}(\infty)$ that is the crystal of a Verma module. We will content ourselves with only explaining a part of what is actually needed and describe crystals for $\mathcal{O}_{\text{int}}$.

Kashiwara operators

Kashiwara uses the notation e_i and f_i for the generators of $U_q(\mathfrak{g})$, which become operators on any module V. Then he uses $\widetilde{e}_i$ and $\widetilde{f}_i$ for certain modified Kashiwara operators. Instead, we use E_i and F_i for the generators and e_i and f_i for the modified Kashiwara operators.

We will write

$$V = \bigoplus_{\mu} V(\mu)$$

where $V(\mu)$ is the μ weight-space. We have

$$E_i V(\mu) \subseteq V(\mu + \alpha_i), \qquad F_i V(\mu) \subset V(\mu - \alpha_i).$$

Kashiwara operators: construction

Using the divided factorials

$$F^{(n)} = \frac{1}{[n]_q!}F^n,$$

every element of V can be written uniquely as

$$x = \sum_{m=0}^{k} F^{(m)}(u_m)$$

where $E_i(u_m) = 0$, i.e. u_m is a multiple of the highest weight vector.
To prove this, note that E_i, F_i, K_i generate a copy $U_q(\mathfrak{g}_i)$ of $U_q(\mathfrak{sl}_2)$. Hence this reduces to a statement about $U_q(\mathfrak{sl}_2)$ modules, which can be checked. Now define the Kashiwara operators

$$e_i F^{(m)}(u) = F^{(m-1)}(u), \qquad f_i F^{(m)}(u) = F^{(m+1)}(u)$$

Crystal bases

Let A be the local ring consisting of $f \in \mathbb{C}(q)$ that have no pole at $q = 0$. Evaluating f at 0 gives a homomorphism $A \to \mathbb{C}$ so $A/qA \cong \mathbb{C}$.

By a lattice in a finite-dimensional $\mathbb{C}(q)$-vector space V we mean an A-submodule $\mathcal{L}$ of V such that $V \cong \mathbb{C}(\amalg) \otimes_A \mathcal{L}$. This means that $\mathcal{L}$ has an $\mathcal{A}$-basis B that is also a $\mathbb{C}$-basis of V.

Theorem (Kashiwara)

V has a lattice $\mathcal{L}$ such that the Kashiwara operators e_i and f_i map $\mathcal{L}$ to itself. Hence e_i and f_i induce endomorphisms of the complex vector space $\mathcal{L}/q\mathcal{L}$. We may choose a basis B of $\mathcal{L}/q\mathcal{L}$ such that

$$e_i(B) \subset B \cup \{0\}, \qquad f_i(B) \subset B \cup \{0\}.$$

Crystal bases (continued)

The proof of Kashiwara's theorem was very difficult.

The crystal base is a crystal as we defined crystals abstractly. Kashiwara also proved that if V and W are modules with crystal bases then $V \otimes W$ has a crystal base, and he proved the tensor product rule that we gave earlier.

Crystal bases are the most natural context for discussing certain topics in combinatorics, including work that was done before they were invented, such as the Robinson-Schensted-Knuth algorithm and deep work of Lascoux and Schützenberger.

Kashiwara-Nakashima

Kashiwara and Nakashima showed that the set of SSYT of shape λ in $\{1, \cdots, n\}$ could be given the structure of a $\mathrm{GL}(n)$ crystal. All these crystals are subcrystals of tensor powers of the standard crystal.

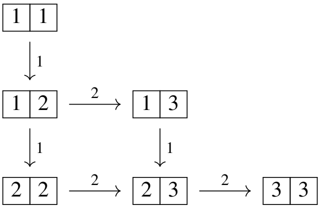
$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}$$

We will describe these crystals of tableaux as subcrystals of $\otimes^n \mathbb{B}$. Recall that $\mathrm{wt}(T) = (\mu_1, \cdots, \mu_n$ where μ_i is the number of i in the tableau T.

Crystals of rows

Let λ be a partition of r, having $\leqslant n$ parts. Our goal is to organize the tableaux of shape λ into a $\mathrm{GL}(n)$ crystal. First we consider the case where $\lambda = (r)$. Then a tableau of shape λ is just a row of length r. The operation f_i changes the rightmost i to $i+1$ if i occurs in a tableau T, otherwise $f_i(T) = 0$.

Example. If $n = 4$ and $r = 2$:



Crystals of rows as subcrystals of $\otimes^r \mathbb{B}$

If we identify the row $\boxed{a}\boxed{b}\boxed{\cdots}\boxed{z}$ of length r with

$$\boxed{a} \otimes \cdots \otimes \boxed{z} \in \otimes^r \mathbb{B}$$

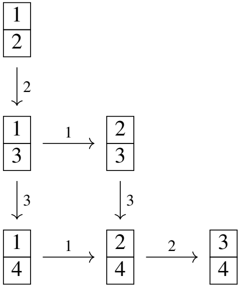
then we get the right tensor operation. It is important that

$$a \leqslant b \leqslant \cdots \leqslant z.$$

Then the elements of $\otimes^r \mathbb{B}$ of this form are closed under e_i and f_i and give a realization of the crystal of rows.

Crystals of columns

For crystals of columns, we recall that a column is strictly increasing. f_i changes i to $i+1$ but only if $i+1$ does not already occur in the tableau T. If $n=4$ and $r=2$, $\mathcal{B}_{(1,1)}$ looks like:



Crystals of columns as subcrystals of $\otimes^r \mathbb{B}$

If we identify $\boxed{a}\boxed{b}\boxed{\cdots}\boxed{z}$ with

$$\boxed{z} \otimes \cdots \otimes \boxed{a}$$

then we get the right tensor operation. It is important that

$$a < b < \cdots < z.$$

Then the elements of $\otimes^r \mathbb{B}$ of this form are closed under e_i and f_i and give a realization of the crystal of columns.

- The crystal $\mathcal{B}_{(r)}$ of rows is the crystal of the r-th symmetric power representation of $\mathrm{GL}(n)$.
- The crystal $\mathcal{B}_{(1^r)} = \mathcal{B}_{(1,\cdots,1)}$ of columns is the crystal of the r-th exterior power representation of $\mathrm{GL}(n)$.

General Kashiwara-Nakashima crystals

If λ is an arbitrary partition of r of length $\leqslant n$ and T is a SSYT of shape λ then we identify T with an element of $\otimes^r \mathbb{B}$ as follows. Read the entries of the tableau from left to right, from bottom to top.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} =$$

$$\boxed{4} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3}$$

It may be seen that e_i and f_i (as defined on $\otimes^r \mathbb{B}$) preserve the set of tableaux of shape λ embedded this way. This is the row reading. There is another systematic way of embedding the crystal of tableaux into $\otimes^r \mathbb{B}$ that gives the same crystal structure.

Review (from Lecture 7): Schur-Weyl-Jimbo duality

Schur-Weyl duality is a relationship between the representations of the symmetric group S_r and the general linear group $\mathrm{GL}(n,\mathbb{C})$.

The groups S_r and $\mathrm{GL}(n,\mathbb{C})$ both act on the same vector space $\otimes^r V$ where $V = \mathbb{C}^n$, the standard module of $\mathrm{GL}(n)$. The group $\mathrm{GL}(n)$ acts diagonally:

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n, \qquad g \in \mathrm{GL}(n).$$

The symmetric group acts by permuting the components.

$$w(v_i \otimes \cdots v_r) = v_{w^{-1}i} \otimes \cdots v_{w^{-1}r}, \qquad w \in S_r.$$

The two actions obviously commute. The problem is to decompose $\otimes^r V$ into irreducibles.

Review: Irreducibles of $\mathrm{GL}(n)$ and S_r

Let λ be a partition of r of length $\leqslant n$. Thus $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_n \geqslant 0$ and $\sum \lambda_i = r$. Then λ indexes both an irreducible representation of S_r and of $\mathrm{GL}(r, \mathbb{C})$.

The irreducible representations of S_r are indexed by partitions of r. We will denote the corresponding irreducible of S_r as $\pi_\lambda^{S_r}$.

The irreducibles of $\mathrm{GL}(n, \mathbb{C})$ are indexed by dominant weights. A weight is a rational character of the diagonal subgroup; the group of weights is in bijection with $\mathbb{Z}^n$ as follows: if $\lambda = (\lambda_1, \cdots, \lambda_n)$, then λ is called dominant if $\lambda_1 \geqslant \cdots \geqslant \lambda_r$. In particular, a partition of length $\leqslant r$ is a dominant weight. Let $\pi_\lambda^{GL(n)}$ be the corresponding irreducible.

Review: Schur-Weyl duality (concluded)

As a $S_r \otimes GL(n)$-module, Schur-Weyl duality is the isomorphism

$$\otimes^r V \cong \bigoplus_{\lambda \vdash r} \pi_\lambda^{S_r} \otimes \pi_\lambda^{GL(n)}.$$

In Lecture 7 we studied Jimbo's generalization, where $\mathrm{GL}(n)$ is replaced by $U_q(\mathfrak{gl}_n)$ and S_r is replaced by its Hecke algebra. In today's lecture we will point out a combinatorial analog due to Robinson, Schensted, Knuth, Lascoux and Schützenberger.

The decomposition of $\otimes^r \mathbb{B}$

Theorem

The $\mathrm{GL}(r)$ *crystal* $\otimes^r \mathbb{B}$ *decomposes into crystals* $\mathcal{B}_\lambda$ *where* λ *runs over the partitions of* r *of length* $\leqslant n$. *The number of times* $\mathcal{B}_\lambda$ *appears equals the degree of the irreducible representation* $\pi_\lambda^{S_r}$ *of* S_r.

Needless to say there is much more to be said about these matters, but I'll stop here. These matters are central in combinatorics and predated crystals. But crystal theory is a very natural way of organizing tableaux combinatorics, including topics such as the plactic monoid, *jeu de taquin*, the Littlewood-Richardson rule and the famous RSK algorithm.