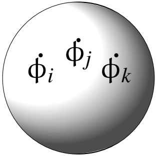
Lecture 17

Daniel Bump

June 3, 2019

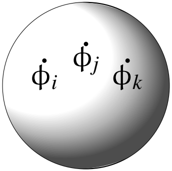


Motivation from CFT

The primary fields in a conformal field theory admit an associative composition law. If ϕ_i are a basis of primary fields then the composition is (for certain constants)

$$\phi_i \star \phi_j = N_{i,j}^{k}\phi_k.$$

If $i \to *i$ is conjugation then $N_{i,j,k} = N_{i,j}^{*k}$ will be symmetric under all three parameters. In the physical interpretation, these will be the 3-point correlation functions at 3 distinct points on a punctured sphere.



Affine Lie algebras

Verlinde and Moore and Seiberg showed that the WZW theories lead to a composition in terms of the representation theory of affine Lie algebras. This was turned into a purely mathematical operation by Kazhdan and Lusztig. The fields are interpreted as level k integrable representations of an affine Lie algebra. Kazhdan and Lusztig, Finkelberg, Andersen and Paradowski and Sawin related these fusion rings to the ones constructed from representations of quantum groups in Lecture 16.

The modest goal of todays lecture is to describe the fusion ring structure on integrable highest weight representations of an affine Lie algebra level k, without proofs. We will not give the definition of Kazhdan and Lusztig, but instead describe the S-matrix and define the fusion coefficients by the Verlinde formula.

Infinite-dimensional Lie algebras

In the 1970's, people became aware that much of classical Lie theory had an important infinite-dimensional generalizations. For example consider the Weyl denominator formula. If $\mathfrak{g}$ is a complex semisimple Lie algebra, say the Lie algebra of the Lie group G, and if $\mathfrak{h}$ is the Lie algebra of a maximal torus T of G, we have the Weyl character formula

$$\chi(\mathbf{z}) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)}$$

where (Weyl denominator formula):

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\rho)} = \mathbf{z}^{\rho} \prod_{\alpha \in \Delta^+} (1 - \mathbf{z}^{-\alpha}).$$

Similar identities were noticed by Macdonald (1972) in which W is replaced by the affine Weyl group.

References

- Kac, Infinite-Dimensional Lie algebras, particularly Chapter 13.
- Kac and Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms (1984).
- Affine Root System Basics [Web link]
- Di Francesco, Mathieu and Sénéchal, Conformal Field Theory, Chapter 14.
- Bakalov and Kirillov, Chapter 7.

The affine Weyl group

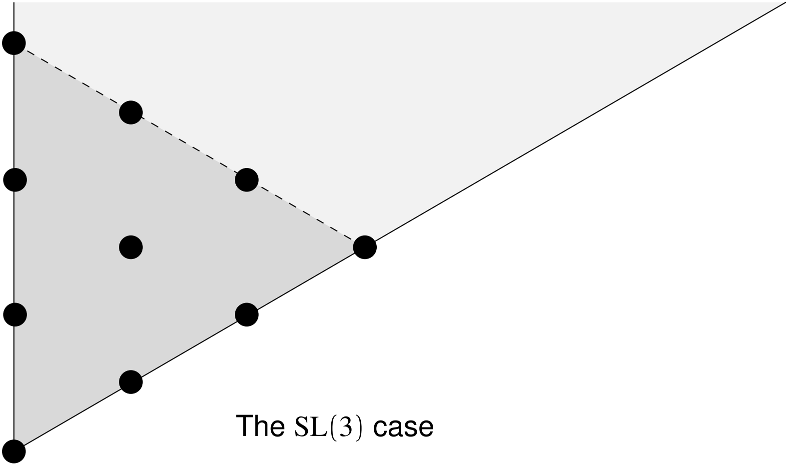
The affine Weyl group is the group generated by reflections in the hyperplanes $\langle \alpha^{\mathfrak{v}}, x\rangle = k$. The connected components of the complement of these hyperplanes are called alcoves. There is a unique alcove that is contained in the positive Weyl chamber that is adjacent to the origin, namely

$$\mathcal{F}_k = \{x | \langle \alpha_i^\vee, x\rangle \geqslant 0,\ \langle \theta^\vee, x\rangle \leqslant k\}.$$

Here α_i are the simple roots and θ is the longest root.

The affine Weyl group is a Coxeter group, generated by the reflections s_i in the hyperplanes bounding the fundamental alcove. The reflection in $\langle \theta^\vee, x\rangle = k$ is denoted s_0.

The fundamental alcove inside the positive Weyl chamber



The $\mathrm{SL}(3)$ case

The Jacobi triple product identity

Jacobi proved

$$\sum_{-\infty}^{\infty}(-1)^n q^{n^2} t^n = \prod_{m=1}^{\infty}(1-q^{2m})(1-tq^{2m-1})(1-t^{-1}q^{2m-1}).$$

The sum on the left can be considered a sum over the $\widehat{\mathfrak{sl}}_2$ affine Weyl group. Macdonald noted a similarity to the Weyl denominator formula and gave similar formulas for other affine root systems.

Meanwhile, infinite-dimensional affine Lie algebras began to appear in mathematical physics as current algebras. Kac (and independently Moody) developed a general theory that contains both the finite-dimensional and affine Lie algebras as special cases.

Generalized Cartan Matrices

Let $A = (a_{ij})$ be a generalized Cartan matrix. The entries are integers with $a_{ii} = 2$ and $a_{ij} \leqslant 0$ if $i \neq j$. Things work best if DA is symmetric for some diagonal matrix D. For simplicity we will assume $D = I$ so A is a symmetric matrix. This is the simply-laced case.

Now there exists a finite-dimensional vector space $\mathfrak{h}^*$ with $\mathfrak{h}$ its dual space, and linearly independent subsets

$$\Pi = \{\alpha_1, \cdots, \alpha_s\} \subset \mathfrak{h}^*, \qquad \Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_s^\vee\}$$

(Roots and coroots) such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$$

and $\dim(\mathfrak{h})$ is as small as possible. These data are called the realization of A.

The Kac-Moody Lie algebra

Now there exists a Lie algebra $\mathfrak{g}(A)$ with generators and relations as follows. We need $\mathfrak{h}$, which will be a commutative subalgebra, and for each $1 \leqslant i \leqslant s$ generators E_i, F_i subject to

$$[H, E_i] = \langle \alpha_i, H \rangle E_i, \qquad [H, F_i] = -\langle \alpha_i, H \rangle F_i,$$

$$[E_i, F_j] = \delta_{ij} \alpha_i^\vee$$

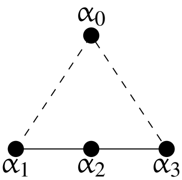
and Serre relations

$$\mathrm{ad}(E_i)^{1-\alpha_{ij}} E_j = \mathrm{ad}(F_i)^{1-\alpha_{ij}} F_j = 0.$$

Such a Lie algebra has a Weyl group, which is a (possibly infinite) Coxeter group.

Extended Dynkin diagrams

A class of Kac-Moody Lie algebras that appears in real life is the class of (untwisted) affine Lie algebras. If $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra of rank r, then there is an "extended Dynkin diagram" with $r+1$ nodes, from which may be read off a Cartan matrix. For example if $\mathfrak{g} = \mathfrak{sl}_4$ then the extended Dynkin diagram looks like



Eliminating the α_0 node gives the ordinary Dynkin diagram.

The extended Dynkin-diagram continued

If the root system Δ of $\mathfrak{g}$ is known we may describe the extended Dynkin-diagram as follows. Let θ be the highest root and $\alpha_0 = -\theta$.

Note: the meaning of α_0 will change later when we embed the roots in a larger space $\mathfrak{h}^*$.

To define the extended Dynkin diagram, we note that for every $i, j \in \{0, 1, \cdots, r\}$ either $i = j$ or $\langle \alpha_0, \alpha_j \rangle \leqslant 0$. The extended Dynkin diagram is made by the usual recipe, namely α_i and α_j are joined by an edge if they are not orthogonal.

The Cartan matrix (finite case)

Starting with the Dynkin diagram we may reconstruct the Cartan matrix A of $\mathfrak{g}$.

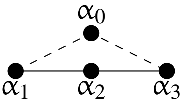
$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } \alpha_i,\, \alpha_j \text{ are adjacent;} \\ 0 & \text{if } \alpha_i,\, \alpha_j \text{ are not adjacent.} \end{cases}$$

The corresponding Kac-Moody algebra is the original finite-dimensional $\mathfrak{g}$, in this example $\mathfrak{sl}_4$.

$$A^\circ = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The Cartan matrix (affine case)

Now the extended Cartan matrix is made the same way from the extended Dynkin diagram:



This adds one row and column to the Cartan matrix.

$$\widehat{A} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

The affine Lie algebra

We may now apply the Kac-Moody construction to both Cartan matrices.

- Applying it to A recovers $\mathfrak{g}$.
- Applying it to $\widehat{A}$ gives the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$.
- The dimension of the Cartan algebra $\widetilde{\mathfrak{h}}$ is $r+2$.

There is another way of constructing these untwisted affine Lie algebras as the Lie algebra of a loop group. Start with the Lie algebra of vector fields on the circle taking values in $\mathfrak{g}$ is

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

$$[X \otimes t^n], [Y \otimes t^m] = [X, Y] \otimes t^{n+m}.$$

Central extensions

Suppose $\mathfrak{g}$ is a Lie algebra and $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear map such that

$$\psi(X, Y) = -\psi(Y, X),$$

$$\psi([X, Y], Z) + \psi([Y, Z], X) + \psi([Z, X], Y) = 0.$$

Such a map is called a 2-cocycle. Given such data we may construct a central extension which is $\mathfrak{g} \oplus \mathbb{C} \cdot K$ and the bracket operation is

$$[X + aK, Y + bK] = [X, Y] + (a + b + \psi(X, Y))K$$

and this is a Lie algebra.

A cocycle for the loop agebra

Start with $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. We define a 2-cocycle by

$$\psi(X \otimes t^n, Y \times t^m)x = \delta_{n,-m}\, n\langle X, Y\rangle$$

in terms of an invariant inner product on $\mathfrak{g}$. We may then construct a central extension $\widehat{\mathfrak{g}}'$ of $\mathfrak{g}$. It is possible to enlarge this Lie algebra one dimension further by adjoining a derivation d to obtain the affine Lie algebra $\widehat{\mathfrak{g}}$.

We will denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\mathfrak{g}$. It can be realized as $\mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ where K is the central element and d is the derivation.

Roots and weights

The set of $\lambda \in \widehat{\mathfrak{h}}$ such that $\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}$ for all α_i are called weights.

For example the simple roots α_i are roots, and one particular root

$$\delta = \sum_{i=0}^{r} a_i \alpha_i$$

called the nullroot is orthogonal to all the others in an invariant inner product on $\widehat{\mathfrak{h}}^*$. The coefficient $a_0 = 1$ and the other a_i can be looked up in Kac, Infinite Dimensional Lie Algebras; for $\mathfrak{g} = \mathfrak{sl}_{r+1}$ they are all equal to 1.

Weight space decompositions

We have $\Lambda_i \in \widehat{\mathfrak{h}}^*$ such that $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$. Then the weight lattice is the linear span of Δ and the Λ_i. The Λ_i are the fundamental weights.

We have a triangular decomposition

$$\widehat{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- .$$

We require modules to have a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda .$$

A module V is called a highest weight module with highest weight λ if it has a vector v_λ annihilated by $\mathfrak{n}_+$ such that

$$X \cdot v_\lambda = \langle X, \lambda \rangle v_\lambda , \qquad X \in \mathfrak{h}.$$

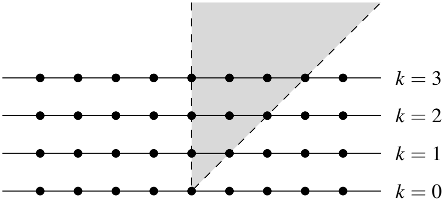
The affine Weyl group

The Weyl group of a general Kac-Moody Lie algebra $\mathfrak{g}$ is be the group of linear operations on $\widehat{\mathfrak{h}}^*$ generated by the simple reflections

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i. \tag{1}$$

Note that this action fixes a point (the origin) unlike the affine Weyl group defined earlier, which included a translations. Let us relate the two actions. If λ is a weight, we call $k = (\delta|\lambda)$ the level of λ. Now we consider the action of the Weyl group W (defined by s_i as in (1).

The affine Weyl group (continued)



The affine Weyl group fixes the origin. But parallel to the level 0 plane, it induces affine linear maps on each level (except level 0). The fundamental alcove on the k level set is the intersection with the shaded area, consisting of dominant weights.

Kac-Moody root systems

For every Kac-Moody Lie algebra, there is a root system. In contrast with finite root systems, infinite Kac-Moody root systems include are two kinds of roots. Roots may be classified as as real or imaginary.

Macdonald invented affine root systems before it was realized that they were root systems of infinite-dimensional Lie algebra. The imaginary roots appeared as factors $1-q^{2m}$ in the Jacobi triple product identity, which Macdonald greatly generalized.

$$\sum_{-\infty}^{\infty}(-1)^n q^{n^2} t^n = \prod_{m=1}^{\infty}(1-q^{2m})(1-tq^{2m-1})(1-t^{-1}q^{2m-1}).$$

For general Kac-Moody root systems, the imaginary roots remain somewhat mysterious.

The Affine Root System

We started with a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ of rank r having a root system Δ. We produced an infinite-dimensional (untwisted) Lie algebra $\widehat{\mathfrak{g}}$ with root system $\widetilde{\Delta}$.

The real roots are

$$\Delta_{\text{re}} = \{\alpha + n\delta \,|\, \alpha \in \Delta, n \in \mathbb{Z}\}.$$

The imaginary roots are

$$\Delta_{\text{im}} = \{n\delta \,|\, n \in \mathbb{Z}, n \neq 0\}.$$

There is a Weyl vector ρ though it cannot be defined as half the sum of the positive roots.

Integrable highest weight modules

A weight $\lambda \in \mathfrak{h}^*$ is dominant if $\langle \alpha_i^\vee, \lambda \rangle$ is a nonnegative integer. We may assume that $\lambda = \sum n_i \Lambda_i$ where n_i are nonnegative integers. In this case the unique irreducible highest weight representation $L(\lambda)$ is called integrable because the representation can be exponentiated to a representation of the loop group. One implication is that the weight multiplicities are invariant under the action of the Weyl group.

Let μ be a weight, and we consider the multiplicity $m(\mu)$ of μ in $L(\lambda)$. This is nonzero uness μ lies in a cone

$$(\lambda + \rho|\lambda + \rho) - (\mu + \rho, \mu + \rho) \geqslant 0.$$

The Weyl-Macdonald identities

The results in this slide are valid for an arbitrary Kac-Moody Lie algebra, though we are interested mainly in the affine case. Recall that there is a Weyl vector ρ such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for all simple coroots α_i. Then

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}).$$

We are using e^{μ} as a formal symbol. Although this is a formal identity, specializing e^{μ} can give an identity between convergent expressions.

For example, in $\widehat{\mathfrak{sl}}_2$ let $\Delta = \{\pm\alpha\}$ be the usual $\mathfrak{sl}_2$ roots, and $\alpha_0 = \delta - \alpha_1$.

The triple product identity

The $\widehat{\mathfrak{sl}}_2$ positive roots are:

$$\Delta_{\text{re}} = \{\alpha_i + n\delta \,|\, n \geqslant 0, i = 0, 1\}$$

$$\Delta_{\text{im}} = \{n\delta \,|\, n \geqslant 1\}$$

Let $q, t \in \mathbb{C}$ with $t \neq 0$. To get a convergent series we want $|q| < 1$. Specialize e^{μ} so that

$$e^{\alpha_1} = tq, \qquad e^{\alpha_0} = t^{-1}q, \qquad e^{\alpha_1} = tq, \qquad e^{\delta} = q^2.$$

Then the Weyl-Macdonald identity

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$$

becomes the Jacobi triple product identity

$$\sum_{-\infty}^{\infty} (-1)^n q^{n^2} t^n = \prod_{m=1}^{\infty} (1 - q^{2m})(1 - tq^{2m-1})(1 - t^{-1}q^{2m-1}).$$

The Kac-Weyl character formula

They Weyl character formula is valid for the integrable highest-weight representation $L(\lambda)$ of an arbitrary Kac-Moody Lie algebra. The character

$$\chi_\lambda = \sum_\mu m(\mu) e^\mu$$

equals:

$$D^{-1} \sum_w (-1)^{\ell(w)} e^{w(\lambda+\rho)}$$

where D is the denominator from the last slide:

$$D = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = e^\rho \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$$

String functions

In the affine case, the character χ_λ of $L(\lambda)$ can be written out in terms of more basic functions called string functions.

Let μ be a weight. The function $m(\mu - n\delta)$ is monotone, and zero outside the cone

$$(\lambda + \rho|\lambda + \rho) - (\mu + \rho, \mu + \rho) \geqslant 0.$$

Therefore we may find a smallest n such that $m(\mu - n\delta)$ is nonzero. We call $\mu - n\delta$ a maximal weight. Define the string function

$$b_\mu^\lambda = \sum_k m(\mu - k\delta) e^{-k\delta}.$$

replacing μ by $\mu - n\delta$ just multiplies the series by $e^{k\delta}$ so there is no loss of generality in assuming that μ is maximal.

String functions (continued)

The character

$$\chi_\lambda = \sum_{\mu \text{ maximal}} b_\mu^\lambda.$$

Moreover using the fact that $\delta = w(\delta)$ for all $w \in W$ we may have $w(b_\mu^\lambda) = b_{w\mu}^\lambda$ and therefore we may rewrite b_μ^λ by finding a Weyl group element such that $w\mu$ is dominant. There are only a finite number of dominant maximal weights. Then

$$\chi_\lambda = \sum_{\mu \text{ maximal, dominant}} \sum_{w \in W/W_\mu} e^{w\mu} b_\mu^\lambda.$$

(To avoid overcounting W_μ is the stabilizer of the dominant maximal weight μ.)

Modular characteristics

Define the modular characteristic (modular anamoly)

$$s_\lambda = \frac{|\lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2k},$$

We are using the notation $(\ |\ )$ for an invariant inner product on Λ. Here $h^\vee$ is the dual Coxeter number and $k = (\delta|\lambda)$. Also

$$s_\lambda(\mu) = s_\lambda - \frac{|\mu|^2}{2k}.$$

Now define

$$c^\lambda_\mu = e^{-s_\lambda(\mu)\delta} b^\lambda_\mu = e^{-s_\lambda(\mu)\delta} \sum_{n \geqslant 0} m(\lambda - n\delta) e^{-n\delta}.$$

These modified string functions are theta functions which are modular forms, as was proved by Kac and Peterson. Because of this, the character χ_λ is a modular form.

The fusion product following Kazhdan and Lusztig

We have already mentioned that if ϕ_i are the primary fields in a WZW conformal field theories the fusion coefficients $N_{ijk} = N_{ij}^{*k}$ are the three-point correlation functions for primary fields ϕ_i on the Riemann sphere. Note that the group of conformal maps of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ is 3-transitive, so there is no loss in assuming the points are 0, ∞ and -1.

Therefore the construction as abstracted by Kazhdan and Lusztig though purely algebraic, also uses the sphere punctured on three points, an affine variety whose affine algebra is $R = \mathbb{C}[t, t^{-1}, (1+t)^{-1}]$.

The fusion product following Kazhdan and Lusztig (continued)

In Kazhdan and Lusztig (though modifying their notation), the affine algebra is a central extension

$$0 \to \mathbb{C} \cdot 1 \to \widehat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0,$$

where $\mathbb{C}((t))$ is the field of formal power series $\sum_{k=-N}^{\infty} a_k t^k$. If V, W are modules of level k, restrict their action to the preimage $(\mathfrak{g} \otimes R)^{\wedge}$ in $\widehat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$. (Recall $R = \mathbb{C}[t, t^{-1}, (1+t)^{-1}]$.) This is a Lie algebra. Now there are two homomorphisms $R \to \mathbb{C}((t))$ in which $t \mapsto t-1$ and $t \mapsto t^{-1}-1$. If $\xi \in (\mathfrak{g} \otimes R)^{\wedge}$ let ξ' and ξ'' be the images of ξ under these two homomorphisms.

The fusion product following Kazhdan and Lusztig (continued)

Consider the space of bilinear forms $\lambda : V \otimes W \to \mathbb{C}$ in which ξ acts by

$$\xi\lambda(x, y) = \lambda(\xi' x, y) - \lambda(x, \xi'' y).$$

Now considering the subspace in which the subalgebra of $\widehat{\mathfrak{g}}$ generated by the E_i (including E_0) acts nilpotently, this gives a module $V \circ W$. The composition law actually is $(V, W) \to (V \circ W)^*$.

We will not follow up on this description. Instead, we will give a different description of the fusion product, based on the S-matrix, modularity and the Verlinde formula.

The group $\mathrm{SL}(2,\mathbb{Z})$

Let $q = e^{2\pi i\tau}$ where $\mathcal{H} = \{\tau = x + iy \in \mathbb{C} | y > 0\}$. The group $\mathrm{SL}(2,\mathbb{R})$ and its discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ act on $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

The subgroup $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ acts discontinuously.

A modular form is a function f that satisfies

$$f(z) = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ or a subgroup of finite index.

Two important elements

Note that $-I \in \mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathcal{H}$, so the action is not faithful. Let

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

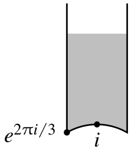
These generate $\mathrm{SL}(2,\mathbb{Z})$. Since $S^2 = -I$, S has order 4 as an element of the group but order 2 in its action on $\mathcal{H}$. Note $S : \tau \mapsto -\frac{1}{\tau}$.

$T : \tau \to \tau + 1$ is the translation by 1. If a holomorphic function f is invariant under $\mathrm{SL}(2,\mathbb{Z})$ it is invariant under T and so it has a Fourier expansion:

$$f(\tau) = \sum a_n q^n, \qquad q = e^{2\pi i \tau}.$$

Digression on $\mathrm{SL}(2,\mathbb{Z})$ (continued)

Here is the well-known fundamental domain for $SL(2,\mathbb{Z})$:



We have marked the fixed points i and $e^{2\pi i/3}$ of S and ST.

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

$$S : z \to -\frac{1}{z}, \qquad T : z \to z + 1.$$

Jacobi forms

One also encounters modular forms involving both the modular parameter τ and one or more auxiliary complex parameters $\mathbf{z} = (z_1, \cdots, z_r)$. This leads to the notion of Jacobi modular forms in which the relevant group is a semidirect product of $\mathrm{SL}(2,\mathbb{Z})$ and a Heisenberg group of degree $2r+1$. The action is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, \mathbf{z}) \mapsto \left(-\frac{1}{\tau}, \frac{\mathbf{z}}{c\tau + d}\right).$$

Examples are found in the theory of theta functions. We consider this action on $\mathbb{C} \otimes \mathfrak{h}^*$.

The S-matrix

Now the S-matrix is essentially the scattering matrix for the characters of the level k. The integrable highest weight representations of level k are parametrized by dominant weights of level k, which lie in the fundamental alcove of level k.

We use the following coordinates on $\hat{\mathfrak{h}}$ and its complexification. Choose an orthonormal basis $v_1, \cdots, v_r$ of $\mathfrak{h}$ and then we will denote

$$(\tau, z, u) = \sum z_i v_i - \tau\Lambda_0 + u\delta$$

where Λ_0 is the affine fundamental weight. We will write the character χ_λ in these coordinates. Then

$$\chi_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{|z|^2}{2\tau}\right) = \sum_{\mu} S_{\lambda,\mu}\chi_\mu(\tau, z, u).$$

Verlinde formula

The identity element in the fusion ring is the representation with highest weight $k\Lambda_0$. So we will denote $S_{k\Lambda_0,\lambda}$ as just $S_{0,\lambda}$. Knowledge of the S-matrix is sufficient to reconstruct the fusion coefficients $N^{\nu}_{\lambda,\mu}$ by the Verlinde formula

$$N^{\nu}_{\lambda,\mu} = \sum_{\sigma} \frac{S_{\lambda,\sigma} S_{\mu,\sigma} \overline{S_{\nu,\sigma}}}{S_{0,\sigma}}$$