Lecture 16

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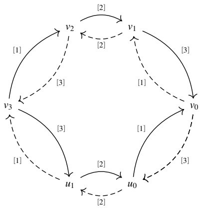


Andersen and Paradowski

In Lecture 15 we described a certain modular tensor category $\mathcal{C}(\mathfrak{g}, k)$ associated with the following data: a complex semisimple Lie algebra $\mathfrak{g}$ and a "level" k. The simple objects are in bijection with weights λ in the fundamental alcove of level k. Thus λ dominant and satisfy $\langle \theta^\vee, \lambda \rangle \leqslant k$, where θ is the highest root and $\theta^\vee$ is the associated coroot.

We claimed that this MTC could be constructed from the quantum group $U_q(\mathfrak{g})$ where q is a suitable root of unity, but the category of modules for $U_q(\mathfrak{g})$ is not semisimple (though it is ribbon), as we will see very soon.

Andersen and Paradowski gave a description of the semisimple category $\mathcal{C}(\mathfrak{g}, k)$ from $U_q(\mathfrak{g})$ which discuss today.

Groups over $\mathbb{Z}$: Chevalley and Kostant

Chevalley (1955) gave a construction of groups of Lie type associated with a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. A key step was to prove that the Lie algebra $\mathfrak{g}$ has a basis with integer structure constants, leading to an algebraic group over $\mathbb{Z}$. This produces a group scheme G over $\mathbb{Z}$ such that $G(F)$ contains a simple group in its composition series for most finite fields F.

Since a group scheme is a commutative Hopf algebra, there should be a dual Hopf algebra that is a $\mathbb{Z}$-form of $U(\mathfrak{g})$. This was constructed in 1966 by Kostant.

Lusztig's quantum group

There are different version of $U_q(\mathfrak{g})$. Lusztig's version of $U_q(\mathfrak{g})$, which we will describe, is related Kostant's enveloping algebra over $\mathbb{Z}$. This version is defined over $\mathcal{A} = \mathbb{C}[v, v^{-1}]$, and gives rise to versions defined over other rings, particularly $\mathbb{C}(q)$ where q can be a root of unity, by extension of scalars. The key idea is to include divided powers in the generating set.

For simplicity we will restrict ourselves to the simply-laced case, i.e. we assume all roots have the same length. Let α_i ($i = 1, \cdots, r$) be the simple roots and $a_{ij} = \langle \alpha_i, \alpha_j \rangle$ with a W-invariant inner product on the weight lattice normalized so $a_{ii} = 2$. If α_i, α_j are not orthogonal then $a_{ij} = -1$.

The generic case

We review the construction of $U_v(\mathfrak{g})$ when v is an indeterminate. This is the $\mathbb{C}(v)$-algebra with generators E_i, F_i and K_i ($i = 1, \cdots, r$) with K_i invertible and relations:

$$K_i E_j K_i^{-1} = v^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = v^{-a_{ij}} F_j, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}},$$

and the quantum Serre relations: if α_i and α_j are orthogonal then E_i and E_j commute, otherwise

$$E_i^2 E_j - [2]_v E_i E_j E_i + E_j E_i^2 = 0$$

where $[2]_v = v + v^{-1}$, and similar relations for the F_i. Let $U_v(\mathfrak{g})$ be the algebra generated by the E_i, F_i and K_i.

The generic case (continued)

We then define the comultiplication by

$$\Delta(E_i) = E_i \otimes 1 + K \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

$$\Delta(K_i) = K_i \otimes K_i,$$

with a suitable counit and antipode.

This differs from our previous definition, the comultiplication being conjugated by $K_i \otimes K_i$.

We are following Lusztig, Modular Representations and Quantum Groups (1989). See also Chari and Pressley, A guide to Quantum groups Section 9.3.

The restricted quantum group

To define the quantum group at a root of unity, it is necessary to have a version in which the indeterminate v can be specialized. We make use of the divided powers

$$E_i^{(k)} = \frac{E_i^k}{[k]_v!}, \qquad F_i^{(k)} = \frac{F_i^k}{[k]_v!}.$$

Let $\mathcal{A} = \mathbb{C}[v, v^{-1}]$.

Let $U_v^{\mathrm{res}}(\mathfrak{g})$ be the $\mathcal{A}$-subalgebra of $U_v(\mathfrak{g})$ generated by the K_i, $E_i^{(k)}$ and $F_i^{(k)}$ for all $k \geqslant 1$. This is closed under the comultiplication, so it is a Hopf algebra, defined over $\mathcal{A}$.

It may be shown that the $\mathcal{A}$-algebra $U_v^{\mathrm{res}}(\mathfrak{g})$ has a basis that is also a $\mathbb{C}(v)$-basis of $U_v(\mathfrak{g})$. Thus

$$U_v(\mathfrak{g}) = \mathbb{C}(v) \otimes_{\mathcal{A}} U_v^{\mathrm{res}}(\mathfrak{g}).$$

Braid group action

There is an action of the braid group on $U_v^{\mathrm{res}}(\mathfrak{g})$. This comes through operators T_i (satisfying the braid relations) in which

$$T_i(E_j) = \begin{cases} -F_jK_j & \text{if } i = j, \\ E_j & \text{if } \alpha_i, \alpha_j \text{ are orthogonal}, \\ -E_iE_j + v^{-1}E_jE_i & \text{if } a_{ij} = -1. \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_j^{-1}E_j & \text{if } i = j, \\ F_j & \text{if } \alpha_i, \alpha_j \text{ are orthogonal}, \\ -F_jF_i + vF_iF_j & \text{if } a_{ij} = -1. \end{cases}$$

$$T_i(K_j) = \begin{cases} K_i^{-1} & \text{if } i = j, \\ K_j & \text{if } \alpha_i, \alpha_j \text{ are orthogonal}, \\ K_iK_j & \text{if } a_{ij} = -1. \end{cases}$$

Certain important elements

Lusztig defines certain elements that are analogous to the Gaussian binomial coefficients

$$\begin{bmatrix} c \\ t \end{bmatrix} = \frac{[c]_v!}{[t]_q![c-t]_q!} = \prod_{s=1}^{t} \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}} .$$

If $t \geqslant 0$ and c are integers let

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

Proposition (Lusztig)

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} \in U_v^{\mathrm{res}}(\mathfrak{g}).$$

Proof

The proof of this fact appears in another paper, Quantum deformations of certain modules over enveloping algebras. By induction assume true for smaller values of t. Use

$$\left[\begin{array}{c} K_i; c \\ t \end{array} \right] - v^t \left[\begin{array}{c} K_i; c+1 \\ t \end{array} \right] = K_i^{-1} v^{-(c+1)} \left[\begin{array}{c} K_i; c \\ t-1 \end{array} \right].$$

to reduce to the case $c = 0$. Then we may use

$$E_i^{(r)} F_i^{(r)} = \sum_{t=0}^{r} F_i^{(r-t)} \left[\begin{array}{c} K_i; 2t-2r \\ t \end{array} \right] E_i^{(r-t)}$$

and another induction.

Quantum groups at roots of unity

For simplicity let ℓ be an odd integer $\geqslant 3$ and let $q = e^{i\pi/\ell}$. Then the specialization $v \to q$ gives a ring homomorphism $\mathcal{A} \to \mathbb{C}$, and we may define

$$U_q(\mathfrak{g}) = \mathbb{C} \otimes_{\mathcal{A}} U_v^{\mathrm{res}}(\mathfrak{g})$$

by extension of scalars.

We denote by K_i, $E_i^{(n)}$, $F_i^{(n)}$ the images of these elements under the natural map from $U_v^{\mathrm{res}}(\mathfrak{g})$. Lusztig proves that K_i^ℓ is central and $K_i^{2\ell} = 1$.

We will restrict ourselves to finite-dimensional modules in which K_i^ℓ acts as the identity. Such modules are called Type 1.

Why we need tilting modules

All modules of

$$U_q(\mathfrak{g}) := \mathbb{C} \otimes_{\mathcal{A}} U_v^{\text{res}}(\mathfrak{g})$$

will be assumed finite-dimensional and Type 1. Such modules form a ribbon category $\mathfrak{Mod}(U_q(\mathfrak{g}))$ that is not semisimple.

Andersen and Paradowski gave a description of a semisimple category derived from $\mathfrak{Mod}(U_q(\mathfrak{g}))$. The objects will be tilting modules, and the morphisms will have to be described. We also recommend Chari and Pressley, A guide to Quantum Groups Chapter 11 and Sawin, Quantum Groups and Modularity for this subject.

Weight space decomposition

Let V be a $U_q(\mathfrak{g})$-module (always Type 1). We will assume that V has a decomposition

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda,$$

where the sum is over the weight lattice Λ, where K_i acts on V_λ by the scalar $q^{\langle \alpha_i^\vee, \lambda \rangle}$.

Unfortunately this is not a good description of V_λ because $q^{\langle \alpha_i^\vee, \lambda \rangle}$ does not determine λ, so we can't just define V_λ to be the eigenspace of the K^i. Instead we define V_λ to be the set of vectors such that

$$K_i v = q^{\langle \alpha_i^\vee, \lambda \rangle} v, \qquad \begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix} v = \begin{bmatrix} \langle \alpha_i^\vee, \lambda \rangle \\ \ell \end{bmatrix} v\,.$$

Weyl modules: the generic case

With this definition,

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda,$$

and

$$E_i^{(k)} V_\mu \subset V_{\mu + k\alpha_i}, \qquad F_i^{(k)} V_\mu \subset V_{\mu - k\alpha_i}.$$

Note that without the Lusztig quantum group, we would not be able to decompose over Λ, only $\Lambda/\ell\Lambda$.

If λ is a dominant weight, there is a unique irreducible Type 1 $U_v(\mathfrak{g})$-module W_v^λ with highest weight λ. Its character is given by the Weyl character formula, that is, its weight spaces have the same dimensions as its classical counterpart.

Weyl modules: root of unity case

We may extend scalars to obtain Weyl modules for quantum groups at roots of unity.

If x is a highest weight vector then $W^\lambda_{\mathcal{A}} = U^{\mathrm{res}}_v(\mathfrak{g}) \cdot x$ is a $\mathcal{A}$-form of W^λ_v in that

$$W^\lambda_v \cong \mathbb{C}(v) \otimes_{\mathcal{A}} W^\lambda_{\mathcal{A}}.$$

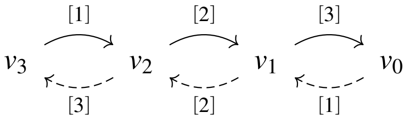
Now let q be an arbitrary nonzero complex number. We make $\mathbb{C}$ into an $\mathcal{A}$-module $\mathbb{C}_q$ via the homomorphism $v \to q$. Then $W^\lambda_q = \mathbb{C}_q \otimes_{\mathcal{A}} W^\lambda_{\mathcal{A}}$ is called a Weyl module for $U_q(\mathbf{g})$.

Example: $\mathfrak{sl}_2$

Let $\alpha = \alpha_1$ be the unique positive root, so $\rho = \frac{1}{2}\alpha$. If $\lambda = n\rho$ is a positive weight we describe W_q^λ. Let v be a highest weight vector and $v_k = F^{(k)}v$. It has weight $\lambda - i\alpha$. We have

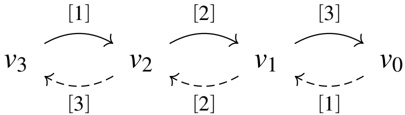
$$Kv_k = q^{n-2k}, \qquad Ev_k = [n-k+1]v_{k-1}, \qquad Fv_k = [k+1]v_{k+1}.$$

For example if $n = 3$, using solid lines to denote E and dashed lines to denote F:



Note that if $\ell = 3$ then $[3] = 0$ so the module is reducible. That is, v_1 and v_2 span an irreducible submodule L, and $M = W_q^{3\rho}/\langle v_1, v_2\rangle$ is the direct sum of two one-dimensional submodules $\mathbb{C}\overline{v}_0$ and $\mathbb{C}\overline{v}_3$.

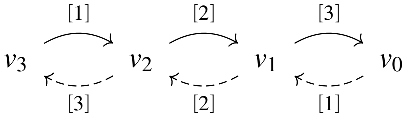
Example (continued)



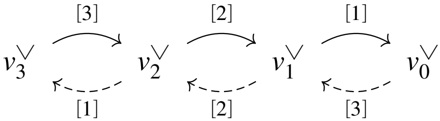
Assuming as before that $l = 3$ so $[3] = 0$ we have remarked that there is a submodule $L \simeq W_q^\rho$ spanned by v_1 and v_2. It might seem that the quotient W_q^λ/L is the direct sum of two one-dimensional submodules, since E and F both annihilate $\bar{v}_0$ and $\bar{v}_3$ in W_q^λ/L.

However $E^{(3)}v_3 = v_0$ and $F^{(3)}v_0 = v_3$, so taking the divided powers into account, the two-dimensional module W^λ/L is irreducible.

Example (continued)



Above: $W_q^{3\rho}$. Below: its contragredient representation.



Assuming as before that $l = 3$ so $[3] = 0$, $W_q^{3\rho}$ is not self-dual. The dual has the two-dimensional module spanned by v_0 and v_3 as a submodule and W_q^{ρ} as a quotient.

Simple modules

Theorem

For every dominant weight λ, there is a unique irreducible finite-dimensional representation $L(\lambda)$ with highest weight λ. It is the unique irreducible quotient of W_q^λ.

Proof: It follows from standard category $\mathcal{O}$ arguments that for every weight λ there is a unique irreducible highest weight module $L(\lambda)$ with highest weight λ. This module is a quotient of any highest weight module with highest weight λ, in particular of W_q^λ, if λ is dominant. Therefore $L(\lambda)$ is finite-dimensional if λ is dominant.

Irreducible Weyl modules

Proposition (Andersen, Polo and Wen)

The module W_q^λ is irreducible if and only if either λ is in the fundamental alcove $\mathcal{F}$ consisting of dominant weights such that

$$\langle \lambda + \rho, \theta \rangle < \ell,$$

or $\lambda + \rho \in \ell\Lambda$.

Here θ is the highest root. There are two types of irreducible Weyl modules. The second type, W_q^λ where $\lambda + \rho$ is divisible by ℓ, are different because their quantum dimensions are zero. The quantum dimension is given by the Weyl dimension formula:

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{[\langle \lambda + \rho, \alpha \rangle]_q}{[\langle \rho, \alpha \rangle]_q}.$$

Tilting modules

We define a Weyl filtration of a module V to be a sequence of submodules

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V,$$

where the successive quotients V_{i+1}/V_i are Weyl modules. If both V and its contragredient $V^\vee$ have Weyl filtrations, V is called a tilting module.

As an example of a module that is not tilting, consider with $\ell = 3$ the four-dimensional module $W_q^{3\rho}$. This is a Weyl module, so trivially it has a Weyl filtration. But its dual, we have seen, is not dual, and a subquotient $L(3\rho)$ occurs in its (unique) composition series

$$0 \to L(3\rho) \to W_q^{3\rho} \to W_q^{\rho} \to 0.$$

It has no Weyl filtration, so $W_q^{3\rho}$ is not a tilting module.

History

Tilting theory has a long history going back to Brenner and Butler (1980).

Tilting modules exist in different categories, for example category $\mathcal{O}$, where a tilting module is defined to be one in which both V and $V^{\vee}$ have filtrations whose quotients are Verma modules.

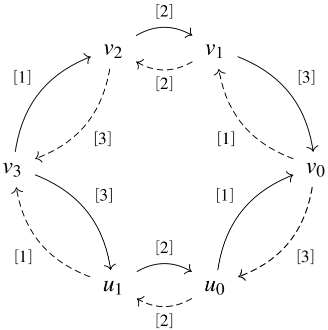
Tilting modules reached quantum groups after prior work in the algebraic category by many, including Auslander and Reites, Ringel and Donkin. In quantum groups they were introduced independently by Andersen and by S. Gelfand and Kazhdan in 1992.

Quotation from Brenner and Butler (1980)

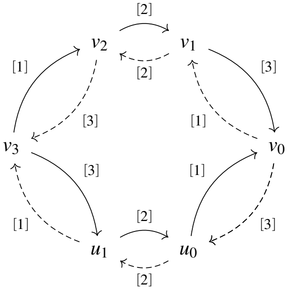
It turns out that there are applications of our functors which make use of the analogous transformations which we like to think of as a change of basis for a fixed root-system—a tilting of the axes relative to the roots which results in a different subset of roots lying in the positive cone .... For this reason, and because the word 'tilt' inflects easily, we call our functors tilting functors or tilts.

Example $\ell = 3$

Since with $\ell = 3$ the $U_q(\mathfrak{sl}_2)$ module $W_q^{3\rho}$ is not tilting, let us describe a unique tilting module $T_q^{3\rho}$ with highest weight 3ρ.



Example $\ell = 3$ (continued)



This module is self-dual, so to verify that it is tilting, v_1 andd v_2 span a submodule $L \cong W_q^{\rho}$, and the quotient $T_q^{3\rho}/W_q^{\rho} \cong W_q^{3\rho}$. Thus it has a Weyl filtration. Since $[4] = [1] = -[2]$:

$$\dim T_q^{3\rho} = \dim W_q^{3\rho} + \dim W_q^{\rho} = [4] + [2] = 0.$$

Properties of tilting modules

Andersen (1992) and Paradowski (1992) proved:

- Any direct sum or tensor product of tilting modules is tilting.
- The dual of a tilting module is tilting.
- Any direct summand in a tilting module is tilting.

The indecomposable tilting modules are in bijection with dominant weights. If λ is a dominant weight, then there is a unique indecomposable tilting module T_q^λ that contains a vector v_λ of weight λ, unique up to scalar multiple. The weight λ is maximal in the sense that $\lambda \succcurlyeq \mu$ for every weight of T_q^λ. The weights of T_q^λ are contained in the convex hull of the W-orbit of λ.

Negligible morphisms and objects

If $f : X \to Y$ are morphisms in a ribbon category we say that f is negligible if $\mathrm{tr}(fg) = 0$ for every $g : Y \to X$. An object X is negligible if 1_X is negligible. Obviously this implies that $\dim(X) = 0$.

With $\ell = 3$, the Weyl module $W_q^{2\rho}$ is an irreducible Weyl module of quantum dimension $[3] = 0$. It is a tilting module. It is negligible.

The tilting module $T_q^{3\rho}$ is negligible even though in its Weyl filtration:

$$0 \to W_q^{\rho} \to T_q^{3\rho} \to W_q^{3\rho} \to 0$$

$\dim(W_q^{\rho})$ and $\dim(W_q^{3\rho})$ are both not negligible.

Negligible tilting modules

Theorem (Andersen 1992)

The irreducible Weyl modules that are not negligible are the W_q^λ such that λ lies in the fundamental alcove $\mathcal{F}$, that is, $\langle \lambda + \rho, \theta \rangle < \ell$.

Now the plan is to consider the category of tilting modules and somehow "quotient out" the negligible objects. One concrete way of doing this is consider the category $\mathcal{C}$ of tilting modules all of whose indecomposable direct summands are T_q^λ where $\lambda \in \mathcal{F}$.

Monoidal structure

This category is semisimple, generated by the modules $W_q^\lambda = T_q^\lambda$, with $\lambda \in \mathcal{F}$. The tensor product rule is to discard negligible tilting modules in the decomposition. That is, if A and B are tilting modules, the $A \otimes B$ is a tilting module, so we may uniquely write $A \otimes B = C \oplus Z$ with C in $\mathcal{C}$ and Z negligible. We discard the Z and $\mathcal{C}$ becomes a monoidal category. It inherits the braiding and twist from the category of all finite-dimensional Type 1 $U_q(\mathfrak{g})$-modules, and it is a ribbon category.

Example

Let us consider the case where $\ell = 5$ and $\mathfrak{g} = \mathfrak{sl}_2$. We need to know some indecomposable tilting modules. First, we have the Weyl modules W^λ with

$$\lambda \in \mathcal{F} = \{\lambda | \langle \alpha + \rho, \theta \rangle < \ell\}.$$

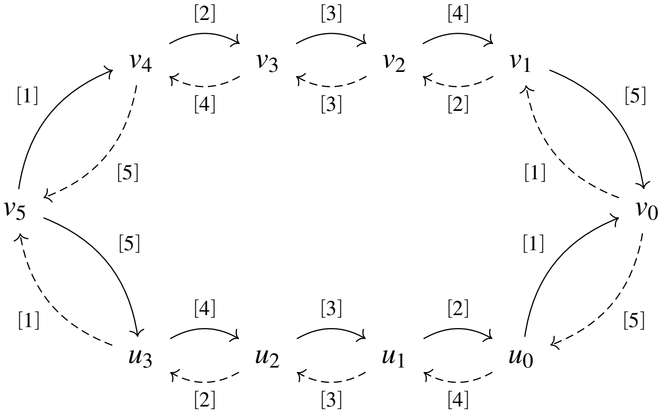
These are the simple objects in the fusion category.

$$W_q^0, \qquad W_q^\rho, \qquad , W_q^{2\rho}, \qquad W_q^{3\rho}.$$

There are other irreducible Weyl modules $W^{k\rho}$ where $k \equiv 4$ mod 5, that is, $\lambda + \rho \in \ell\Lambda$. These are tilting modules, but they are negligible.

Now some other indecomposable tilting modules may be found.

Example $\ell = 5$ (continued)



With $\ell = 5$, the quantum dimension is $[4] + [6] = 0$.

$$0 \to W_q^{3\rho} \to T_q^{5\rho} \to W_q^{5\rho} \to 0$$

Example $\ell = 5$ (continued)

Similarly:

$$0 \to W_q^{2\rho} \to T_q^{6\rho} \to W_q^{6\rho} \to 0$$

$$0 \to W_q^{1\rho} \to T_q^{7\rho} \to W_q^{7\rho} \to 0$$

$$0 \to W_q^{0} \to T_q^{8\rho} \to W_q^{8\rho} \to 0$$

These tilting modules all have quantum dimension 0, and ordinary dimension 10. After which we have another irreducible Weyl module $T_q^{9\rho} = W_q^{9\rho}$.

Example $\ell = 5$ (continued)

If $\lambda \in \{0, \rho, 2\rho, 3\rho\}$ let x_λ denote the class of the irreducible Weyl module W_0^λ, which is also a non-negligible indecomposable tilting module. We wish to compute some products in two ways, using the Kac-Walton formula from the last lecture, and using tilting modules.

You can do compute these products in Sage as follows:

```
sage: A13=FusionRing("A1",3)
sage: A13.fusion_labels(['x0','x1','x2','x3'])
sage: x1*x3
x2
```

Racah Speiser method

We review the analogous computation for a Lie group G. The method Racah-Speiser method decomposes $\pi_\lambda\pi_\nu$ into irreducibles. We need the character of one factor, say

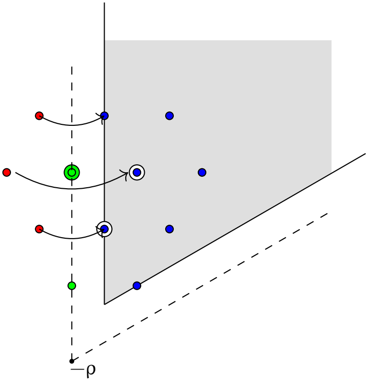
$$\chi_\lambda(\mathbf{z}) = \sum K(\lambda, \mu)\mathbf{z}^\mu.$$

Then provided all $\mu + \nu$ are in the positive Weyl chamber $\mathcal{C}$:

$$\chi_\lambda\chi_\nu = \sum_\mu K(\lambda, \mu)\chi_{\mu+\nu}(\mathbf{z}).$$

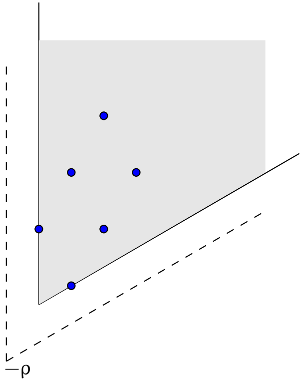
If some of the $\mu + \nu$ are not dominant, we may still try to move them into the positive Weyl chamber using the dot action $w \cdot \mu = w(\mu + \rho) - \rho$. We have to take into account a sign, and there will be cancellations.

Review: the Racah-Speiser method ($SL(3,\mathbb{C})$)



The green weights lie on a hyperplane through $-\rho$ so they are discarded. Three other red weights to the left of the hyperplane are reflected and subtracted.

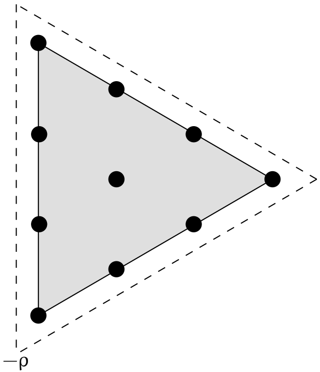
Review: after cancellation



$$\chi_{(3,1,0)}\chi_{(3,3,0)} = \chi_{(4,3,3)}+\chi_{(4,4,2)}+\chi_{(5,3,2)}+\chi_{(5,4,1)}+\chi_{(6,3,1)}+\chi_{(6,4,0)}$$

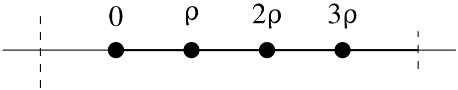
The Kac Walton Formula ($\mathrm{SL}(3)$)

The Kac Walton formula proceeds similarly but we use the fundamental alcove:



The Kac Walton Formula ($\ell = 5$)

For SL_2 the fundamental alcove of level $\ell - h^\vee = \ell - 2$ is a line segment:



Now we are allowed to reflect on either line dashed lines, at $-\rho$ and 4ρ. When we have reflected $\mu + \nu$ if it lies on one of the dashed lines it is zero and we may discard it. Otherwise it reflects in and we may try to make cancellations.

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Example ($\ell = 5$) continued

Let

$$\mathbf{z} = \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}.$$

We will try to compute $W_q^{\rho} \otimes W_q^{3\rho}$ in the fusion category. We write

$$\chi_\rho(\mathbf{z}) = z + z^{-1} = \mathbf{z}^{(1/2,-1/2)} + \mathbf{z}^{(-1/2,1/2)}.$$

If we are considering these as characters of $SL(2,\mathbb{C})$ then the Racah-Speiser method gives

$$\chi_\rho \chi_{3\rho} = \chi_{4\rho} + \chi_{2\rho}$$

For the Kac-Walton formula there is the complication that 4ρ lies outside the fundamental alcove; in fact, it lies on the line of reflection, so its contribution is zero. Thus in the fusion ring:

$$\chi_\rho * \chi_{3\rho} = \chi_{2\rho}.$$

Using Tilting Modules

Now let us do the same computation using tilting modules. We know that $W_q^{\rho} \otimes W_q^{3\rho}$ is a tilting module, so it decomposes into a direct sum of indecomposable tilting modules. But the characters of these are linearly independent and so it is easy to compute the decomposition.

$$W_q^{\rho} \otimes W_q^{3\rho} = W_q^{2\rho} \oplus W_q^{4\rho}.$$

We remember that $W^{4\rho}$ is negligible and so it is to be discarded. We obtain the same formula:

$$W_q^{\rho} \otimes W_q^{3\rho} \sim W_q^{2\rho}$$

meaning that the two differ by a negligible tilting module.

Example ($\ell = 5$) continued

Similarly if we compute $\chi_{2\rho} \otimes \chi 3\rho$ we write

$$\chi_{2\rho}(\mathbf{z}) = z^2 + 1 + z^{-2} = \mathbf{z}^{2\rho} + 1 + \mathbf{z}^{-2\rho}$$

So we have three contributions corresponding to weights

$$\mu + \nu \in \{\rho, 3\rho, 5\rho\}.$$

This time when we reflect 5ρ we get a term that cancels the 3 row term and so the Kac-Walton formula predicts

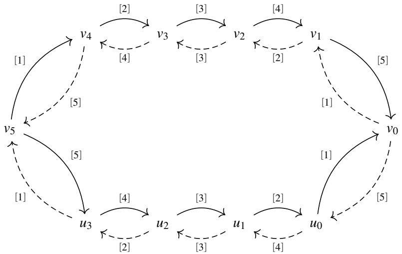
$$\chi_{2\rho} * \chi_{3\rho} = \chi_{\rho}.$$

We wish to understand this in terms of tilting modules. To decompose $W_q^{2\rho} \otimes W_q^{3\rho}$ it is enough to identify a tilting module with the same character.

Tilting ($\ell = 5$)

Remember that the negligible tilting module $T_q^{5\rho}$ has character

$$z^{5/2} + 2z^{3/2} + 2z^{1/2} + 2z^{-1/2} + 2z^{-3/2} + z^{5/2}.$$



More Tilting

Because $W^{2\rho} \otimes W^{3\rho}$ and $T^{5\rho} + W^{\rho}$ have the same character

$$z^{5/2} + 2z^{3/2} + 2z^{1/2} + 2z^{-1/2} + 2z^{-3/2} + z^{5/2},$$

and both are tilting modules, they are equal. Of course in the fusion ring, we discard the negligible tilting module $T^{5\rho}$ and obtain the same answer as with the Kac-Walton fomrula. Let us denote by x_i the class of $W^{i\rho}$ in the fusion ring, for $i = 0, 1, 2, 3$. We have computed in two different ways

$$x_1 x_3 = x_2, \qquad x_2 x_3 = x_1.$$

Origin in Physics

Chari and Pressley cite the following to show how this kind of fusion multiplication for quantum groups was understood by physicists.

- Pasquier and Saleur (1990). Common structures between finite systems and conformal field theories through quantum groups, Nucl. Phys. B. 330, 523-56.
- Fröhlich and Kerber (1993), Quantum Groups, Quantum Categories and Quantum Field Theory, Springer LNM 1542.