Lecture 15

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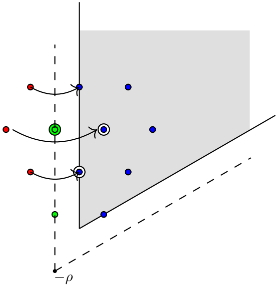


Motivation

Let $\mathfrak{g}$ be a semisimple Lie algebra and let k be an integer that we will sometimes call the level. There is associated with these data a modular tensor category $\mathcal{C}(\mathfrak{g}, k)$ which we will describe without trying to give complete proofs.

The category can be constructed in at least two ways,

- Motivated by CFT, from integrable representations of level k of the affine Lie algebra $\widehat{g}$;
- From the representation theory of $U_q(\mathfrak{g})$ where $q = e^{2\pi i/\varkappa}$

Next week we will discuss the second construction. Meanwhile we will describe (without proof) the simple objects, ribbon element and monoidal operation algorithmically (the Kac-Walton formula).

Motivation (continued)

However the Kac-Walton formula is very similar to a method of decomposing tensor products of irreducible representations of a Lie group, so we review that first.

This method was periodically discovered by different people. In my Lie groups book it is called the Brauer-Klimyk algorithm but in physics literature it is often called the Racah-Speiser algorithm. We will begin with that, then show how to modify it to compute fusion products in the Kac-Walton formula.

Weights

Let G be a complex reductive Lie algebra such as $\mathrm{GL}(r)$ or $\mathrm{SL}(r)$, and let T be a maximal torus. Thus if $G = \mathrm{GL}(r)$ or $\mathrm{SL}(r)$ we may take T to be the diagonal subgroup. The group $X^*(T)$ of rational characters of T is a torus: for $\mathrm{GL}(r)$ we may identify $X^*(T) = \mathbb{Z}^r$, with $\lambda = (\lambda_1, \cdots, \lambda_r)$ being the character

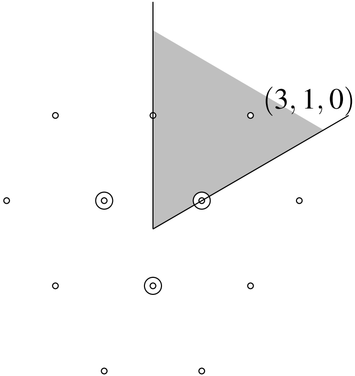
$$\mathbf{z} = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix} \mapsto \mathbf{z}^\lambda = \prod_{i=1}^{r} z_i^{\lambda_i}.$$

For $\mathrm{SL}(r)$ the group $X^*(T)$ is the quotient $\mathbb{Z}^r/\mathbb{Z}$ where $\mathbb{Z}$ is embedded diagonally in $\mathbb{Z}^r$.

Let π be an irreducible representation of G. Since T is abelian, restricting G to T it decomposes into one-dimensional representations, that is weights, with certain multiplicities.

Example

Let $G = GL(3)$. Then G has a 15-dimensional irreducible representation whose weights are as in the following table.



The doubly circled weights have multiplicity 2.

Roots

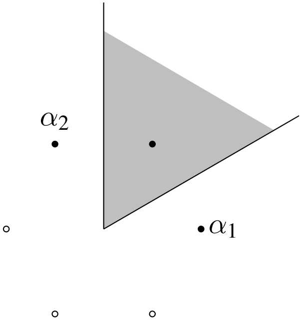
The group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation. The root system Φ consists of the nonzero weights α in the adjoint representation. The roots are partitioned into positive and negative ones. For $\mathrm{GL}(3)$ the positive roots are:

$$(1,-1,0), \qquad (0,1,-1), \qquad (1,0,-1).$$

The Weyl vector ρ which is half the sum of the positive roots plays a special role. For $\mathrm{GL}(3)$ it is $(1,0,-1)$. The simple roots are the positive roots that can't be decomposed into other positive roots. For $\mathrm{GL}(3)$ the simple roots are

$$\alpha_1 = (1,-1,0), \qquad (0,1,-1).$$

The $\mathrm{GL}(3)$ root system



Here is the $\mathrm{GL}(3)$ root system. Positive roots: $\bullet$. Negative roots: $\circ$. The simple roots are α_1 and α_2. The shaded region is the positive Weyl chamber

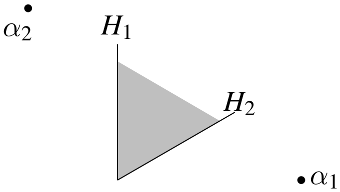
$$\mathcal{C} = \{\lambda | \langle \lambda, \alpha_i \rangle \geqslant 0 \text{ for simple roots } \alpha_i\}$$

Dominant weights

The positive Weyl chamber:

$$\mathcal{C} = \{\lambda | \langle \lambda, \alpha_i \rangle \geqslant 0 \text{ for simple roots } \alpha_i\}$$

We are embedding the weight lattice in an ambient real vector space $\mathcal{V}$ and $\mathcal{C}$ is a subset of $\mathcal{V}$. Here $\langle\ ,\ \rangle$ is a W-invariant inner product on Λ, normalized so $\langle \alpha, \alpha \rangle = 2$ if α is a short root. The Weyl group W is generated by the reflections s_i in the hyperplanes H_i perpendicular to the α_i, which are the walls of $\mathcal{C}$.



The highest weight

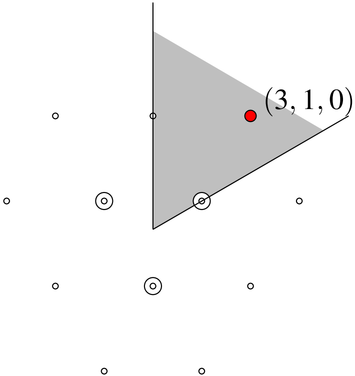
We define a partial order on Λ, in which $\lambda \succcurlyeq \mu$ if $\lambda - \mu$ is a sum with nonnegative coefficients of the simple roots.

If π is an irreducible representation, there is a unique weight λ of π that is maximal or "highest" with respect to this partial order. The weight λ is dominant. We denote $\pi = \pi_\lambda$.

Theorem (Weyl)

The association $\lambda \leftrightarrow \pi_\lambda$ is a bijection between the dominant weights and the irreducible representations of G.

The highest weight



The highest weight vector $(3, 1, 0)$ of this 15-dimensional representation is drawn in red.

The Weyl Character formula

Let $\mathbf{z} \in T$. Let $\chi_\lambda(\mathbf{z}) = \mathrm{tr}\ \pi_\lambda(\mathbf{z})$ be the character of π_λ.

Theorem (Weyl)

We have

$$\chi_\lambda(\mathbf{z}) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)}$$

where the Weyl denominator is

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{\rho} = \mathbf{z}^{\rho} \prod_{\alpha \in \Phi^+} (1 - \mathbf{z}^{-\alpha}).$$

We will omit the proof, but we point out an important interpretation.

BGG Category $\mathcal{O}$

Let $B = TU$ be the positive Borel subgroup of G, where T is the given maximal torus, and U is the maximal unipotent of U. Its Lie algebra $\mathfrak{u}$ is generated by the positive roots.

Let λ be a weight of G, not necessarily dominant. We call a possibly infinite-dimensional Π representation of G a highest weight representation with highest weight λ if it is generated by a vector v_λ for some weight λ that is fixed by U, and such that

$$\Pi(\mathbf{z})v_\lambda = \mathbf{z}^\lambda v_\lambda$$

for $\mathbf{z} \in T$. There is a nice abelian category $\mathcal{O}$ that contains all highest weight representations, due to Bernstein, Gelfand and Gelfand.

Interpretation of the Weyl Character formula

If λ is a weight, not necessarily dominant, there is a universal highest weight module $M(\lambda)$ such that every highest weight representation with highest weight λ is a quotient of $M(\lambda)$. It is constructed by inducing λ from B to G. Its character can be computed using the triangular decomposition (Lecture 10):

$$U(\mathfrak{g}) = U(\mathfrak{u}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{u}_-).$$

The character of $M(\lambda)$ is

$$\mathbf{z}^\lambda \prod_{\alpha\in\Phi^+} (1 - \mathbf{z}^{-\alpha})^{-1}.$$

Interpretation of the Weyl Character formula (continued)

Now if λ is dominant, write the Weyl character formula

$$\chi_\lambda(\mathbf{z}) = \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)-\rho} \prod_{\alpha \in \Phi^+} (1 - \mathbf{z}^{-\alpha})^{-1}.$$

So in the Grothendieck group of category $\mathcal{O}$ this expresses π_λ as an alternating sum of Verma modules.

The Brauer-Klimyk or Racah-Speiser algorithm

Let χ_λ be the character of π_λ. We expand it in terms of weight μ, each with multiplicity $K(\lambda, \mu)$:

$$\chi_\lambda(\mathbf{z}) = \sum_\mu K(\lambda, \mu)\mathbf{z}^\mu.$$

for $\mathbf{z} \in T$. Then we can try to decompose $\chi_\lambda \chi_\nu$, which is the character of $\pi_\lambda \otimes \pi_\nu$.

The simplest case is where $\nu + \mu$ is dominant for all weights μ of λ. Then

$$\chi_\lambda * \chi_\nu = \sum_\mu K(\lambda, \mu)\chi_{\mu+\nu},$$

$$\pi_\lambda * \pi_\nu = \bigoplus_\mu K(\lambda, \mu)\pi_{\mu+\nu},$$

Proof

We substitute the weight expansion for χ_λ and the Weyl character formula for χ_ν:

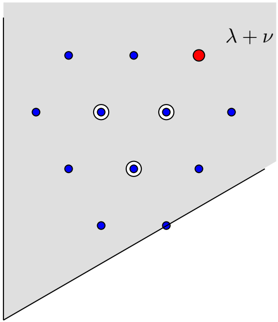
$$\chi_\lambda \chi_\nu = \Delta^{-1} \sum_\mu K(\lambda,\mu)\,\mathbf{z}^\mu \sum_w (-1)^{\ell(w)}\,\mathbf{z}^{w(\nu+\rho)}.$$

Interchange the order of summation, so that the sum over ν is the inner sum, and make the variable change $\nu \longrightarrow w(\nu)$. Since $K(\lambda,\mu) = K(\lambda, w\mu)$, we get

$$\Delta^{-1} \sum_w \sum_\nu K(\lambda,\mu)\,(-1)^{\ell(w)}\,\mathbf{z}^{w(\nu+\mu+\rho)}.$$

Now we may interchange the order of summation again and apply the Weyl character formula to obtain $\sum K(\lambda,\mu)\chi_{\nu+\mu}$.

Example



As long as $\mu + \nu$ is dominant for every weight μ of λ, we get the decomposition of $\pi_\lambda \otimes \pi_\mu$ as

$$\sum_{\mu} K(\lambda, \mu) \pi_{\nu+\mu}.$$

What if $\mu + \nu$ is not dominant?

The proof of the Brauer-Klimyk-Steinberg-Racah-Speiser formula goes through but we have to reinterpret

$$\Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{\mu+\nu+\rho}$$

Proposition

Let λ be given, not assumed dominant. Write $\lambda = w\xi$ where $w \in W$ and ξ is dominant. Let $\eta = \xi + w^{-1}\rho - \rho$. Then

$$\Delta^{-1} \sum_{w} (-1)^{w} \mathbf{z}^{w(\lambda+\rho)} = \begin{cases} (-1)^{\ell(w)} \chi_\eta & \text{if } \eta \text{ is dominant} \\ 0 & \text{otherwise.} \end{cases}$$

This is easily proved by making a change of variables in the Weyl character formula. It may be seen that η is dominant unless ξ lies on a wall of the positive Weyl chamber.

The algorithm, continued

We may describe the algorithm as follows. There is a modified action of the Weyl group known as the “dot” action in which

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

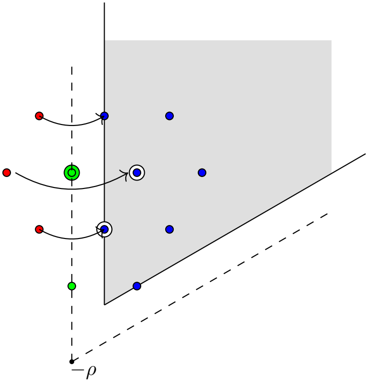
The point fixed by this action is $-\rho$. Let $\mathcal{H}$ be the set of hyperplanes perpendicular to the roots. It includes the walls of $\mathcal{C}$. If $\mu + \nu$ lies on one of these hyperplanes, it contributes zero to the sum:

$$\sum_{\mu} K(\lambda, \mu) \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{\mu+\nu+\rho}$$

But if $\mu + \nu$ does not lie on one of these hyperplanes, then $\eta = w \cdot (\mu + \nu)$ is dominant for some $w \in W$ and we get a term

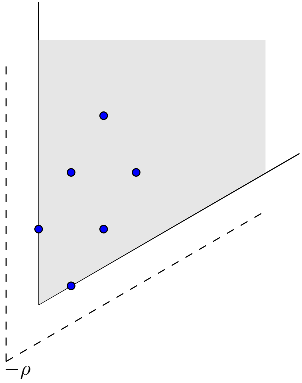
$$\pm K(\lambda, \mu) \chi_{\eta}.$$

Example



The green weights lie on a hyperplane through $-\rho$ so they are discarded. Three other red weights to the left of the hyperplane are reflected and subtracted.

After cancellation



$$\chi_{(3,1,0)}\chi_{(3,3,0)} = \chi_{(4,3,3)} + \chi_{(4,4,2)} + \chi_{(5,3,2)} + \chi_{(5,4,1)} + \chi_{(6,3,1)} + \chi_{(6,4,0)}$$

The affine Weyl group (I)

The Weyl group W is generated by the reflections in the hyperplanes H_α perpendicular to the roots, or just reflections s_i in the hyperplanes H_{α_i} perpendicular to the simple roots. It is a Coxeter group.

The set of hyperplanes H_α is stable under W. The positive Weyl chamber $\mathcal{C}$ is a connected component of the complement of the H_α, and it is a fundamental domain for W.

These facts have analogs for the affine Weyl group which is an infinite Coxeter group that we will now describe.

The affine Weyl group (II)

Fix a positive integer k, to be called the level.

The root system Φ has a highest root θ. For $\mathrm{GL}(r)$ it is $(1, 0, \cdots, 0, -1)$. Let $\alpha_0 = -\theta$. The roots $\{\alpha_0, \alpha_1, \cdots, \alpha_r\}$ have the property that $\langle \alpha_i, \alpha_j \rangle \leqslant 0$ if $i \neq j$.

Consider the family of affine linear maps $r_{\alpha,m}$ on $\mathcal{V}$ defined by $r_{\alpha,m}(v) = \langle \alpha, v \rangle + m$. Let $H_{\alpha,m} = \{v | r_{\alpha,m} = 0\}$.

Definition

The group generated by the set of reflections in the hyperplanes $H_{\alpha,m}$ is the affine Weyl group of level k.

The fundamental alcove

The affine Weyl group ${W_{\text{aff}}}^k$ is a Coxeter group. Recall this means that W has generators $s_0, s_1, \cdots, s_r$ with relations $s_i^2 = 1$ and braid relations of the type

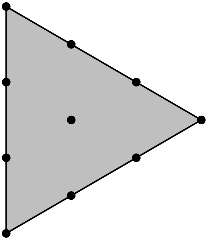
$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

where the number of factors on both sides is the order of $s_i s_j$.

A fundamental domain for ${W_{\text{aff}}}^k$ consists of the fundamental alcove (of level k):

$$\mathcal{F}_k = \langle \alpha_i, v \rangle \geqslant 0, \quad (i = 1, \cdots, r), \qquad \langle \alpha_0, v \rangle \geqslant -k.$$

Example



Here is the $\mathrm{SL}(3)$ fundamental alcove of level 3, and the weights that are contained in it.

There exists modular tensor category $\mathcal{C}(\mathfrak{g}, k)$ whose simple objects are in bijection with the weights λ inside $\mathcal{F}_k$. The monoidal operation, called fusion is given by the Kac-Walton algorithm.

The Kac-Walton algorithm

The Kac-Walton algorithm is the affine analog of the Racah-Speiser algorithm. We compute the decomposition

$$[\lambda] * [\mu] = \sum_{\nu} N_{\lambda,\mu}^{\nu}[\nu].$$

We begin with the character

$$\chi_\lambda = \sum_{\mu} K(\lambda,\mu)\mathbf{z}^{\mu}$$

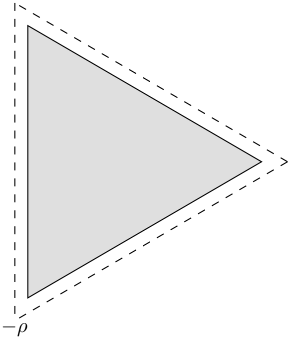
which is known from the Weyl character formula. If the weights $\mu+\nu$ are all in the fundamental alcove, then the fusion

$$[\lambda] * [\mu] = \sum K(\lambda,\mu)[\mu+\nu].$$

So in this case the fusion product coincides with the tensor product.

The Kac-Walton algorithm

If some $\mu + \nu$ are not in $\mathcal{F}_k$ imitate the Speiser-Racah method using reflections in hyperplanes parallel to the walls of the fundamental alcove, offset by 1.



This describes the monoidal composition in $\mathcal{C}(\mathfrak{g}, k)$.

The dual Coxeter number

The dual Coxeter number $h^\vee$ appears in many formulas. This can be defined as $h^\vee = \langle \rho, \theta^\vee \rangle$ where $\theta^\vee$ is the longest coroot.

Cartan type	$h^\vee$
A_r	$r+1$
B_r	$2r-1$
C_r	$r+1$
D_r	$2r-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

The modular category

This modular tensor category arises two different ways.

- Motivated by CFT, from integrable representations of level k of the affine Lie algebra $\widehat{\mathfrak{g}}$;
- From the representation theory of $U_q(\mathfrak{g})$ where $q = e^{\pi i/\ell}$

Here $\ell = k + h^\vee$ in the simply-laced case, and can be made explicit in every case.

In the first construction, the monoidal structure on these representations was found by physicists and made rigorous by Kazhdan and Lusztig. In both cases one actually obtains a ribbon category that is not semisimple but can be semisimplified by a simple procedure.

The twist and quantum dimension

The twist is given by the formula

$$\theta_\lambda = q^{\langle \lambda, \lambda + 2\rho \rangle} .$$

The quantum dimension is

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{[\langle \lambda + \rho, \alpha \rangle]_q}{[\langle \rho, \alpha \rangle]_q}.$$

Here as usual $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. This can be compared with the usual Weyl dimension formula, in the Lie group case:

$$\dim(\pi_\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

The S-matrix

The S-matrix can be computed from the formula (Lecture 13):

$$\widetilde{s}_{\lambda,\mu} = \sum_k \frac{N^{\nu}_{\lambda*,\mu}\theta_\nu d_\nu}{\theta_\lambda\theta_\mu}.$$

There is another formula:

$$\widetilde{s}_{\lambda\mu} = \frac{\sum_w (-1)^{\ell(w)} q^{2\langle\lambda+\rho, w(\mu+\rho)\rangle}}{\sum_w (-1)^{\ell(w)} q^{2\langle\rho, w(\rho)\rangle}}$$

We hope to cover further interesting formulas (such as the Verlinde formula) that are valid in a general MTC in our last few lectures.