Lecture 14

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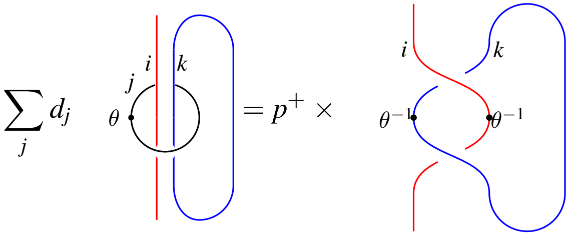
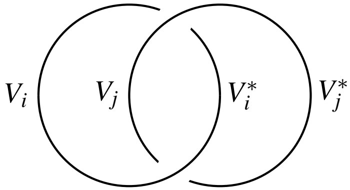
May 29, 2019



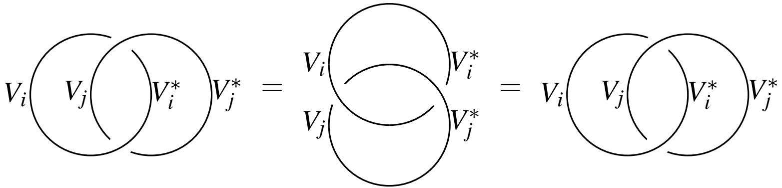
Review: Modular tensor categories

We recall the definition of a modular tensor category. This is a semisimple ribbon category with a finite number of simple objects V_i ($i \in I$). We assume that the ground ring $K = \mathrm{End}(I)$ is a field, where I is the unit object. This implies that I is itself a simple object. Moreover, we assume that $\mathrm{End}(V_i) = \mathrm{End}(K)$ for all simple objects V_i. Finally, we assume that the S-matrix $(\widetilde{s}_{ij})$ is invertible, where $\widetilde{s}_{ij}$ is the scalar (i.e. endomorphism of I) defined by the Hopf link:



Review: Symmetry of the S-matrix

We now observe that $\widetilde{s}_{ij} = \widetilde{s}_{ji}$. To see this use the crossing identities to move the link around:



The Fusion Ring

The monoidal structure gives the Grothendieck group of the category a multiplication that makes it into a ring. This ring is a free abelian group on the isomorphism classes of simple modules. Thus if $i \in I$ let $[i] = [V_i]$ be the class of a representative simple module V_i. We can decompose $V_i \otimes V_j$ into simple modules V_k with structure constants N_{ij}^k. Thus

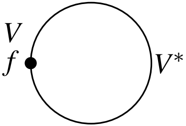
$$[i][j] = \sum_k N_{ij}^k [k], \qquad V_i \otimes V_j = \bigoplus_k N_{ij}^k V_k.$$

The N_{ij}^k are nonnegative integers. Take the quantum dimension, which is multiplicative by Lecture 4:

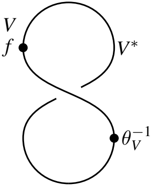
$$d_i d_j = \sum_k N_{ij}^k d_k, \qquad d_i = \dim(V_i).$$

Review: Alternative trace principle

Let $f : V \to V$ be a morphism in a ribbon category. We can compute the trace in two different ways:



To see this, replace ev^V by $\mathrm{ev}_V(1 \otimes \theta_V^{-1})c_{V,V^*}$ in the first figure, and in the second, replace coev^V by $c_{V,V^*}(\theta_V \otimes 1)\,\mathrm{coev}_V$.

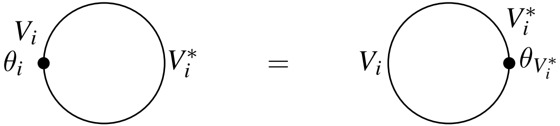


Review: Consequences of the alternative trace principle

If $f : V_i \to V_i$ is any morphism, since V_i is simple, $\mathrm{End}(V_i) = K$. Thus f is just a scalar. In particular, θ_{V_i} is a scalar which we will denote θ_i.

We will denote the quantum dimension of V_i as d_i. Applying the alternative trace principle to $1_V : V \to V$ gives $d_i = d_{i*}$, where $i*$ is the index such that $V_{i*} = V_i^*$.

Remembering that $\theta_{V_i^*} = \theta_{V_i}^*$, the alternative trace principle implies that $\theta_i = \theta_{i*}$.

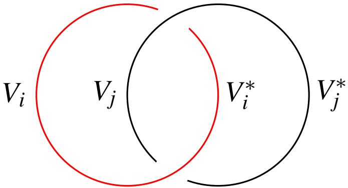


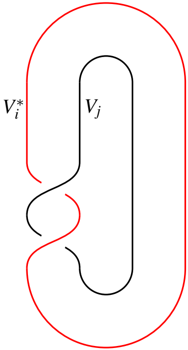
Review: Alternative definition of $\widetilde{s}_{ij}$

As in Lecture 13

$$\widetilde{s}_{ij} = \mathrm{tr}\big(c^{-1}_{V_{i*},V_j} c^{-1}_{V_j,V_{*i}}\big).$$

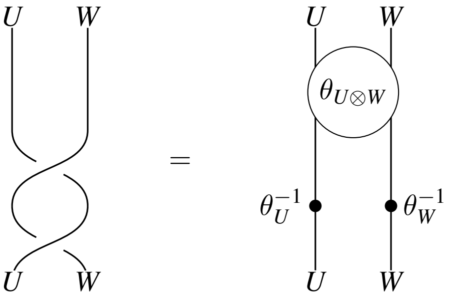
Indeed use the principle of the alternative trace to flip one circle:





Reminder: the ribbon axiom

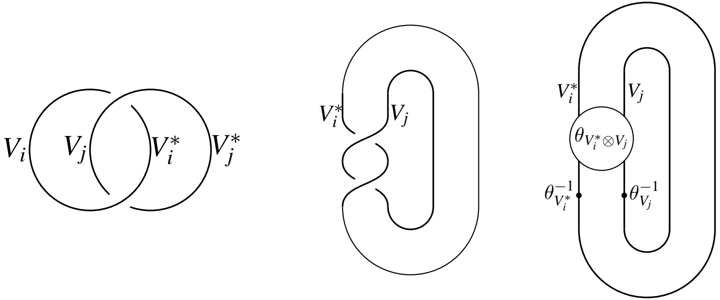
$$c_{U,W}^{-1} \circ c_{W,U}^{-1} = (\theta_U^{-1} \otimes \theta_W^{-1}) \circ \theta_{U\otimes W} = \theta_{U\otimes W} \circ (\theta_U^{-1} \otimes \theta_W^{-1})$$



Review: Another formula for the S-matrix

Proposition

$$\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1}\,\mathrm{tr}(\theta_{V_{i*}\otimes V_j}) = \theta_i^{-1}\theta_j^{-1}\sum_k N_{i*,j}^k \theta_k d_k.$$



For the second formula use $V_i^* \otimes V_j = \sum_k N_{i*,j}^k V_k$ and $\theta_{i*} = \theta_i$.

Review: The group $\mathrm{SL}(2,\mathbb{Z})$

The group $\mathrm{SL}(2,\mathbb{Z})$ is the mapping class group of the torus, i.e. the group of group of homeomorphisms modulo isotropy. The larger group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathcal{H}=\{z=x+iy\in\mathbb{C}|y>0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

The subgroup $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ acts discontinuously.

A modular form is a function f that satisfies

$$f(z)=(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ or a subgroup of finite index.

Review: Two important elements

Note that $-I \in \mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathcal{H}$, so the action is not faithful. Let

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

These generate $\mathrm{SL}(2,\mathbb{Z})$. Since $S^2 = -I$, S has order 4 as an element of the group but order 2 in its action on $\mathcal{H}$.

$T : z \to z + 1$ is the translation by 1. If a holomorphic function f is invariant under $\mathrm{SL}(2,\mathbb{Z})$ it is invariant under T and so it has a Fourier expansion:

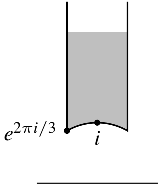
$$f(z) = \sum a_n q^n, \qquad q = e^{2\pi i z}.$$

Although T has infinite order, ST has order 3 in its action on $\mathcal{H}$ or 6 as an element of $\mathrm{SL}(2,\mathbb{Z})$.

$$S^2 = -I, \qquad (ST)^3 = -I.$$

Review: the fundamental domain

Here is the well-known fundamental domain for $SL(2,\mathbb{Z})$:



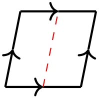
We have marked the fixed points i and $e^{2\pi i/3}$ of S and ST.

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

$$S : z \to -\frac{1}{z}, \qquad T : z \to z + 1.$$

The mapping class group of a torus

The group $\mathrm{SL}(2,\mathbb{Z})$ has another significance: it is the mapping class group of a torus. A torus M may be obtained by gluing opposite edges of a parallelogram:



A homeomorphisms may be obtained by obtained by cutting the torus apart along the dotted line twisting and regluing. Another complementary twist will serve to generate the mapping class group. These Dehn twists correspond to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Relations in a MTC (Following Bakalov and Kirillov Chapter 3)

Let

$$p^+ = \theta_i d_i^2, \qquad p^- = \theta_i^{-1} d_i^2$$

and define matrices $t = (\delta_{ij}\theta_i)$, $c = (\delta_{i,j*})$. We will prove:

Proposition

$$ct = tc, \qquad c\widetilde{s} = \widetilde{s}c, \qquad c^2 = 1,$$
$$(\widetilde{s}t)^3 = p^+\widetilde{s}^2, \qquad (\widetilde{s}t^{-1})^3 = p^-\widetilde{s}^2 c,$$
$$\widetilde{s}^2 = p^+p^-c.$$

The relation $ct = tc$ follows from $\theta_i = \theta_{i*}$, a consequence of $\theta_V^* = \theta_{V^*}$ and the alternative trace principle. The identity $c^2 = 1$ is obvious and $c\widetilde{s} = \widetilde{s}c$ is not hard.

Significance

From Lecture 13, $\mathrm{SL}(2,\mathbb{Z})$ has generators S and T

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

with relations

$$S^2 = (ST)^3 = -I, \qquad (-I)S = S(-I), \qquad (-I)T = T(-I).$$

Compare this with $s = c\tilde{s}$, where $c = (\delta_{i,j*})$ is the "charge conjugation matrix" and $t = (\delta_{i,j}\theta_i)$ is the matrix of twists. The matrix c is central and

$$s^2 = \tilde{s}^2 = \text{constant} \times c, \qquad (st)^3 = \text{constant} \times c,$$

and c is central. So we have a projective representation $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}_{|I|}(K)$ such that

$$\pi(S) = s, \qquad \pi(T) = t, \qquad \pi(-I) = c.$$

Significance (continued)

- Turaev showed that there is an action of the mapping class group of an orientable surface on the fusion category. The mapping class group of a torus is $\mathrm{SL}(2,\mathbb{Z})$.
- In WZW conformal field theories, MTC categories are constructed in which the fields are modular forms. Hence there is an action of $\mathrm{SL}(2,\mathbb{Z})$. These categories are associated with integrable representations of affine Lie algebras, whose characters are modular forms by Kac and Peterson (1984).

In this class we will be concerned with MTC coming from representations of quantum groups. The two constructions are related in papers of Finkelberg, Andersen and Paradowski, and Sawin.

Significance (continued)

Before we launch into the construction of the $\mathrm{SL}(2, \mathbb{Z})$ action, we explain further this quote from Bakalov and Kirillov.

The appearance of the modular group in tensor categories may seem mysterious; however there is a simple geometrical explanation, based on the fact that to each modular tensor category one can associate a $2+1$ dimensional TQFT. This shows that in fact we have an action of the mapping class group of any oriented 2-dimensional surface on the appropriate objects in MTC. This is the key idea in [Turaev's book].

To explain this further, both Turaev's book and the book of Bakalov and Kirillov are largely concerned with 3 equivalent things: Modular Tensor Categories, 3D TQFT and 2D Modular Functors.

Modular Functors

The notion of a modular functor is due to Moore and Seiberg and Graeme Segal. Consider the category of 2-dimensional orientable manifolds with morphisms homotopy classes of homeomorphisms. This is a monoidal category, the composition law being disjoint union. To oversimplify, a 2D modular functor is a monoidal functor τ from this category to the category of finite-dimensional vector spaces. (See Bakalov and Kirillov Chapter 5.) If M is an object in the category then the mapping class group which is the group of automorphisms of M in this category acts on $\tau(M)$.

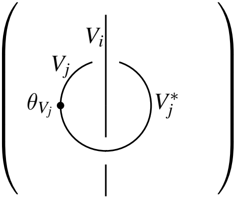
Following Bakalov and Kirillov

Recall that

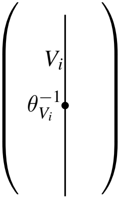
$$p^{+} = \theta_i d_i^2, \qquad p^{-} = \theta_i^{-1} d_i^2 .$$

We will show that

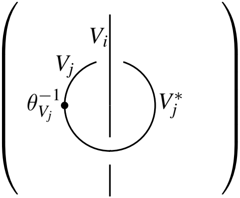
$\sum_j d_j$



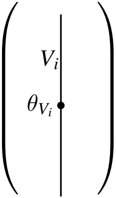
$= p^{+}$



$\sum_j d_j$



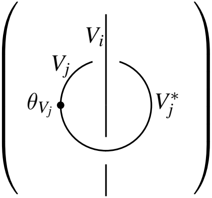
$= p^{-}$



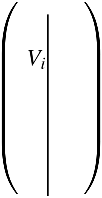
Proof

We prove the first one only. Multiply both sides by θ_i. We need

$$\sum_j d_j \theta_i$$

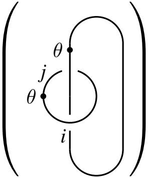


$$= p^+$$



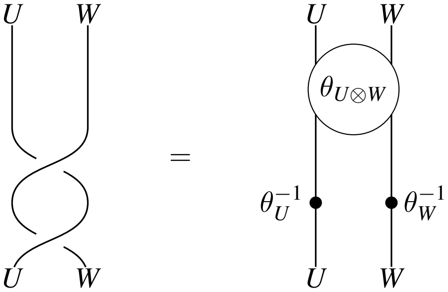
Since V_i is simple, $\mathrm{Hom}(V_i, V_i)$ is one-dimensional so the two sides are proportional. Trace of the left side:

$$\sum_j d_j$$



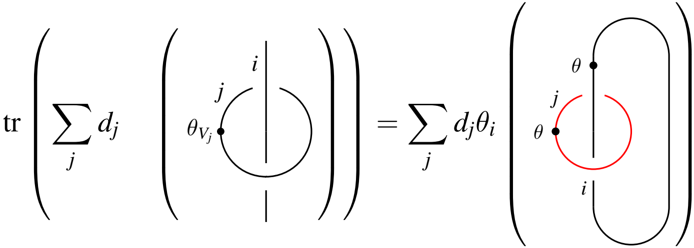
Reminder: the ribbon axiom

$$c_{U,W}^{-1} \circ c_{W,U}^{-1} = (\theta_U^{-1} \otimes \theta_W^{-1}) \circ \theta_{U\otimes W} = \theta_{U\otimes W} \circ (\theta_U^{-1} \otimes \theta_W^{-1})$$



Proof, continued

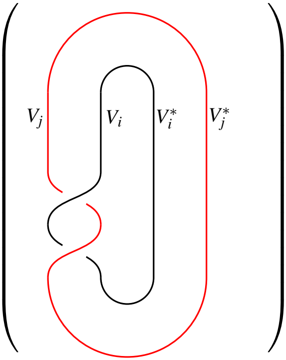
We are computing the trace:



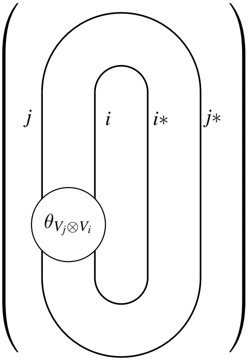
Proof, continued

Using the principle of the alternative trace to flip the red circle, then the ribbon axiom:

$\sum_j d_j$



$= \sum_j d_j$



Proof, continued

Now we substitute

$$V_j \otimes V_i = \sum_k N_{ji}^k V_k$$

to obtain

$$\sum_{j,k} d_j N_{ji}^k \,\mathrm{tr}(\theta_k V_k) = \sum_{j,k} d_j N_{ji}^k \theta_k d_k.$$

Now N_{ji}^k is the multiplicity of V^k in $V_j \otimes V_i$, or the multiplicity of the unit object in $V_{*k} \otimes V_j \otimes V_i$. Thus $N_{ji}^k = N_{k*,i}^{j*}$. Now remembering

$$\sum_k N_{ij}^k d_k = d_i d_j, \qquad d_j = d_{j*}, \qquad d_k = d_{k*},$$

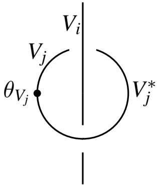
we get

$$\sum_{j,k} d_{j*} N_{k*,i}^{j*} \theta_k d_k = d_i \sum_k \theta_k d_k^2 = d_i p^+.$$

Proof, concluded

To summarize, we showed that

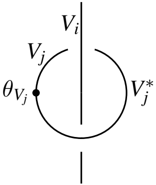
$$\sum_j \operatorname{tr}\left(d_j\theta_i \left(\right.\right.$$



$$\left.\left.\right)\right) = d_i p^+ = p^+ \operatorname{tr}(1_{V_i}).$$

Since $\mathrm{End}(V_i) = K$, this implies

$$\sum_j d_j \left(\right.$$

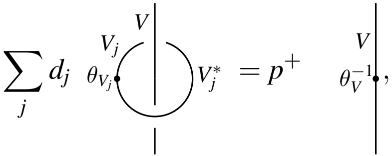


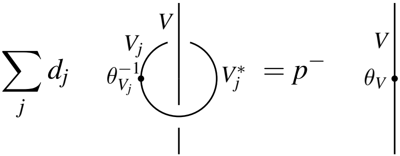
$$\left.\right) = p^+$$

and the other identity is similar.

Our story so far

With $p^+ = \theta_i d_i^2, \qquad p^- = \theta_i^{-1} d_i^2$ and $V = V_i$ irreducible,



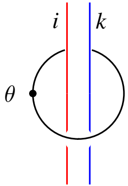


By linearity, this identity is true even if V is not irreducible.

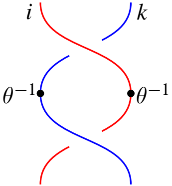
This implies ...

We will show that this implies

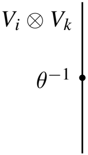
$$\sum_j d_j$$



$$= p^+$$



Indeed by the last slide, LHS equals



and the statement follows using the ribbon axiom.

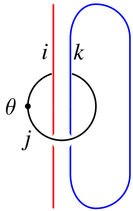
Towards $(st)^3$

Remember we need:

$$ct = tc, \qquad c\widetilde{s} = \widetilde{s}c, \qquad c^2 = 1,$$
$$(\widetilde{s}t)^3 = p^+\widetilde{s}^2, \qquad (\widetilde{s}t^{-1})^3 = p^-\widetilde{s}^2 c,$$
$$\widetilde{s}^2 = p^+p^-c.$$

and the first formula is easy. We now prove $(\widetilde{s}t)^3 = p^+\widetilde{s}^2$, following Bakalov and Kirillov. We prove this by evaluating

$$E = \sum_j d_j$$

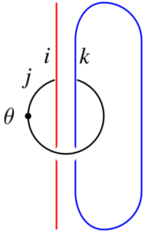


two ways.

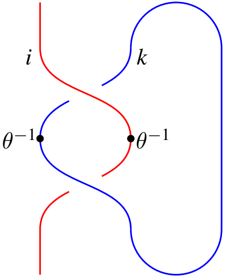
First way

By the calculation we just did

$$E = \sum_j d_j$$

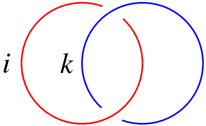


$$= p^+ \times$$



To evaluate this scalar take the trace and divide by $\dim(V_i)$:

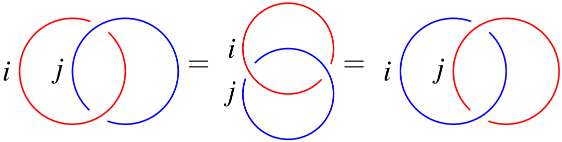
$$E = p^+ d_i^{-1}\theta_i^{-1}\theta_k^{-1} \times$$



$$= \boxed{d_i^{-1}\theta_i^{-1}\theta_k^{-1}\tilde{s}_{ik}}$$

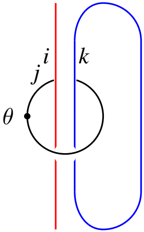
Second way

Remember how we proved the symmetry $\widetilde{s}_{ij} = \widetilde{s}_{ij}$:

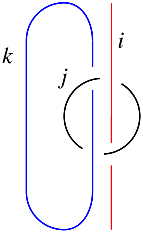


Similarly:

$E = \sum_j d_j$

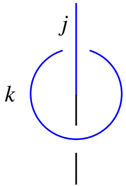


$= \sum_j d_j \theta_j$

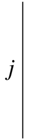


Second evaluation, continued

Now



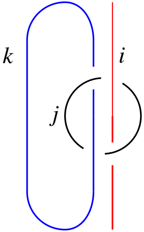
$$= d_j^{-1}\widetilde{s}_{kj} \times$$



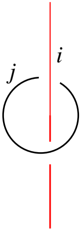
$$= d_j^{-1}\widetilde{s}_{kj}$$

as we see by taking the trace. Using this twice:

$$E = \sum_j d_j \theta_j$$



$$= \sum_j d_j \theta_j d_j^{-1} \widetilde{s}_{kj}$$



$$= \boxed{\sum_j \theta_j \widetilde{s}_{kj} \widetilde{s}_{ji} d_i^{-1}}$$

conclusion

Comparing our two evaluations for E:

$$p^+ d_i^{-1}\theta_i^{-1}\theta_k^{-1}\widetilde{s}_{ik} = \sum_j \theta_j \widetilde{s}_{kj}\widetilde{s}_{ji} d_i^{-1}.$$

Multiplying by d_i and writing in terms of the matrices $\widetilde{s} = (\widetilde{s}_{ij})$, $t = \mathrm{diag}(t_i)$:

$$p^+ t^{-1}\widetilde{s}t^{-1} = \widetilde{s}t\widetilde{s}.$$

Therefore

$$(\widetilde{s}t)^3 = p^+\widetilde{s}.$$

A similar calculation gives

$$(\widetilde{s}t^{-1})^3 = p^-\widetilde{s}c,$$

where $c = (\delta_{i*,j})$ is the charge conjugation matrix.

Concluding the proof of the proposition

So far we have:

$$ct = tc, \qquad c\widetilde{s} = \widetilde{s}c, \qquad c^2 = 1,$$
$$(\widetilde{s}t)^3 = p^+\widetilde{s}^2, \qquad (\widetilde{s}t^{-1})^3 = p^-\widetilde{s}^2 c,$$

We want to deduce:

$$\widetilde{s}^2 = p^+p^-c. \tag{1}$$

First let us show that $\widetilde{s}^2$ commutes with t. It is enough to show that t commutes with $p^+\widetilde{s}^2$. We have

$$t(p^+\widetilde{s}^2)t^{-1} = t(\widetilde{s}t)^3\,t^{-1} = (t\widetilde{s})^3 = \widetilde{s}^{-1}(\widetilde{s}t)^3\,\widetilde{s} = \widetilde{s}^{-1}(p^+\widetilde{s}^2)\widetilde{s} = p^+\widetilde{s}^2.$$

A step in the last calculation gives $(t\widetilde{s})^3 = p^+\widetilde{s}^2$. So

$$p^+p^-\widetilde{s}^4c = (t\widetilde{s})^3(\widetilde{s}t^{-1})^3 = t\widetilde{s}t\widetilde{s}t\widetilde{s}\widetilde{s}t^{-1}\widetilde{s}t^{-1}\widetilde{s}t^{-1}.$$

Since $\widetilde{s}^2$ commutes with t, rearrange and get $\widetilde{s}^6$, whence (1).