Lecture 13

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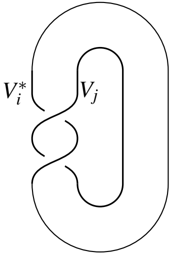


Review: The quantized enveloping algebra

Today we will show how $U_q(\mathfrak{sl}_n)$ can be obtained from the Drinfeld double construction.

In Lecture 10 we gave generators and relations for this algebra, but we did not give motivation for them or prove that the algebra as described was correct. But we want to have this in mind so let us recall this presentation from Lecture 10.

$U_q(\mathfrak{sl}_n)$ has generators K_i (invertible) together with E_i and F_i. The K_i together with the E_i generate a Hopf subalgebra $U_q(\mathfrak{b})$ (corresponding to the positive Borel subgroup of $SL(n)$) and the $K - I$ with the F_i generate the Hopf subalgebra $U_q(\mathfrak{b}_-)$ corresponding to the negative Borel.

Review: the Cartan matrix

Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } j = i \pm 1; \\ 0 & \text{if } |j - i| \geqslant 2, \end{cases}$$

These entries make up the Cartan matrix. So for $SL(4)$:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The algebra $U_q(\mathfrak{sl}_n)$ has generators K_i (invertible), E_i and F_i subject to the relations

$$K_iE_jK_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle}E_j, \qquad K_iF_jK_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

and the quantum Serre relations,

$$E_iE_j = E_jE_i \qquad \text{if } |i - j| > 1,$$

$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0.$$

The comultiplication is given on generators by:

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

Hopf pairings

Let A and B be Hopf algebras, together with a pairing $A \times B \to K$ (the ground field) that satisfies the following conditions.

$$\langle a, bb' \rangle = \langle a_{(1)}, b \rangle \langle a_{(2)}, b' \rangle$$

and

$$\langle aa', b \rangle = \langle a, b_{(2)} \rangle \langle a', b_{(1)} \rangle.$$

Note the reversal in the second identity. We also require

$$\langle a, 1 \rangle = \varepsilon(a), \qquad \langle 1, b \rangle = \varepsilon(b), \qquad \langle S(a), b \rangle = \langle a, S^{-1}b \rangle.$$

This is called a Hopf pairing. For example, the dual pairing between A and $A^{\mathrm{op}} = (A^*)^{\mathrm{cop}}$ satisfies these conditions.

Generalized double construction

Given two Hopf algebras A, B with a Hopf pairing, we may define the double $A \boxtimes B$ by the formulas of Lecture 12. The comultiplication is the same as $A \otimes B$, and the multiplication is given by:

$$(a \boxtimes 1)(1 \boxtimes \lambda) = a \boxtimes \lambda,$$

$$(1 \boxtimes \lambda)(a \boxtimes 1) = \langle \lambda_{(1)}, S^{-1}a_{(1)} \rangle (a_{(2)} \boxtimes \lambda_{(2)}) \langle \lambda_{(3)}, a_{(3)} \rangle.$$

This is a Hopf algebra. (We omit giving a formula for the antipode.)

We should be cautious about asserting quasitriangularity, but in our application we will end up with a QTHA.

Application to quantized enveloping algebras

Following Kassel, Rosso and Turaev, we construct $U_q(\mathfrak{sl}_n)$. Roughly the idea is construct the double of $U_q(\mathfrak{b})$ where $\mathfrak{b}$ is the Borel subalgebra. This has generators K_i (invertible) and E_i $(i \leqslant n-1)$ subject to relations

$$K_iK_j = K_jK_i, \qquad K_iE_jK_i^{-1} = q^{a_{ij}}E_j \tag{1}$$

where the Cartan matrix entries are a_{ij} and the quantum Serre relations. However we omit the Serre relation and denote by U_+ the free Lie algebra with the relations (1). Similarly U_- is generated by $\widetilde{K}_i$ and F_i subject to

$$\widetilde{K}_i\widetilde{K}_j = \widetilde{K}_j\widetilde{K}_i, \qquad \widetilde{K}_iF_j\widetilde{K}_i^{-1} = q^{-a_{ij}}F_j.$$

The Hopf pairing (continued)

The comultiplication in U_+ is defined by

$$\Delta(K_i) = K_i \otimes K_i, \qquad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,$$

It is easy to see that $K_i \otimes K_i$ and $E_i \otimes K_i + 1 \otimes E_i$ satisfy the same relations as K_i and E_i, so such an algebra homomorphism $\Delta : U_+ \to U_+ \otimes U_+$. Also $S(K_i) = K_i^{-1}$, $S(E_i) = -E_i$ is an antipode, so U_+ is a Hopf algebra.

To recapitulate U_+ is the quotient of a free Hopf algebra on generators K_i, K_i^{-1} E_i modulo the relations

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j = q^{a_{ij}} E_j K_i, \qquad K_i^{-1} E_j = q^{-a_{ij}} E_j K_i^{-1},$$

The Hopf pairing

Proposition

There is a Hopf pairing between U_+ and U_- such that

$$\langle K_i, \widetilde{K}_j \rangle = q^{-a_{ij}}, \qquad \langle E_i, F_j \rangle = \frac{\delta_{ij}}{q - q^{-1}},$$

$$\langle E_i, \widetilde{K}_j \rangle = \langle K_i, F_j \rangle = 0 \ .$$

We only sketch the proof. Consider a pair of Hopf algebras A and B that are free algebras on generators $\{a_i\}$ free algebras and $\{b_j\}$. Assume that $\Delta(a_i)$ is a linear combination of $a_j \otimes a_k$ and similarly for $\Delta(b_k)$. Then a Hopf pairing on $A \times B$ may be defined arbitrarily on generators $\langle a_i, b_j \rangle$, and extended all of $A \times B$ using the comultiplications.

The Hopf pairing (continued)

Thus we define the Hopf pairing on the free Lie algebras having U_+ and U_- as quotients, and there are only a few relations to be checked for compatibility.

For example we need to check that $K_iE_j - q^{a_{ij}}E_jK_i$ has trivial pairing with the generators $\widetilde{K}_j$ and F_j.

Now we may construct the double $U_+ \boxtimes U_-$.

Let I_+ be the kernel of the pairing, that is,

$$I_+ = \{t \in U_+ \mid \langle t, U_- \rangle = 0\}.$$

It is a two-sided ideal of U_+.

The quantum Serre relations

It may be checked the quantum Serre relations:

$$E_i^2 E_j - (q+q^{-1})E_i E_j E_i + E_j E_i^2, \qquad j = i \pm 1,$$

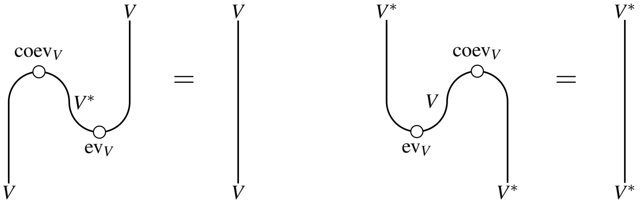
$$E_i E_j - E_j E_j \qquad (|i-j| > 1)$$

are in the kernel I_+ of the Hopf pairing. (It is enough to check that they pair trivially with the generators $\widetilde{K}_i$ and F_i. Therefore we may divide by the ideal that they generate This almost gives $U_q(\mathfrak{sl}_n)$ except that we have two copies of the Cartan subgroup, K_i and $\widetilde{K}_i$. However we may further divide by the ideal generated by $K_i - \widetilde{K}_i$ to obtain $U_q(\mathfrak{sl}_n)$, with all generators and relations implemented.

Reminder (Lecture 9) Dual Objects

Let V be an object in a rigid monoidal category. We recall the assumptions we made of the dual, which we call the left dual V^*. It comes with morphisms $\mathrm{ev}_V : V^* \otimes V \to I$ and $\mathrm{coev}_V : I \to V \otimes V^*$ subject to

$$(1_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes 1_V) = 1_V, \qquad (\mathrm{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \mathrm{coev}_V) = 1_{V^*}.$$

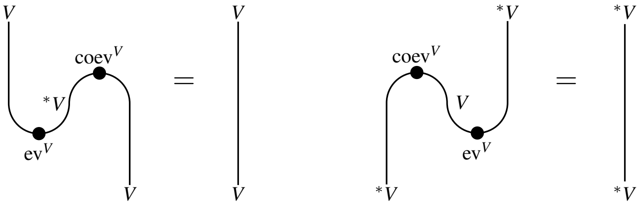


Reminder (Lecture 9): The right dual

Dually, we can ask for a right dual ${}^*V$ with morphisms $\mathrm{ev}^V : V \otimes {}^*V \to I$ and $\mathrm{coev}^V : I \to {}^*V \otimes V$ subject to

$$(\mathrm{ev}^V \otimes 1_V) \circ (1_V \otimes \mathrm{coev}^V) = 1_V,$$

$$(1_{^*V} \otimes \mathrm{ev}^V) \circ (\mathrm{coev}^V \otimes 1_{^*V}) = 1_{V^*}.$$

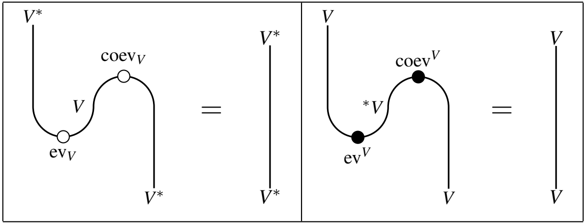


Reminder (Lecture 9): Left and right duals

The notions of left and right dual are not really different. V^* is a right dual of V if and only if V is a left dual of V^*. If every object in the category has both a right and a left dual, then by definition

$$^*(V^*) = V, \qquad (^*V)^* = V,$$

since the only difference between the defining properties



is the labelling of V, V^* and $^*V$.

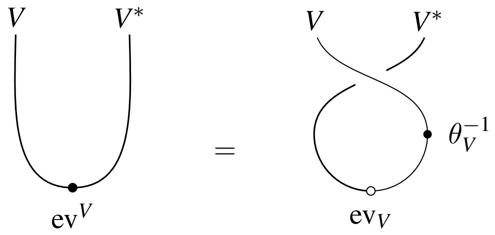
Reminder (Lecture 9): Duals in ribbon categories

Proposition

In a ribbon category, every left dual is also a right dual.

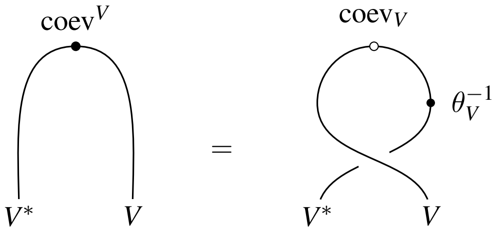
We describe the morphisms $\mathrm{ev}^V : V \otimes V^* \to I$ and $\mathrm{coev}^V : I \to V^* \otimes V$.

$$\mathrm{ev}^V = \mathrm{ev}_V \circ (1_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*}$$



Duals in ribbon categories (continued)

$$\mathrm{coev}^V = c_{V,V^*} \circ (1_{V^*} \otimes \theta_V^{-1}) \circ \mathrm{coev}_V$$



References for Modular Tensor Categories

Last time I gave some references. In addition to Turaev's book:

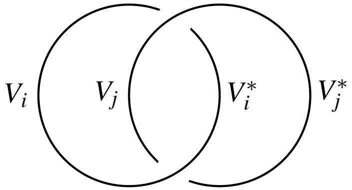
- Bakalov and Kirllov: Lectures on tensor categories and modular functions
- Fuchs: fusion rules in conformal field theory
- Di Francesco, Mathieu and Sénéchal, Conformal Field Theory, chapters 15 and 16.

This list of references is infinitely expandable.

- Erik Verlinde, Fusion rules and modular transformations in 2D conformal field theory. Nuclear Phys. B 300 (1988), no. 3, 360-376.
- Moore and Seiberg, Lectures on RCFT (on-line somewhere)
- Fuchs, Runkel and Schwegert, Construction of RCFT Correlators I: Parition Functions

Review: Modular tensor categories

We recall the definition of a modular tensor category. This is a semisimple ribbon category with a finite number of simple objects V_i ($i \in I$). We assume that the ground ring $K = \mathrm{End}(I)$ is a field, where I is the unit object. This implies that I is itself a simple object. Moreover, we assume that $\mathrm{End}(V_i) = \mathrm{End}(K)$ for all simple objects V_i. Finally, we assume that the S-matrix $(\widetilde{s}_{ij})$ is invertible, where $\widetilde{s}_{ij}$ is the scalar (i.e. endomorphism of I) defined by the Hopf link:



The Fusion Ring

The monoidal structure gives the Grothendieck group of the category a multiplication that makes it into a ring. This ring is a free abelian group on the isomorphism classes of simple modules. Thus if $i \in I$ let $[i] = [V_i]$ be the class of a representative simple module V_i. We can decompose $V_i \otimes V_j$ into simple modules V_k with structure constants N_{ij}^k. Thus

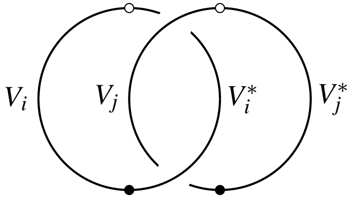
$$[i][j] = N_{ij}^k [k], \qquad V_i \otimes V_j = \bigoplus_k N_{ij}^k V_k.$$

We sum the repeated subscript k. The N_{ij}^k are nonnegative integers. Taking the quantum dimension, which is multiplicative by Lecture 4:

$$d_i d_j = N_{ij}^k d_k, \qquad d_i = \dim(V_i).$$

Remark about notation

It is understood that we use coev_{V_i} and ev^{V_i} in this definition. Thus:



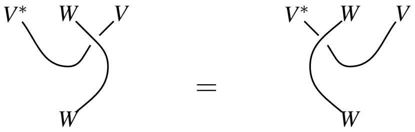
Remark

In a MTC if V_i are the objects ($i \in I$) then V_i^* is V_{i*} for some $i* \in I$. So the duality (conjugation) is implemented as a permutation of the index set.

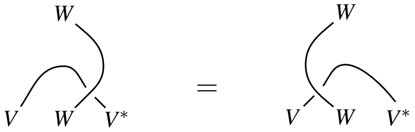
Reminder of Lecture 3

In Lecture 3 we proved that

$$(\mathrm{ev}_V \otimes 1_W)(1_{V^*} \otimes c_{W,V}) = (1_W \otimes \mathrm{ev}_V)(c^{-1}_{W,V^*} \otimes 1_V)$$

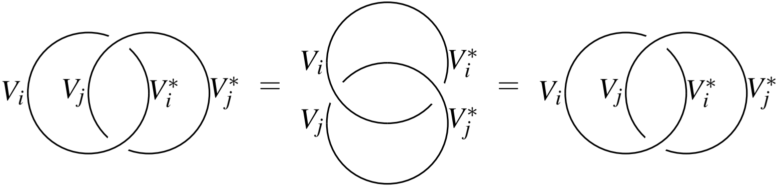


We will call this the coevaluation crossing identity. The evaluation crossing identity is similar:



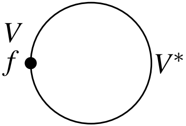
Symmetry of the S-matrix

We now observe that $\widetilde{s}_{ij} = \widetilde{s}_{ji}$. To see this use the crossing identities to move the link around:

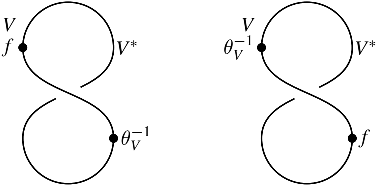


Alternative trace principle

Let $f : V \to V$ be a morphism in a ribbon category. We can compute the trace in two different ways:



To see this, replace ev^V by $\mathrm{ev}_V(1 \otimes \theta_V^{-1})c_{V,V^*}$ in the first figure, and in the second, replace coev^V by $c_{V,V^*}(\theta_V \otimes 1)\,\mathrm{coev}_V$.

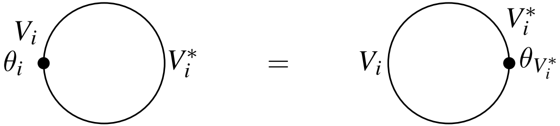


Consequences of the alternative trace principle

If $f : V_i \to V_i$ is any morphism, since V_i is simple, $\mathrm{End}(V_i) = K$. Thus f is just a scalar. In particular, θ_{V_i} is a scalar which we will denote θ_i.

We will denote the quantum dimension of V_i as d_i. Applying the alternative trace principle to $1_V : V \to V$ gives $d_i = d_{i*}$, where $i*$ is the index such that $V_{i*} = V_i^*$.

Remembering that $\theta_{V_i^*} = \theta_{V_i}^*$, the alternative trace principle implies that $\theta_i = \theta_{i*}$.

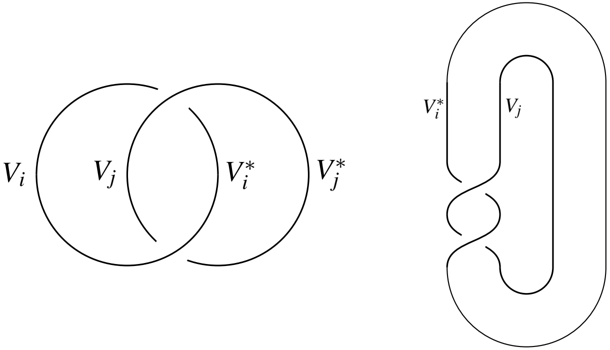


Alternative definition of $\widetilde{s}_{ij}$

We prove

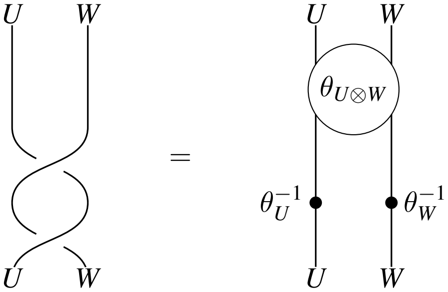
$$\widetilde{s}_{ij} = \mathrm{tr}\big(c^{-1}_{V_{i*},V_j} c^{-1}_{V_j,V_{*i}}\big).$$

Indeed use the principle of the alternative trace to flip one circle:



Reminder: the ribbon axiom

$$c_{U,W}^{-1} \circ c_{W,U}^{-1} = (\theta_U^{-1} \otimes \theta_W^{-1}) \circ \theta_{U\otimes W} = \theta_{U\otimes W} \circ (\theta_U^{-1} \otimes \theta_W^{-1})$$



Another formula for the S-matrix

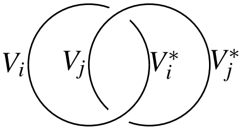
We will prove:

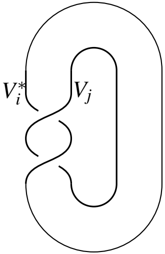
Proposition

We have

$$\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1}\,\mathrm{tr}(\theta_{V_{i*}\otimes V_j})$$
$$= \theta_i^{-1}\theta_j^{-1}N_{i*,j}^{k}\theta_k d_k.$$

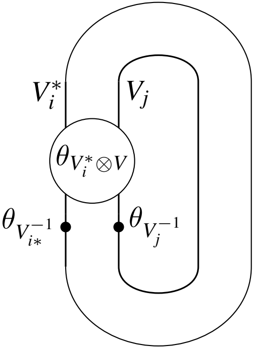
Remember:





Proof

Use the ribbon axiom:



Thus

$$\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1}\,\mathrm{tr}(\theta_{V_{i*}\otimes V_j}).$$

The first relation

We just proved

$$\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1}\,\mathrm{tr}(\theta_{V_{i*}\otimes V_j}).$$

Now

$$V_{i*}\otimes V_j = \oplus_k N_{i*,j}^k V_k$$

and $\mathrm{tr}(\theta_{V_{i*}\otimes V_j})$ can be computed by summing the traces on these summands. Therefore

$$\widetilde{s}_{ij} = \theta_i^{-1}\theta_j^{-1} N_{i*,j}^k \theta_k d_k.$$

Modular Meanings

Bakalov and Kirillov say:

The appearance of the modular group in tensor categories may seem mysterious; however there is a simple geometrical explanation, based on the fact that to each modular tensor category one can associate a $2+1$ dimensional TQFT. This shows that in fact we have an action of the mapping class group of any oriented 2-dimensional surface on the appropriate objects in MTC. This is the key idea in [Turaev's book].

Thus $\mathrm{SL}(2,\mathbb{Z})$ is the mapping class group of the torus.

More Modular Meanings

In Chapter IV Section 5, Turaev defines a more general action. Concerning this, Turaev (p.190) states:

It is this relationship which suggested the terms modular functor and modular category.

Yet this is not the whole story since in certain “rational” conformal field theories the fields are actually modular forms!

The group $\mathrm{SL}(2,\mathbb{Z})$

The group $\mathrm{SL}(2,\mathbb{Z})$ mentioned in the last quote is the mapping class group of the torus, i.e. the group of group of homeomorphisms modulo isotropy. The larger group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathcal{H} = \{z = x + iy \in \mathbb{C} | y > 0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

The subgroup $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ acts discontinuously.

A modular form is a function f that satisfies

$$f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ or a subgroup of finite index.

Two important elements

Note that $-I \in \mathrm{SL}(2,\mathbb{Z})$ acts trivially on $\mathcal{H}$, so the action is not faithful. Let

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

These generate $\mathrm{SL}(2,\mathbb{Z})$. Since $S^2 = -I$, S has order 4 as an element of the group but order 2 in its action on $\mathcal{H}$.

$T : z \to z+1$ is the translation by 1. If a holomorphic function f is invariant under $\mathrm{SL}(2,\mathbb{Z})$ it is invariant under T and so it has a Fourier expansion:

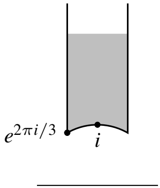
$$f(z) = \sum a_n q^n, \qquad q = e^{2\pi i z}.$$

Although T has infinite order, ST has order 3 in its action on $\mathcal{H}$ or 6 as an element of $\mathrm{SL}(2,\mathbb{Z})$.

$$S^2 = -I, \qquad (ST)^3 = -I.$$

Digression on $\mathrm{SL}(2,\mathbb{Z})$ (continued)

Here is the well-known fundamental domain for $SL(2,\mathbb{Z})$:



We have marked the fixed points i and $e^{2\pi i/3}$ of S and ST.

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

$$S : z \to -\frac{1}{z}, \qquad T : z \to z + 1.$$

Looking ahead

Let $c = (\delta_{i,i*})$ be the conjugation map, $s = c\widetilde{s} = \widetilde{s}c$.

We will see later that given a modular tensor category, there is a (projective) representation of $\mathrm{SL}(2,\mathbb{Z})$ in which the role of $S \in \mathrm{SL}(2,\mathbb{Z})$ is played by s.

The role of the translation T is played by the matrix $t = (\delta_{ij}\theta_i)$ of twists.