Lecture 12

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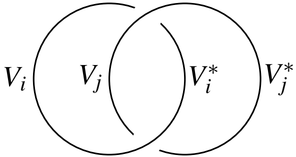


Review: convolution theory

Suppose C is a coalgebra and A is an algebra. Then the space $\mathrm{Hom}(C,A)$ of all linear maps $C \to A$ has an algebra structure. If $f, g \in \mathrm{Hom}(C,A)$ then $f \star g \in \mathrm{End}(H)$ is defined to be the composition:

$$C \xrightarrow{\quad \Delta \quad} C \otimes C \xrightarrow{\quad f \otimes g \quad} A \otimes A \xrightarrow{\quad \mu \quad} A\,.$$

Associativity follows from the associativity of μ and the coassociativity of Δ.

As a special case, if A is a Hopf algebra then $A \otimes A^* \cong \mathrm{End}(A)$ as a vector space. This is an isomorphism of algebras using the convolution product in $\mathrm{End}(A)$.

Review: Convolution theory (continued)

The counit ε is the unit in the convolution ring. This is a paraphrase of the definition of ε.

If $f \in \mathrm{End}(A)$ then the corresponding element of $A \otimes A^*$ is $f(e_i) \otimes e^i$ (implied summation). In particular the canonical element $T = e_i \otimes e^i \in A \otimes A^*$ corresponds the identity map $I_A \in \mathrm{End}(A)$. Transferring this fact to $A \otimes A^*$ gives the identity

$$\varepsilon(e_i) \otimes e^i = 1_{A \otimes A^*}.$$

The antipode S is the inverse of I in the canonical ring. Again, this is a paraphrase of the definition of S. So transferring this fact to $A \otimes A^*$,

$$T^{-1} = \sum_i S(e_i) \otimes e^i.$$

Review: the canonical element

Let A be a finite-dimensional Hopf algebra with basis e_i, and let e^i be the dual basis of A^*. The canonical element $T = e_i \otimes e^i \in A \otimes A^*$ does not depend on choice of basis.

In Lecture 11 we proved

$$(\Delta \otimes 1)(T) = T_{13}T_{23}, \qquad (1 \otimes \Delta)(T) = T_{12}T_{13}.$$

Explicitly

$$T_{13}T_{23} = e_i \otimes e_j \otimes e^i e^j, \qquad T_{12}T_{13} = e_i e_j \otimes e^i \otimes e^k.$$

(In Lecture 11 we wrote this for $A^{\text{op}} \otimes A^*$ so the second formula appeared differently.)

The story so far

The Drinfeld double was characterized thus by Drinfeld in his 1986 ICM talk.

Let A be a Hopf algebra. Denote by A° the algebra A^* with the opposite comultiplication. It can be shown that there is a unique quastriangular Hopf algebra $(D(A), R)$ such that (1) $D(A)$ contains A and A° as Hopf subalgebras (2) R is the image of the canonical element of $A \otimes A^\circ$ under the embedding $A \otimes A^\circ \to D(A)$ and (3) the linear mapping $A \otimes A^\circ \to D(A)$ given by $a \otimes b \to ab$ is bijective.

In Lecture 11 we constructed the dual Hopf algebra $D(A)^*$ by modifying the comultiplication of $A^* \otimes A^{\mathrm{op}}$, using one of its two canonical elements to twist. Today We will see that the other canonical element provides the R-matrix.

Review: $D(A)^*$

We created $D(A)^*$ by Drinfeld twisting $A^* \otimes A^{\text{op}}$. Thus as an algebra $D(A)^* = A^* \otimes A^{\text{op}}$ but the comultiplication is

$$\Delta_F(x) = F\Delta(x)F^{-1}$$

where

$$F = (1_{A^*} \otimes S^{-1}e_i) \otimes (e^i \otimes 1_A),$$

$$F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A).$$

Then we define $D(A)$ to be the dual of $D(A)^*$. As a coalgebra it is $A \otimes A^\circ$, where $A^\circ = (A^{\text{op}})^* = (A^*)^{\text{cop}}$. Since the comultiplication of $D(A)^*$ differs from the comultiplication in $A^* \otimes A^{\text{op}}$, the multiplication in $D(A)$ is correspondingly modified from $A \otimes A^{\text{op}}$.

Subalgebras of $D(A)$ isomorphic to A, A°

Since the multiplication in $D(A)$ is changed from $A \otimes A^\circ$, we will modify the notation and write $a \boxtimes \lambda$ for the element corresponding to $a \in A$ and $\lambda \in A^c irc$. We will continue to use the notation $\lambda \otimes a$ for the element of $D(A)^*$.

Proposition

The maps $A \longrightarrow D(A)$ and $A^\circ \longrightarrow D(A)$ given by $a \mapsto a \boxtimes 1$ and $\lambda \mapsto 1 \boxtimes \lambda$ are Hopf algebra homomorphisms. Thus $D(A)$ contains isomorphic copies of A and A°.

Since $D(A)$ and $A \otimes A^\circ$ are identified as coalgebras, we have only to show that the maps $a \mapsto a \boxtimes 1$ and $\lambda \mapsto 1 \boxtimes \lambda$ respect multiplication. These are similar and we prove the first.
We must show that for $\nu \otimes x \in D(A)^*$ we have

$$\langle (a \boxtimes 1)(b \boxtimes 1), \nu \otimes x \rangle = \langle ab \boxtimes 1, \nu \otimes x \rangle$$

Proof

Since

$$F = (1_{A^*} \otimes S^{-1} e_i) \otimes (e^i \otimes 1_A), \qquad F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A),$$

we have

$$\Delta_F(\nu \otimes x) = (\nu_{(1)} \otimes S^{-1}(e_i) x_{(1)} e_j) \otimes (e^i \nu_{(2)} e^j \otimes x_{(2)}).$$

Thus $\langle (a \boxtimes 1)(b \boxtimes 1), \nu \otimes x \rangle$ equals

$$\langle a \boxtimes 1, \nu_{(1)} \otimes S^{-1}(e_i) x_{(1)} e_j \rangle \langle b \boxtimes 1, e^i \nu_{(2)} e^j \otimes x_{(2)} \rangle.$$

We now remember

$$\langle 1, x \rangle = \epsilon(x).$$

Also $\varepsilon(S^{-1}(e_i)) = \varepsilon(e_i) S^{-1}(1) = \varepsilon(e_i)$.

Proof (concluded)

From the previous page, $\langle (a \boxtimes 1)(b \boxtimes 1), \nu \otimes x \rangle$ equals

$$\langle a \boxtimes 1, \nu_{(1)} \otimes S^{-1}(e_i) x_{(1)} e_j \rangle \langle b \boxtimes 1, e^i \nu_{(2)} e^j \otimes x_{(2)} \rangle.$$

Thus

$$\langle (a \boxtimes 1)(b \boxtimes 1), \nu \otimes x \rangle = \langle a, \nu_{(1)} \rangle \varepsilon(e_i) \varepsilon(x_{(1)}) \varepsilon(e_j) \langle b, e^i \nu_{(2)} e^j \rangle \varepsilon(x_{(2)}).$$

Since $\varepsilon(e_i) \otimes e^i = 1_{A \otimes A^*}$ we get

$$\langle (a \boxtimes 1)(b \boxtimes 1), \nu \otimes x \rangle = \langle a, \nu_{(1)} \rangle \langle b, \nu_{(2)} \rangle \varepsilon(x) = \langle ab, \nu \rangle \varepsilon(x) = \langle ab \boxtimes 1, \nu \otimes x \rangle.$$

Therefore $(a \boxtimes 1)(b \boxtimes 1) = (ab \boxtimes 1)$ in $D(A)$.

Similarly $(1 \boxtimes \lambda)(1 \boxtimes \mu) = 1 \boxtimes \lambda\mu$.

Multiplication in $D(A)$

Although the multiplication in $D(A)$ is different from $A \otimes A^\circ$, the following formulas can be used.

Proposition

We have

$$(a \boxtimes 1)(1 \boxtimes \lambda) = a \boxtimes \lambda,$$

$$(1 \boxtimes \lambda)(a \boxtimes 1) = \langle \lambda_{(1)}, S^{-1} a_{(1)} \rangle (a_{(2)} \boxtimes \lambda_{(2)}) \langle \lambda_{(3)}, a_{(3)} \rangle.$$

Since

$$\Delta_F(\nu \otimes x) = (\nu_{(1)} \otimes S^{-1}(e_i) x_{(1)} e_j) \otimes (e^i \nu_{(2)} e^j \otimes x_{(2)}),$$

we have

$$\langle (a \boxtimes 1)(1 \boxtimes \lambda), \nu \otimes x \rangle = \langle a \boxtimes 1, \nu_{(1)} \otimes S^{-1}(e_i) x_{(1)} e_j \rangle \langle 1 \boxtimes \lambda, e^i \nu_{(2)} e^j \otimes x_{(2)} \rangle$$

$$= \langle a, \nu_{(1)} \rangle \varepsilon(e_i) \varepsilon(x_{(1)}) \varepsilon(e_j) \varepsilon(e^i) \varepsilon(\nu_{(2)}) \varepsilon(e^j) \langle \lambda, x_{(2)} \rangle.$$

Proof

Since $\varepsilon(e_i) \otimes e^i = 1_{A \otimes A^*}$ we have $\varepsilon(e_i)\varepsilon(e^i) = \varepsilon(1_{A \otimes A^*}) = 1$ and using

$$\nu_{(1)}\varepsilon(\nu_{(2)}) = \nu, \qquad \varepsilon(x_{(1)})x_{(2)} = x,$$

we see

$$\langle (a \boxtimes 1)(1 \boxtimes \lambda), \nu \otimes x \rangle = \langle a, \nu \rangle \langle \lambda, x \rangle = \langle a \boxtimes \lambda, \nu \otimes x \rangle$$

proving $(a \boxtimes 1)(1 \boxtimes \lambda) = a \boxtimes \lambda$.

Proof (continued)

Next consider

$$\langle (1\boxtimes\lambda)(a\boxtimes 1), \nu\otimes x\rangle = \langle 1\boxtimes\lambda, \nu_{(1)}\otimes S^{-1}(e_i)x_{(1)}e_j\rangle\langle a\boxtimes 1, e^i\nu_{(2)}e^j\otimes x_{(2)}\rangle.$$

This equals

$$\varepsilon(\nu_{(1)})\langle\lambda_{(1)}, S^{-1}(e_i)\rangle\langle\lambda_{(2)}, x_{(1)}\rangle\langle\lambda_{(3)}, e_j\rangle\langle e^i, a_{(1)}\rangle\langle\nu_{(2)}, a_{(2)}\rangle\langle\varepsilon^j, a_{(3)}\rangle\varepsilon(x_{(2)})$$

$$= \langle S^{-1}(\lambda_{(1)}), e_i\rangle\langle\lambda_{(2)}, x\rangle\langle\lambda_{(3)}, e_j\rangle\langle a_{(1)}, e^i\rangle\langle a_{(2)}, \nu\rangle\langle a_{(3)}, \varepsilon^j\rangle$$

Now

$$\langle\lambda_{(3)}, e_j\rangle\langle\varepsilon^j, a_{(3)}\rangle =$$

$$\langle\lambda_{(3)}, a_{(3)}\rangle\,\langle S^{-1}(\lambda_{(1)}), e_i\rangle\langle a_{(1)}, e^i\rangle = \langle S^{-1}(\lambda_{(1)}), a_{(1)}\rangle$$

Proof (concluded)

so

$$\langle(1 \boxtimes \lambda)(a \boxtimes 1), \nu \otimes x\rangle =$$

$$\langle S^{-1}(\lambda_{(1)}), a_{(1)}\rangle\langle\lambda_{(2)}, x\rangle\langle\lambda_{(3)}, a_{(3)}\rangle\langle a_{(2)}, \nu\rangle =$$

$$\langle\lambda_{(1)}, S^{-1}a_{(1)}\rangle\langle\lambda_{(3)}, a_{(3)}\rangle\langle a_{(2)} \boxtimes \lambda_{(2)}, \nu \otimes x\rangle.$$

This proves

$$(1 \boxtimes \lambda)(a \boxtimes 1) = \langle\lambda_{(1)}, S^{-1}a_{(1)}\rangle(a_{(2)} \boxtimes \lambda_{(2)})\langle\lambda_{(3)}, a_{(3)}\rangle.$$

The R-matrix

Theorem (Drinfeld)

The double $D(A)$ is quasitriangular with R-matrix

$$R = (e_i \boxtimes 1_{A^\circ}) \otimes (1_A \boxtimes e^i).$$

By convolution theory, R is invertible and

$$R^{-1} = (S(e_i) \boxtimes 1_{A^\circ}) \otimes (1_A \boxtimes e^i).$$

To verify quasitriangularity we need

$$\tau\Delta(x) = R\Delta(x)R^{-1}, \qquad x \in D(A),$$

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R_{13}R_{12}.$$

The braiding axioms

The identities

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R_{13}R_{12} \tag{1}$$

follow from the corresponding identities

$$(\Delta\otimes 1)T = T_{13}T_{23} = e_i\otimes e_j\otimes e^i e^j, \qquad (1\otimes\Delta T) = T_{12}T_{13} = e_i e_j\otimes e^i\otimes e^j$$

for the canonical element. These identities are for $A \otimes A^*$, whereas $D(A)$ is isomorphic as a coalgebra to $A \otimes A^\circ$ with $A^\circ = (A^*)^{\mathrm{cop}}$. So for the second identity we need

$$(1 \otimes \tau\Delta T) = T_{13}T_{12} = e_j e_i \otimes e^j \otimes e^i,$$

and applying this to $R = (e_i \boxtimes 1_{A^\circ}) \otimes (1_A \boxtimes e^i)$, (1) follows.

The coboundary identity

To finish the quasitriangularity proof, we need

$$R(\Delta(u)R^{-1} = \tau\Delta(u), \qquad u \in D(A). \tag{2}$$

It is sufficient to check this for u in a set of generators, so we may assume that $u = a \boxtimes 1_{A^\circ}$ or $u = 1_A \boxtimes \lambda$.

The computation is simplified by using a quasitriangularity criterion due to Radford, Minimal quasitriangular Hopf algebras, J. Algebra 157 (1993), 285-315. I am also following the (possibly hard to find) book Quantum Groups and Knot Invariants by Kassel, Rosso and Turaev.

Radford actually gave 4 equivalent criteria for the coboundary condition.

Radford's criteria

Proposition (Radford)

Each of the following conditions is equivalent to

$$R(\Delta(u)R^{-1} = \tau\Delta(u), \qquad u \in D(A). \tag{3}$$

(i) $R^{(1)}u \otimes R^{(2)} = u_{(2)}R^{(1)} \otimes u_{(1)}R^{(2)}S(u_{(3)}),$

(ii) $uR^{(1)} \otimes R^{(2)} = R^{(1)}u_{(2)} \otimes S(u_{(1)})R^{(2)}u_{(3)},$

(iii) $R^{(1)} \otimes R^{(2)}u = u_{(3)}R^{(1)}S^{-1}(u_{(1)}) \otimes u_{(2)}R^{(2)},$

(iv) $R^{(1)} \otimes uR^{(2)} = u_{(3)}R^{(1)}S^{-1}(u_{(1)}) \otimes u_{(2)}R^{(2)},$

We will use (i) and (iii) but only prove that (iii) implies (3). Thus:

$$R\Delta(u) = R(1 \otimes u_{(2)})(u_{(1)} \otimes 1) = u_{(4)}R^{(1)}S^{-1}(u_{(2)})u_{(1)} \otimes u_{(3)}R^{(2)}$$

$$= \tau\Delta(u)R.$$

The coboundary identity for $D(A)$

It is enough to prove the coboundary condition for a set of generators of $D(A)$. The strategy is to use Radford's criterion (iii) to handle u of the form $a \boxtimes 1$ and criterion (ii) for $u = 1 \boxtimes \lambda$. We will only carry out the first calculation.

We must show

$$R^{(1)} \otimes R^{(2)} u = u_{(3)} R^{(1)} S^{-1}(u_{(1)}) \otimes u_{(2)} R^{(2)}$$

with $u = a \boxtimes 1$, $R^{(1)} = e_i \boxtimes 1$, $R^{(2)} = 1 \boxtimes e^i$.

Proof

We want to verify

$$R(1 \otimes u) = u_{(3)} R^{(1)} S^{-1}(u_{(1)}) \otimes u_{(2)} R^{(2)}$$

with $R^{(1)} = e_i \boxtimes 1$, $R^{(2)} = 1 \boxtimes e^i$ and $u = a \boxtimes 1$. The left-hand side is

$$(e_i \boxtimes 1)(1 \boxtimes 1) \otimes (1 \boxtimes e^i)(a \boxtimes 1)$$

$$= \langle e^i{}_{(1)}, S^{-1} a_{(1)} \rangle \langle e^i{}_{(3)}, a_{(3)} \rangle ((e_i \boxtimes 1) \otimes (a_{(2)} \boxtimes e^i{}_{(2)})).$$

The right-hand side equals

$$(a_{(3)} e_i S^{-1}(a_{(1)}) \boxtimes 1) \otimes (a_{(2)} \boxtimes 1)(1 \boxtimes e^i)$$

$$= (a_{(3)} e_i S^{-1}(a_{(1)}) \boxtimes 1) \otimes (a_{(2)} \boxtimes e^i).$$

Proof (continued)

Let Δ° be the comultiplication in A°. It is the opposite of the comultiplication Δ in A^*. Therefore in A

$$a_{(3)}e_iS^{-1}a_{(1)} = \langle e^j, a_{(3)}e_iS^{-1}a_{(1)}\rangle e_j = \langle e^j{}_{(3)}, a_{(3)}\rangle\langle e^j{}_{(2)}, e_i\rangle\langle e^j{}_{(1)}, S^{-1}a_{(1)}\rangle e_j.$$

Using this the right-hand side equals

$$\langle e^j{}_{(3)}, a_{(3)}\rangle\langle e^j{}_{(2)}, e_i\rangle\langle e^j{}_{(1)}, S^{-1}a_{(1)}\rangle((e_j \boxtimes 1) \otimes (a_{(2)} \boxtimes e^i)),$$

or since $\langle e^j{}_{(2)}, e_i\rangle e^i = e^j{}_{(2)}$

$$\langle e^j{}_{(3)}, a_{(3)}\rangle\langle e^j{}_{(1)}, S^{-1}a_{(1)}\rangle((e_j \boxtimes 1) \otimes (a_{(2)} \boxtimes e^j{}_{(2)})).$$

Switching i and j this equals the left-hand side, and the identity is proved.

For the otherside, use Radford's criterion (ii). We leave this case to the reader.

The plan

Now let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}_-$ be the triangular decomposition of a semisimple Lie algebra. Drinfeld constructed the quantized enveloping algebra as follows.

First of all, we have two Lie algebras

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}, \qquad \mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{u}_-$$

that are very similar to each other. The quantized enveloping algebra $U_q(\mathfrak{u})$ can be constructed directly, though it is easier to work with a "free" algebra U_+ that omits the Serre relations and has $U_q(\mathfrak{u})$ as a quotient, and similarly U_-.

The plan (continued)

It is possible to construct a pairing between U_+ and U_- that that is almost a dual pairing, and hence construct the double $U_+ \boxtimes U_-$. Unfortunately this contains two copies of $\mathfrak{h}$, but a suitable quotient is $U_q(\mathfrak{g})$, and it will be possible to see all the relations, including the Serre relations this way.

Strictly speaking $U_q(\mathfrak{g})$ is not quasitriangular, since the universal R-matrix does not live in $U_q(\mathfrak{g})$ but in a completion. The reason that our theorem does not imply quasitriangularity is U_+ and U_- are infinite dimensional, so $e_i \otimes e^i$ is an infinite sum. So quasitriangularity can be obtained (in a suitable sense) but there are further technical issues.

Modular categories: overview

Modular tensor categories are ribbon categories of a particular sort. They are introduced in Chapter II of Turaev's book, where he writes:

As we know, ribbon categories give rise to invariants of links in Euclidean 3-space. Unfortunately, they are too general to yield similar invariants of links in 3-manifolds. This leads to the concept of modular category which is the key algebraic concept of this monograph.

Modular categories are closely related to topological quantum field theories. They can be constructed using quantum groups at roots of unity or conformal field theory. Interestingly these categories come with an action of $\mathrm{SL}(2,\mathbb{Z})$ which is the origin of the term "modular." Today we will give the definition.

Abelian categories

Around 1955, Buchsbaum and (independently) Grothendieck axiomatized categories in which homological algebra works. Their axiomatizations were similar but slightly different. The notion of an abelian category eventually stabilized to that used in Mac Lane's books Homology and Categories for the Working Mathematician.

The archetypal abelian category is the category of modules over a ring, a comfortable category in which the Snake Lemma is proved by diagram chasing. But other naturally occurring abelian categories such as the category of sheaves of abelian groups over a topological space do not present themselves as module categories. Still the Mitchell-Freyd embedding theorem shows that every abelian category can be embedded in a module category, so proofs using diagram chasing are valid.

Abelian categories (continued)

An additive category is one in which the Hom sets form abelian groups, the composition law being bilinear; there is also assumed to be a 0 element that is both initial and terminal, and finite products that are also coproducts.

A morphism $f : A \to B$ is a monomorphism (generalizing the notion of an injective map) if for morphisms $g, g' : C \to A$ the identity $f \circ g = f \circ g'$ implies $g = g'$. The dual property is epimorphism.

If $f : A \to B$ is a morphism, the kernel of f is a morphism $i : K \to A$ such that $f \circ i = 0$ and for all objects C and $g : C \to A$ if $fg = 0$ then g factors uniquely through K. A kernel is a monomorphism. The dual notion is that of a cokernel.

Abelian categories (continued)

In an abelian category, it is assumed that every morphism have both a kernel and a cokernel.

It is also assumed that every monomorphism is the kernel of its cokernel, and that every epimorphism is the cokernel of its kernel.

Finally we assume that we may factor any morphism $f : A \to B$ as $f = \psi \circ \phi$ where for some object C the morphism $\phi : A \to C$ is an epimorphism and $\psi : C \to B$ is a monomorphism.

Tensor categories

The term tensor category is not used consistently by different authors. But we will use this term to mean an additive category with an additional bilinear bifunctor $\otimes$ that makes it into a monoidal category. Usually we want the category to be abelian.

The bilinearity assumption means that we have natural and additive isomorphisms

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C).$$

Let K be the unit object in the tensor category $\mathcal{C}$. Then $\mathrm{End}(K)$ is a ring; its additive structure comes from the fact that it is an object in an additive category, and the multiplication comes from the monoidal isomorphism $K \cong K \otimes K$. We will call $\mathrm{End}(K)$ the ground ring.

Simple objects

We define a simple object in an abelian tensor category to be object A that is not zero but which has no subobjects. Thus if $i : C \to A$ is morphism then either $i = 0$ or i is an epimorphism.

For example the unit object K is simple if $\mathrm{End}(K)$ is a field. See Deligne and Milne, Tannakian Categories, Proposition 1.17.

▸ Deligne and Milne, Tannakian Categories

If V is any object then since $V \cong K \otimes V$ it becomes a vector space over $k = \mathrm{End}(K)$. If K is an algebraically closed field and V is a simple object that is finite-dimensional over k, then $k = \mathrm{End}(V)$ (Schur's Lemma).

Semisimple categories

We will define a semisimple category to be an abelian tensor category with unit object K in which $k = \mathrm{End}(K)$ is a field, all objects are finite-dimensional vector spaces over K, and if $\{V_i | i \in I\}$ are representatives of the isomorphism classes of simple objects, then every object is a finite direct sum of V_i. In such a decomposition,

$$V \cong \bigoplus n_i V_i$$

the multiplicities n_i are uniquely determined.

Fusion categories

A Modular tensor category is a semisimple ribbon category $\mathbf{C}$ with ground field algebraically closed. Frequently these categories are discussed under the term fusion categories. Such a category can be constructed from representations of a quantum group at a at root of unity, and alternatively as the fusion category of fields in a Wess-Zumino-Witten (WZW) conformal field theory.

▸ Bakalov and Kirllov: Lectures on tensor categories and modular functions

▸ Fuchs: fusion rules in conformal field theory

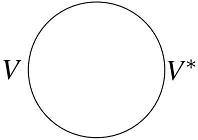
In addition to those references see Di Francesco, Mathieu and Sénéchal, Conformal Field Theory, chapters 15 and 16.

The quantum dimension

We are assuming that the category $\mathcal{C}$ is semisimple. So abelian with a finite number of nonisomorphic simple objects V_i $(i \in I)$ such that every object is uniquely isomorphic to a direct sum of V_i. It is ribbon, so every object has a dual, which is both a right and a left dual.

Let K denote the unit object in the category If $V = V_i$ then we can define the quantum dimension to be the morphism $K \to K$ defined as in previous lectures by the composition

$$K \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{\mathrm{ev}^V} K$$



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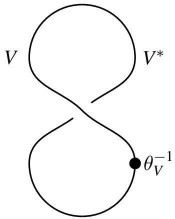
The quantum dimension (continued)

From Lecture 9, this definition has implicitly used the ribbon element, since if we substitute the definition of $\mathrm{ev}^V : V \otimes V^* \to K$, the quantum dimension is actually the quantum trace of $1_V : V \to V$ as defined in Lecture 4:

$$K \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{1 \otimes \theta_V^{-1}} V^* \otimes V \xrightarrow{\mathrm{ev}_V} K$$



The Fusion ring

Let V_i ($i \in I$) be representatives of the isomorphism classes of simple objects in the tensor category. We are assuming that the number of these is finite. We define nonnegative integers N_{ij}^k by

$$V_i \otimes V_j = N_{ij}^k V_k$$

(implied summation).

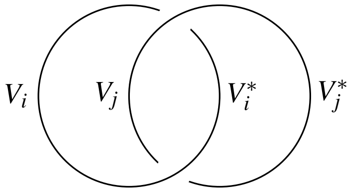
The Grothendieck group of the category is called the Fusion ring. It has generators x_i corresponding to the simple objects in the category, with structure constants N_{ij}^k, so

$$x_i x_j = N_{ij}^k x_k.$$

The ring $\mathfrak{F}$ with these generators has a conjugation operation $c : \mathfrak{F}$ that permutes the x_i so that $c(x_i) = x_{i*}$ where $V_{i*} = V_i^*$ is the dual. Also the twist θ_{V_i} is a scalar θ_i by Schur's Lemma.

The S-matrix

Now let V_i and V_j be simple objects. We define $\widetilde{s}_{i,j}$ to be the scalar that is the morphism $K \to K$ defined by the link:



Now we impose the assumption that the matrix $\widetilde{s} = (\widetilde{s}_{ij})$ is invertible. This is called the S-matrix. ("S" for scattering.) This completes the definition of a modular tensor category.

Modularity

The group $\mathrm{SL}(2, \mathbb{Z})$ has two generators

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

subject to the relations

$$S^4 = 1, \qquad S^2 T = T S^2, \qquad (ST)^3 = S^2.$$

The modularity consists of an action of $\mathrm{SL}(2, \mathbb{Z})$ on the category, or at least its Grothendieck group, known as the fusion ring. The matrix $\widetilde{s}^4$ is not the identity, but it is diagonal, so multiplying it by certain constants gives a matrix s that satisfies $s^4 = 1$. Supplementing it by the matrix $t = (\delta_i \theta_{ij})$, the relations of $\mathrm{SL}(2, \mathbb{Z})$ are satisfied.

Why modularity?

The fact that there is an action of $\mathrm{SL}(2, \mathbb{Z})$ is explained by the fact that the fields in certain conformal field theories can be interpreted as modular forms. See:

Erik Verlinde, Fusion rules and modular transformations in 2D conformal field theory. Nuclear Phys. B 300 (1988), no. 3, 360-376.

The fusion rings can be constructed alternatively from such conformal field theories, or from the representation theory of quantum groups at roots of unity.