Lecture 11

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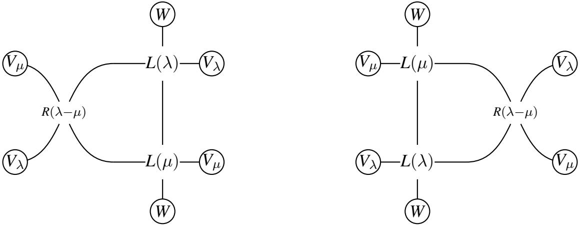


Convolution

Let H be a finite-dimensional Hopf algebra. Let $\mathrm{End}(H)$ be the vector space of all linear transformations of H. Then $\mathrm{End}(H)$ has two completely different ring structures. First, it is a ring in which the multiplication is the composition of endomorphisms. This ring is isomorphic to $\mathrm{Mat}_d(K)$ where $d = \dim(H)$ and K is the ground field.

The second, unrelated ring structure is called convolution. If f and g are endomorphisms of H, define $f \star g \in \mathrm{End}(H)$ to be the composition:

$$H \xrightarrow{\quad\Delta\quad} H \otimes H \xrightarrow{\quad f\otimes g\quad} H \otimes H \xrightarrow{\quad\mu\quad} H\,.$$

Associativity follows from the associativity of μ and the coassociativity of Δ.

The counit and antipode

The map $\eta\varepsilon : H \to H$ serves as a unit in the ring $\mathrm{End}(H)$. Since we are identifying K with its image under the unit map η, we will denote this map as just ε. To see that it is a unit, note that

$$(f \star \varepsilon)(x) = f(x_{(1)})\varepsilon(x_{(2)}) = f(x_{(1)}\varepsilon(x_{(2)})) = f(x)$$

so $f \star \varepsilon = f$ and similarly $\varepsilon \star f = f$.

Now the identity map $I_H \in \mathrm{End}(H)$ has a convolution inverse, and that is the antipode. Indeed

$$(I \star S)(x) = x_{(1)}S(x_{(2)}) = \varepsilon(x)$$

so $I \star S = \varepsilon$ and similarly $S \star I = \varepsilon$, and we have noted that ε is the unit in the convolution ring.

An isomorphism

Recall that $\mathrm{End}(H) \cong H \otimes H^*$. In this isomorphism a pure tensor $x \otimes \lambda$ corresponds to the rank one endomorphism $\Phi_{x\otimes\lambda}$ defined by

$$\Phi_{x\otimes\lambda}(h) = \langle \lambda h \rangle x.$$

This isomorphism is a ring isomorphism. Let us check that it respects multiplication.

$$(\Phi_{x\otimes\lambda} \star \Phi_{y\otimes\mu})(h) = \langle \lambda, h_{(1)} \rangle x \langle \mu, h_{(2)} \rangle y = \langle \lambda\mu, h \rangle xy,$$

so

$$\Phi_{x\otimes\lambda} \star \Phi_{y\otimes\mu} = \Phi_{xy\otimes\lambda\mu}.$$

Thus denoting $\Phi : H \otimes H^* \to \mathrm{End}(H)$ the linear map such that $\Phi(x \otimes \lambda) = \Phi_{x\otimes\lambda}$, we see Φ is an algebra isomorphism.

Fourier expansions

Choose a basis e_i of H, and let e^i be the dual basis of H^*. Then if $f \in \mathrm{End}(H)$, we have

$$f = \Phi\left(\sum_i f(e_i) \otimes e^i\right).$$

This is a kind of Fourier expansion. To check it, apply both sides to a basis vector e_j. We have

$$\sum_j \Phi_{f(e_i)\otimes e^i}(e_j) = \sum_i \langle e^i, e_j\rangle f(e_i) = f(e_j).$$

We will sometimes use the summation convention and write

$$f = \Phi\big(f(e_i) \otimes e^i\big).$$

The canonical element of $H^* \otimes H$

As a special case take f to be the identity map. Then we see that $\Phi(T) = I_H$ where

$$T = e_i \otimes e^i.$$

This will be called the canonical element of $H \otimes H^*$.
We have also seen that the identity map in $\mathrm{End}(H)$ is convolution invertible, and its inverse is the antipode. Taking $f = S$ we have

$$S = \Phi\big(S(e_i) \otimes e^i\big).$$

Thus we have proved

$$\boxed{T^{-1} = \textstyle\sum_i S(e_i) \otimes e^i\ .}$$

A counit identity

Another result that we can prove using convolution theory is

$$\sum_i \varepsilon(e_i) \otimes e^i = 1_{H \otimes H^*}.$$

Indeed, under Φ the left-hand side becomes the counit $\varepsilon : H \to H$ (or $\eta\varepsilon$) which we have noticed is the unit in the convolution ring. Since Φ is a ring isomorphism, this must be the identity element of $H \otimes H^*$.

Interchanging the roles of H and H^*, we have also

$$\sum_i e_i \otimes \varepsilon(e^i) = 1_{H \otimes H^*}.$$

Introduction

In his 1986 ICM talk Drinfeld defined the quantum double thus:

Let A be a Hopf algebra. Denote by A° the algebra A^* with the opposite comultiplication. It can be shown that there is a unique quastriangular Hopf algebra $(D(A), R)$ such that (1) $D(A)$ contains A and A° as Hopf subalgebras (2) R is the image of the canonical element of $A \otimes A^\circ$ under the embedding $A \otimes A^\circ \to D(A)$ and (3) the linear mapping $A \otimes A^\circ \to D(A)$ given by $a \otimes b \to ab$ is bijective.

We will prove this in two lectures. Today, we will construct the dual Hopf algebra $D(A)^*$. The strategy will be to take $A^* \otimes A^{\mathrm{op}}$ and use one of its two canonical elements to twist. (It is the other canonical element that provides the R-matrix.)

Review: Dual Hopf algebra

Let H be a finite-dimensional Hopf algebra. We will assume that the antipode of H is invertible. It can be shown that this is automatic for finite-dimensional Hopf algebras, but if one wishes to generalize the construction of the quantum double the antipode needs to be invertible.
Let H^* be the dual Hopf algebra. With $\lambda, \mu \in H^*$ and $y, y \in H$

$$\langle D\lambda, x \otimes y \rangle = \langle \lambda, \mu(x \otimes y) \rangle = \langle \lambda, xy \rangle.$$

or in Sweedler notation

$$\langle \lambda, xy \rangle = \langle \lambda_{(1)}, x \rangle \langle \lambda_{(2)}, y \rangle.$$

Dually,

$$\langle \lambda\mu, x \rangle = \langle \lambda, x_{(1)} \rangle \langle \mu, x_{(2)} \rangle.$$

Furthermore

$$\langle S(\lambda), x \rangle = \langle \lambda, S(x) \rangle.$$

Review: The opposite Hopf algebra

Let $(H, \mu, \eta, \Delta, \varepsilon)$ denote the Hopf algebra H with multiplication μ, unit η, comultiplication Δ and unit ε.

Now we may reverse the multiplication, and define $\mu^{\text{op}} = \mu \circ \tau$. So $\mu^{\text{op}}(x \otimes y) = \mu(y \otimes x) = yx$. We do not need to reverse the comultiplication. It is easy to check that $(H, \mu^{\text{op}}, \eta, \Delta, \varepsilon)$ is a bialgebra. For example consider the Hopf axiom:

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\Delta\otimes\Delta} & H \otimes H \otimes H \otimes H & \xrightarrow{1_H\otimes\tau\otimes 1_H} & H \otimes H \otimes H \otimes H \\ \downarrow{\scriptstyle \mu^{\text{op}}} & & & & \downarrow{\scriptstyle \mu^{\text{op}}\otimes\mu^{\text{op}}} \\ H & & \xrightarrow{\Delta} & & H \otimes H \end{array}$$

This diagram commutes since, both ways to $x \otimes y$

$$(yx)_{(1)} \otimes (yx)_{(2)} = y_{(1)}x_{(1)} \otimes y_{(1)}x_{(2)}.$$

Review: The opposite Hopf algebra (continued)

Alternatively, we may reverse the comultiplication and denote $\Delta^{\mathrm{op}} = \tau \circ \Delta$. Then Let $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon)$ is also a bialgebra. There are thus four bialgebras altogether:

$$H = (H, \mu, \eta, \Delta, \varepsilon), \qquad H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon),$$
$$H^{\mathrm{cop}} = (H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon), \qquad H^{\mathrm{op\,cop}} = (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \varepsilon).$$

These are Hopf algebras if the antipode is invertible. The antipode for H^{op} and H^{cop} is S^{-1}.

Also if the antipode is invertible, it is an isomorphism from H to $H^{\mathrm{op\,cop}}$, and similarly from H^{cop} to H^{op}.

The algebra $(H^*)^{\mathrm{cop}}$ is important in the quasitriangularity story and we will denote it H°.

About the antipode

Assuming that H is a Hopf algebra, we've exhibited bialgebras H^{op}, H^{cop} and $H^{\mathrm{op\,cop}}$. For H^{op} and H^{cop} to be Hopf algebras, it is necessary for them to have antipodes. If the antipode S of H is invertible, then S^{-1} is an antipode for both H^{op} and H^{cop}.

Let us check this for H^{op}, leaving H^{cop} to the reader.

$$\begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{1_H \otimes S^{-1}} & H \otimes H \\ \downarrow \varepsilon & & & & \downarrow \mu\tau \\ K & & \xrightarrow{\eta} & & H \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{S^{-1} \otimes 1_H} & H \otimes H \\ \downarrow \varepsilon & & & & \downarrow \mu\tau \\ K & & \xrightarrow{\eta} & & H \end{array}$$

The first diagram commutes since $\mu\tau(1 \otimes S^{-1})\Delta(x) = S^{-1}(x_{(2)})\, x_{(1)} = S^{-1}(S(x_{(1)})x_{(2)}) = S^{-1}(\varepsilon(x)) = \varepsilon(x)$. The second is similar.

The canonical element of $A^{\mathrm{op}} \otimes A^*$

Let A be a finite-dimensional Hopf algebra. Let e_i be a basis of A, and e^i the dual basis of A^*. We consider (omitting a summation)

$$T = e_i \otimes e^i$$

which we interpret as an element of $A^{\mathrm{op}} \otimes A^*$. Note that this does not depend on the choice of basis e_i. We will prove

$$\boxed{(\Delta \otimes 1)(T) = T_{13}T_{23} \quad (1 \otimes \Delta)(T) = T_{13}T_{12}}$$

Explicitly

$$T_{13}T_{23} = e_i \otimes e_j \otimes e^i e^j, \qquad T_{13}T_{12} = e_i e_j \otimes e^i \otimes e^j.$$

For the second identity we have used the fact that $T \in A^{\mathrm{op}} \otimes A^*$, because we have to reverse the multiplication:

$$T_{13}T_{12} = (e_j \otimes 1 \otimes e^j)(e_i \otimes e^i \otimes 1) = e_i e_j \otimes e^i \otimes e^j.$$

Proof

With $T = e_i \otimes e^i$, consider $(\Delta \otimes 1)T \in A^{\mathrm{op}} \otimes A^{\mathrm{op}} \otimes A^*$. Expand

$$(\Delta \otimes 1)T = e_i \otimes e_j \otimes \lambda^{ij},$$

where $\lambda^{ij} \in A^*$ are to be determined. Let c_{ij}^k be the coefficients determined by

$$\Delta e_k = c_{ij}^k \, e_i \otimes e_j.$$

We have $\lambda^{ij} = c_{ij}^k e^k$ because

$$c_{ij}^k e_i \otimes e_j \otimes e^k = (\Delta \otimes 1)(e_k \otimes e^k) = (e_i \otimes e_j \otimes \lambda^{ij}).$$

On the other hand

$$c_{ij}^k = \langle e^i \otimes e^j, \Delta(e_k) \rangle = \langle e^i e^j, e_k \rangle$$

so

$$(\Delta \otimes 1)T = c_{ij}^k (e_i \otimes e_j \otimes e^k) = \langle e^i e^j, e_k \rangle (e_i \otimes e_j \otimes e^k) = T_{13} T_{23}.$$

The other identity is proved similarly.

Reminder: the convolution theory and T^{-1}

We proved earlier that if H is a finite-dimensional Hopf algebra and

$$T = e_i \otimes e^i \in H \otimes H^*$$

(implied summation) then $T^{-1} = S(e_i) \otimes e^i$. We would like to apply this result with $H = A^{\mathrm{op}}$. We note that $H^* = (A^{\mathrm{op}})^*$ is the same as A^* as an algebra: it has a different comultiplication than A^* but since A and A^{op} have the same comultiplication, A^* and H^* have the same multiplication, and this result applies in $A^{\mathrm{op}} \otimes A^*$. But we have to remember that the antipode of A^{op} is S^{-1}, and so in $A^{\mathrm{op}} \otimes A^*$

$$T^{-1} = S^{-1}(e_i) \otimes e^i.$$

Review: Drinfeld twisting

Proposition (Proved in Lecture 10)

Let H be a Hopf algebra and let F be an invertible element of $H \otimes H$. Assume that

$$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$$

and that $(1 \otimes \varepsilon)(F) = (\varepsilon \otimes 1)(F) = 1$. Define

$$\Delta_F(x) = F\Delta(x)F^{-1}.$$

Then we may replace the comultiplication in H by Δ_F to obtain another Hopf algebra with the same algebra structure.

If $H = A^* \otimes A^{\mathrm{op}}$ where A is a finite-dimensional Hopf algebra we will exhibit an invertible F that allows us to twist $H = A^\circ \otimes A$. This will give us the dual of the Drinfeld quantum double $D(A)$.

The twist

We will now work inn $H = A^* \otimes A^{\mathrm{op}}$. We have just proved some identities in $A^{\mathrm{op}} \otimes A^*$, but we will apply those in $H \otimes H = A^* \otimes A^{\mathrm{op}} \otimes A^* \otimes A^{\mathrm{op}}$, which, we note, contains a copy of $A^{\mathrm{op}} \otimes A^*$.

Our goal is to exhibit an element F of $H \otimes H$ that satsifies the hypotheses of the Drinfeld twisting proposition, particularly

$$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F).$$

We define:

$$F = (1_{A^*} \otimes S^{-1} e_i) \otimes (e^i \otimes 1_A)$$

By convolution theory

$$F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A).$$

Proof

We want to show:

$$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$$

We have

$$(\Delta \otimes 1)(e_i \otimes e^i) = T_{13}T_{23}, \qquad (1 \otimes \Delta)T = T_{13}T_{12}$$

and since $F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A)$,

$$(\Delta \otimes 1)F^{-1} = (F^{-1})_{13}(F^{-1})_{23}, \qquad (1 \otimes \Delta)F^{-1} = (F^{-1})_{13}(F^{-1})_{12}$$

Therefore

$$(\Delta \otimes 1)F = F_{23}F_{13}, \qquad (1 \otimes \Delta)F = F_{12}F_{13}.$$

Proof (concluded)

Now F_{23} and F_{12} commute since

$$F_{23} = (1_{A^*} \otimes 1_A) \otimes (1_{A^*} \otimes S^{-1} e_i) \otimes (e^i \otimes 1_A),$$

$$F_{12} = (1_{A^*} \otimes S^{-1} e_i) \otimes (e^i \otimes 1_A) \otimes (1_{A^*} \otimes 1_A).$$

So

$$F_{12}(\Delta \otimes 1)(F) = F_{12}F_{23}F_{13} = F_{23}F_{12}F_{13} = F_{23}(1 \otimes \Delta)(F).$$

We also need $(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1$, but this may be deduced from the identity

$$\varepsilon(e_i) \otimes e^i = e_i \otimes \varepsilon(e^i) = 1$$

in $A^{\mathrm{op}} \otimes A^*$, which follows from convolution theory.

Summary

The ring that we have constructed is not $D(A)$, but its dual $D(A)^*$. As a ring, it is $A^* \otimes A^{\mathrm{op}}$. The comultiplication has been modified by twisting, that is:

$$\Delta_F(x) = F\Delta(x)F^{-1}$$

where

$$F = (1_{A^*} \otimes S^{-1}e_i) \otimes (e^i \otimes 1_A).$$

The property of quasitriangularity is not self-dual. So $D(A)$ will turn out to be quasitriangular, and its category of modules is braided. The ring we have constructed, $D(A)^* = (A^* \otimes A^{\mathrm{op}})^F$ (where the notation connotes twisting by F is not quasitriangular but dual quasitriangular. This implies that its category of comodules is braided.

Historical origins

The origin of quantum groups came out of the Quantum Inverse Scattering Method, a technique for studying integrable systems developed in St. Petersburg by Faddeev and his students, including Kulish, Sklyanin, Reshetikhin, Takhtajan, Korepin, Izergin and Semenov-Tian-Shansky. They found an algebraic structure underlying applications of the Yang-Baxter equation. For an informative account see the following paper of Faddeev.

▸ Faddeev: History and Perspectives of Quantum Groups

A key feature of this story is the RTT equation, a kind of parametrized Yang-Baxter equation. Faddeev calls it the Fundamental commutation relation and introduces it by the example of the XXX Heisenberg spin chain Hamiltonian.

The RTT equation

The equation in question can be written

$$R(\lambda-\mu)L_1(\lambda)L_2(\mu)=L_2(\mu)L_1(\lambda)R(\lambda-\mu).$$

Here R there are vector spaces V and W such that R acts on $V\otimes V$ and L acts on $V\otimes W$. Both sides act on $V\otimes V\otimes W$. $L_1(\lambda)$ is the operator $L(\lambda)$ applied to the first and third component, and $L_2(\mu)$ is the operator $L(\mu)$ acting on the second and third component.

In our usual notation we might write this identity

$$R_{12}(\lambda-\mu)L_{13}(\lambda)L_{23}(\mu)=L_{23}(\mu)L_{13}(\lambda)R_{12}(\lambda-\mu).$$

The parametrized Yang-Baxter equation

In addition to the RTT equation

$$R_{12}(\lambda-\mu)L_{13}(\lambda)L_{23}(\mu) = L_{23}(\mu)L_{13}(\lambda)R_{12}(\lambda-\mu). \quad (1)$$

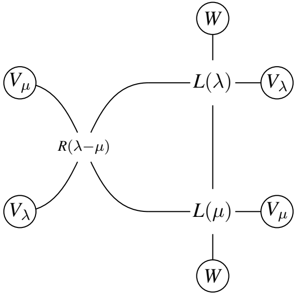
We will have another Yang-Baxter equation:

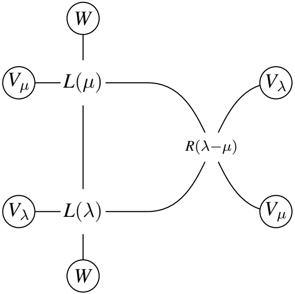
$$R_{12}(\lambda-\mu)R_{13}(\lambda-\nu)R_{23}(\mu-\nu) = R_{23}(\mu-\nu)R_{13}(\lambda-\nu)R_{12}(\lambda-\mu) \quad (2)$$

We have written the parameter group additively, though there is one unusual case that we are aware of where it is nonabelian. Usually the parameter group is $\mathbb{C}$, $\mathbb{C}^\times$ or an elliptic curve. The elliptic curve cases arise in the eight-vertex model, or the XYZ Hamiltonian (Baxter).

The RTT relation

We already saw a version of the RTT relation in the parametrized Yang-Baxter equation that was used in Lecture 6 on solvable lattice models. We recall that there are two vector spaces V and W. It is sometimes convenient to label each copy of V by a parameter.





The case where $W = K$

We have written the relation (1) in the form

$$R_{12}(\lambda-\mu)L_{13}(\lambda)L_{23}(\mu) = L_{23}(\mu)L_{13}(\lambda)R_{12}(\lambda-\mu).$$

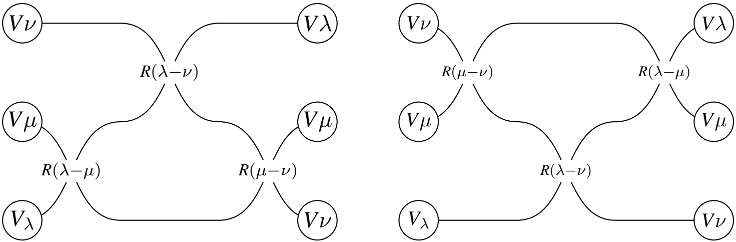
where Faddeev just writes

$$R_{12}(\lambda-\mu)L_{1}(\lambda)L_{2}(\mu) = L_{2}(\mu)L_{1}(\lambda)R_{12}(\lambda-\mu).$$

There are 3 vector spaces involved: V_λ, V_μ and W. So omitting the 3 subscript means treating W as less imporant. If $W = K$ we definitely omit it and the RTT relation looks like:

The parametrized Yang-Baxter equation

The other parametrized Yang-Baxter equation (2), which does not involve $L(\lambda)$ can be diagrammed this way.



Quantum group interpretation

In the quantum group interpetation, the V_λ and W should be objects in a braided category. Potentially this is the category of finite-dimensional modules of a quantum group. In Lecture 6, the quantum group was $U_q(\widehat{\mathfrak{sl}}_2)$, where the "hat" denotes affinization. In this case, there is one two dimensional module V_λ for each $\lambda \in \mathbb{C}^\times$. Also in this case, the module W is chosen from this family, but in other cases, it might not be.

In a limiting case, the modules V_λ may all be the same. Then the RTT relation and the Yang-Baxter equation look like this:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \qquad R_{12}T_1T_2 = T_2T_1R_{12},$$

or

$$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}.$$

The FRT construction

Faddeev, Reshetikhin and Takhtajan solved the following problem. Given a solution of the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

produce a quasitriangular Hopf algebra with R-matrix R. Actually they constructed the dual Hopf algebra as follows. They considered the RTT relation in the form:

$$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}$$

to be an identity in involving two copies of a matrix T. They took the entries in T to be noncommuting indeterminates, subject to the relations implied by this identity. They showed that the ring generated by these indeterminates may be given the structure of a dual quasitriangular Hopf algebra. This is an important construction of quasitriangular Hopf algebras.

Hopf algebra interpretation

We proved earlier today, for an arbitrary Hopf algebra A, the following identities. Let $T = e_i \otimes e^i$ be the canonical element of $A \otimes A^*$. Then

$$(\Delta \otimes 1)(T) = T_{13}T_{23}, \qquad (1 \otimes \Delta)(T) = T_{13}T_{12}$$

That is,

$$T_{13}T_{23} = e_i \otimes e_j \otimes e^i e^j, \qquad T_{13}T_{12} = e_i e_j \otimes e^i \otimes e^j.$$

In the special case where A is quasitriangular, will prove

$$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}.$$

Thus the canonical element satisfies the same identity as the Lax operator that appears in the RTT equation.

Proof

Apply τ to the first two components in

$$(\Delta \otimes 1)(T) = T_{13}T_{23} = e_i \otimes e_j \otimes e^i e^j.$$

We get:

$$(\tau\Delta \otimes 1)(T) = T_{23}T_{13} = e_j \otimes e_i \otimes e^i e^j.$$

Remember $\tau\Delta(x) = R\Delta(x)R^{-1}$ or

$$(\tau\Delta \otimes 1)(x \otimes y) = R_{12}\Delta(x \otimes y)R_{12}^{-1}.$$

So

$$T_{23}T_{13} = R_{12}T_{13}T_{23}R_{12}^{-1}$$

proving

$$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}.$$