Lecture 10

Daniel Bump

May 29, 2019

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_q E_i^{1-a_{ij}-k} E_j E_i^k,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_q F_i^{1-a_{ij}-k} F_j F_i^k,$$

Borel subgroups

A special role in the theory of Lie groups is played by the Borel subgroups. For $\mathrm{GL}(r)$ these are the subgroups conjugate to the group B of upper triangular matrices.

If T is a maximal torus, then the Lie algebra $\mathfrak{g}$ decomposes into T-eigenspaces, called root spaces. The roots are decomposed into positive and negative roots. The Lie algebra of B consists of just the positive root spaces, together with the Lie algebra of T.

There is an opposite Borel subgroup B_- that includes T and the negative roots. Although B has many conjugates, these two (B and B_-) work together and we will take a closer look at them, using $\mathrm{GL}(3)$ as an example.

The Borel subgroup and its relatives $\mathrm{GL}(3)$

Any complex reductive group G will have Borel subgroups, of which we will pick a particular "positive" one B, and its negative B_-. The intersection of these will be a maximal torus T. Moreover B is a semidirect product TU where U is a unipotent subgroups, and similarly $B_- = TU_-$. For $\mathrm{GL}(3)$:

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \qquad T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix},$$

$$B_- = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}, \qquad U_- = \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix}.$$

Note that U is normal in B and $T \cong B/U$.

The Bruhat decomposition

Let $N(T)$ be the normalizer of T. Then $W = N(T)/T$ is the Weyl group, famous for acting on everything in sight. We have the Borel decomposition:

$$G = \bigcup_{w \in W} BwB \qquad \text{(disjoint)}.$$

In theis decomposition, each “Bruhat cell” BwB or BwB_- has dimension $\ell(w)$, the length of w in the Coxeter group W. This is the smallest number k such that w can be written as a product of k simple reflections.

Let w_0 be the long Weyl group element, having maximal length. Then Bw_0B is open and dense in G.

The Big Cell

We have $w_0 B w_0 = B_-$. We have another decomposition

$$G = \bigcup_{w \in W} B w B_- \qquad \text{(disjoint)}$$

which may be deduced from the first. In this decomposition it is BB_- that is the open cell. We may write $B = UT$ and $B_- = TU_-$.

So $UTU_- = BB_-$ is dense in G. In fact the multiplication map:

$$U \times T \times U_- \longrightarrow G$$

is a homeomorphism of $U \times T \times U_-$ onto its image, which is a dense open subset of G.

The triangular decomposition (continued)

This fact has an analog for the Lie algebras. Let

$$\mathfrak{u} = \mathrm{Lie}(U), \qquad \mathfrak{h} = \mathrm{Lie}(T), \qquad \mathfrak{u}_- = \mathrm{Lie}(U_-).$$

Then

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}_-.$$

This direct sum decomposition is fundamental. Together with the Poincaré-Birkhoff-Witt theorem, it implies

$$U(\mathfrak{g}) = U(\mathfrak{u}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{u}_-).$$

This decomposition is also fundamental.

In which $U_q(\mathfrak{u})$ is not a Hopf algebra

Let $B = TU$ be the standard Borel subgroup of a complex Lie group G. Here T is the standard maximal torus of B and U is the maximal unipotent subgroup. Then U is normal in B but T is not.

Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{u}$ and $\mathfrak{b}$ be the Lie algebras of G, T, U and B. Then each of the enveloping algebras $U(\mathfrak{h})$, $U(\mathfrak{t})$ and $U(\mathfrak{b})$ are all Hopf subalgebras of $U(\mathfrak{g})$, which are closed under both multiplication and comultiplication.

For example, if $G = \mathrm{SL}(2)$ then $\mathfrak{u}$ is spanned by E and since $\Delta(E) = 1 \otimes E + E \otimes 1$ it is closed under comultiplication. This fails for $U_q(\mathfrak{u})$ since

$$\Delta(E) = E \otimes K + I \otimes E.$$

Coideal subalgebras of a Hopf algebra

Dualizing the notion of an ideal of an algebra, a left coideal I in a coalgebra A is a subspace such that $\Delta(I) \subset A \otimes I$.

Thus $U_q(\mathfrak{u})$, if we define it to be the subalgebra of $U_q(\mathfrak{g})$ generated by E is not a Hopf algebra, but it is a right coideal in the Hopf algebra $U_q(\mathfrak{g})$.

Thus closed Lie groups of G correspond to Lie subalgebras of $\mathfrak{g}$, or to Hopf subalgebras of $U(\mathfrak{g})$, but when we pass to the quantum group, the corresponding subalgebra of $U_q(\mathfrak{g})$ is not always a Hopf subalgebra. But it may be a coideal subalgebra, and coideal subalgebras are the key to quantum analogs of symmetric spaces.

Letzter: Coideal subalgebras and quantum symmetric pairs

Borel subgroups and quasitriangularity

It is easy enough to construct $U_q(\mathfrak{sl}_2)$ from scratch. Similar constructions "by hand" for more general quantum groups can be given. But such constructions do not produce the R-matrix.

Drinfeld realized that it is possible to create quasitriangular the quantized enveloping algebra of G from the enveloping algebra of G by a doubling method.

One starts with the quantized enveloping algebra of B, doubles it, notices that there are two copies of $U(\mathfrak{h})$ and removes one of them. This procedure has the advantage of proving quasitriangularity.

Axioms of a Hopf Algebra are self-dual

The axioms of a Hopf algebra are self-dual. That is, each axiom becomes another axiom on the reversing the direction of the arrows. For example the associativity of the multiplication μ and the coassociativity of the comultiplication Δ are dual:

$$
\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\mu \otimes 1} & H \otimes H \\
\big\downarrow {\scriptstyle 1 \otimes \mu} & & \big\downarrow {\scriptstyle \mu} \\
H \otimes H & \xrightarrow{\quad \mu \quad} & H
\end{array}
\qquad\qquad
\begin{array}{ccc}
H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
\big\downarrow {\scriptstyle \Delta} & & \big\downarrow {\scriptstyle \Delta \otimes 1} \\
H \otimes H & \xrightarrow{1 \otimes \Delta} & H \otimes H \otimes H
\end{array}
$$

The Hopf axiom itself is self-dual:

$$
\begin{array}{ccccc}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H & \xrightarrow{1_H \otimes \tau \otimes 1_H} & H \otimes H \otimes H \otimes H \\
\big\downarrow {\scriptstyle \mu} & & & & \big\downarrow {\scriptstyle \mu \otimes \mu} \\
H & & \xrightarrow{\qquad\qquad \Delta \qquad\qquad} & & H \otimes H
\end{array}
$$

The dual of a finite-dimensional Hopf algebra

An immediate consequence is that if H is a finite-dimensional Hopf algebra then the dual vector space H^* is also a Hopf algebra. If $\mu : H \otimes H \to H$ is the multiplication, the dual map $\mu^* : H^* \to H^* \otimes H^*$ becomes the comultiplication in H^* and similarly Δ^* is the multiplication in H^*.

To avoid confusion, let us avoid using the notations μ^* and Δ^* for comultiplication and multipliction in H^*. Instead we will write $m = \Delta^* : H^* \otimes H^* \to H^*$ and $D = \mu^* : H^* \to H^* \otimes H^*$. We will denote the dual pairing $H^* \otimes H \to K$ by $\langle\ ,\ \rangle$.

Implicit in this is the identification $(H \otimes H)^* = H^* \otimes H^*$. So if $\lambda, \mu \in H^*$ then $\lambda \otimes \mu \in H^* \otimes H^*$ is the functional

$$\langle \lambda \otimes \mu, x \otimes y \rangle = \langle \lambda, x \rangle \langle \mu, y \rangle.$$

Duality in Sweedler notation

Now remembering that $D : H^* \to H^* \otimes H^*$ is dual to $\mu : H \otimes H$ gives the identity

$$\langle D\lambda, x \otimes y \rangle = \langle \lambda, \mu(x \otimes y) \rangle = \langle \lambda, xy \rangle.$$

Now we employ Sweedler notation and write $D(\lambda) = \lambda_{(1)} \otimes \lambda_{(2)}$. (As usual, there is an implied summation.) Then

$$\langle \lambda, xy \rangle = \langle \lambda_{(1)}, x \rangle \langle \lambda_{(2)}, y \rangle.$$

Dually, let us remember that the multiplication m in H^* is defined to be dual to the comultiplication in H. This gives us the companion formula

$$\langle \lambda\mu, x \rangle = \langle \lambda, x_{(1)} \rangle \langle \mu, x_{(2)} \rangle.$$

More general dual Hopf algebras

More generally suppose we have two Hopf algebras H and A over a field K with a bilinear pairing $A \times H \to K$. Changing notation slightly we will write the multiplication and comultiplication in both Hopf algebras as μ and Δ. We assume

$$\langle \lambda, xy \rangle = \langle \lambda_{(1)}, x \rangle \langle \lambda_{(2)}, y \rangle, \qquad \langle \lambda\mu, x \rangle = \langle \lambda, x_{(1)} \rangle \langle \mu, x_{(2)} \rangle.$$

We also require

$$\langle 1_A, x \rangle = \varepsilon(x), \qquad \langle \lambda, 1_H \rangle = \varepsilon(\lambda), \qquad \langle S(\lambda), S(x) \rangle = \langle \lambda, x \rangle.$$

Then we say H and A are in duality.

For example let G be an affine algebraic group over $\mathbb{C}$. Let $\mathfrak{g}$ be the Lie algebra of the complex Lie group $G(\mathbb{C})$. We may take $H = U(\mathfrak{g})$ and $A = \mathcal{O}(G)$ to be the affine ring of rational functions on G. Then H and A are Hopf algebras in duality.

The opposite Hopf algebra

Let $(H, \mu, \eta, \Delta, \varepsilon)$ denote the Hopf algebra H with multiplication μ, unit η, comultiplication Δ and unit ε.

Now we may reverse the multiplication, and define $\mu^{\mathrm{op}} = \mu \circ \tau$. So $\mu^{\mathrm{op}}(x \otimes y) = \mu(y \otimes x) = yx$. We do not need to reverse the comultiplication. It is easy to check that $(H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon)$ is a Hopf algebra. For example consider the Hopf axiom:

$$
\begin{array}{ccccc}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H & \xrightarrow{1_H \otimes \tau \otimes 1_H} & H \otimes H \otimes H \otimes H \\
\downarrow{\scriptstyle \mu^{\mathrm{op}}} & & & & \downarrow{\scriptstyle \mu^{\mathrm{op}} \otimes \mu^{\mathrm{op}}} \\
H & & \xrightarrow{\qquad\qquad\Delta\qquad\qquad} & & H \otimes H
\end{array}
$$

This diagram commutes since, both ways to $x \otimes y$

$$(yx)_{(1)} \otimes (yx)_{(2)} = y_{(1)}x_{(1)} \otimes y_{(1)}x_{(2)}.$$

The opposite Hopf algebra (continued)

Alternatively, we may reverse the comultiplication and denote $\Delta^{\mathrm{op}} = \tau \circ \Delta$. Assume that the antipode of H is invertible. Then $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon)$ is also a Hopf algebra. There are thus four Hopf algebras altogether:

$$
\begin{aligned}
H &= (H, \mu, \eta, \Delta, \varepsilon), & H^{\mathrm{op}} &= (H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon), \\
H^{\mathrm{cop}} &= (H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon), & H &= (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \varepsilon).
\end{aligned}
$$

The algebra $(H^*)^{\mathrm{cop}}$ is important in the quasitriangularity story and we will denote it H°. (Again S must be invertible.)

Tensor product of Hopf algebras

If A and B are algebras, then $A \otimes B$ is an algebra with multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. This multiplication $\mu_{A\otimes B}$ is

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{1_A \otimes \tau \otimes 1_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B \quad .$$

Similarly if A, B are coalgebras, so is $A \otimes B$ with comultiplication

$$A \otimes B \xrightarrow{\Delta \otimes \Delta} A \otimes A \otimes B \otimes B \xrightarrow{1_A \otimes \tau \otimes 1_B} (A \otimes B) \otimes (A \otimes B)$$

And if A, B are Hopf algebras, so is $A \otimes B$.

Drinfeld Twisting

Twisting modifies the comultiplication of a Hopf algebra.

Proposition

Let H *be a Hopf algebra and let* F *be an invertible element of* $H \otimes H$. *Assume that*

$$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$$

and that $(1 \otimes \varepsilon)(F) = (\varepsilon \otimes 1)(F) = 1$. *Define*

$$\Delta_F(x) = F\Delta(x)F^{-1}.$$

Then we may replace the comultiplication in H *by* Δ_F *to obtain another Hopf algebra with the same algebra structure.*

Proof (I): coassociativity

The assumption $F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$ implies

$$(\Delta_F \otimes 1)(F)F_{12} = (1 \otimes \Delta_F)(F)F_{23}.$$

For coassociativity we want

$$(\Delta_F \otimes 1)(F\Delta(x)F^{-1}) = (1 \otimes \Delta_F)(F\Delta(x)F^{-1}).$$

Since $\Delta_F : H \to H \otimes H$ is a homomorphism, this is equivalent to

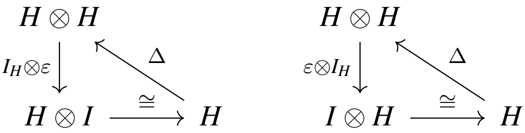
$$(\Delta_F \otimes 1)(F) \cdot F_{12}(\Delta \otimes 1)(\Delta(x))F_{12}^{-1} \cdot (\Delta_F \otimes 1)(F^{-1}) =$$

$$(1 \otimes \Delta_F)(F) \cdot F_{23}(1 \otimes \Delta)(\Delta(x))F_{23}^{-1} \cdot (1 \otimes \Delta_F)(F^{-1}).$$

We see that these are equal.

Proof (concluded)

Note that $\Delta_F(x) = F\Delta(x)F^{-1}$ is an algebra homomorphism $H \to H \otimes H$ because Δ is. The other assumption $(1 \otimes \varepsilon)F = (\varepsilon \otimes 1)(F)$ implies the counit axioms



Indeed, apply $\varepsilon \otimes 1$ to $\Delta_F(x) = F\Delta(x)F^{-1}$. Since the counit $\varepsilon : H \to K$ is a homomorphism we get

$$(\varepsilon \times 1)(F) \cdot (\varepsilon \times 1)(\Delta(x)) \cdot (\varepsilon \times 1)(F)^{-1} = 1 \cdot x \cdot 1^{-1} = x,$$

and similarly $(1 \otimes \varepsilon)\Delta_F(x) = x$.

Reminder of A°

Let A be a finite-dimensional Hopf algebra. Its dual A^* is a Hopf algebra, and $A^\circ = (A^*)^{\mathrm{cop}}$ is obtained from A^* by reversing the comultiplication. Denote the comultiplication in A° by Δ°.

Since A^* and A are in duality if $\lambda, \mu \in A^*$ and $x, y \in A$ we have

$$\langle \lambda\mu, x\rangle = \langle \lambda \otimes \mu, \Delta(x)\rangle, \qquad \langle \Delta\lambda, x \otimes y\rangle = \langle \lambda, xy\rangle \ .$$

These formulas may be taken to be the definitions of the multiplication and comultiplication in A^*. For A°, the multiplication is the came as A^*, but the comultiplication is reversed.

$$\langle \Delta^\circ\lambda, x \otimes y\rangle = \langle \lambda, yx\rangle \ .$$

In Sweedler notation, if $\Delta(\lambda) = \lambda_{(1)} \otimes \lambda_{(2)}$ then

$$\Delta^\circ(\lambda) = \tau\Delta(\lambda) = \lambda_{(2)} \otimes \lambda_{(1)}.$$

A duality theorem

The following result may be due to Drinfeld but we are following Reshetikhin and Semenov-Tian-Shansky, Quantum R-matrices and factorization problems. We will prove:

Theorem

Let A be a finite-dimensional quasi-triangular Hopf algebra with R-matrix $R = R^{(1)} \otimes R^{(2)}$. Define a map $\rho : A^\circ \to A$ by

$$\rho(\lambda) = \langle \lambda, R^{(1)} \rangle R^{(2)}.$$

Then ρ is a Hopf algebra homomorphism.

The homomorphism ρ may not be an isomorphism. For example if A is cocommutative and $R = 1$ then $\rho = \varepsilon$.

Proof (I)

Reminder:

$$\rho(\lambda) = \langle \lambda, R^{(1)} \rangle R^{(2)}.$$

First we prove that $\rho(\lambda\mu) = \rho(\lambda)\rho(\mu)$. Write the axiom $(\Delta \otimes 1)R = R_{13}R_{23}$ in the form

$$\Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes \widetilde{R}^{(1)} \otimes R^{(2)}\widetilde{R}^{(2)}$$

where we've used two copies R on the right-hand side. Now

$$\rho(\lambda\mu) = \langle \lambda\mu, R^{(1)} \rangle R^{(2)}$$

$$= \langle \lambda \otimes \mu, \Delta R^{(1)} \rangle R^{(2)} = \langle \lambda \otimes \mu, R^{(1)} \otimes \widetilde{R}^{(1)} \rangle R^{(2)}\widetilde{R}^{(2)}$$
$$= \langle \lambda, R^{(1)} \rangle R^{(2)} \langle \mu, \widetilde{R}^{(1)} \rangle \widetilde{R}^{(2)} = \rho(\lambda)\rho(\mu).$$

Thus $\rho : A^\circ \to A$ is an algebra homomorphism.

Proof (II)

Now let us show that $(\rho \otimes \rho)\Delta^{\circ}(\lambda) = \Delta\rho(\lambda)$, so that ρ is a coalgebra homomorphism. This time we will use $(1 \otimes \Delta)R = R_{13}R_{12}$, that is

$$R^{(1)} \otimes \Delta R^{(2)} = R^{(1)}\tilde{R}^{(1)} \otimes \tilde{R}^{(2)} \otimes R^{(2)}.$$

Apply $a \otimes b \otimes c \mapsto \langle \lambda, a\rangle (b \otimes c)$ to both sides:

$$\Delta\rho(\lambda) = \langle \lambda, R^{(1)}\rangle \Delta R^{(2)} = \langle \lambda, R^{(1)}\tilde{R}^{(1)}\rangle (\tilde{R}^{(2)} \otimes R^{(2)}) =$$

$$\langle \lambda_{(1)}, R^{(1)}\rangle \langle \lambda_{(2)}, \tilde{R}^{(1)}\rangle (\tilde{R}^{(2)} \otimes R^{(2)})$$

$$= \langle \lambda_{(2)}, \tilde{R}^{(1)}\rangle \tilde{R}^{(2)} \otimes \langle \lambda_{(1)}, R^{(1)}\rangle R^{(2)}$$

$$= \rho(\lambda_{(2)}) \otimes \rho(\lambda_{(1)}) = (\rho \otimes \rho)\Delta^{\circ}(\lambda)$$

since $\Delta^{\circ}(\lambda) = \lambda_{(2)} \otimes \lambda_{(1)}$.

A look ahead to the Drinfeld Double

Let A be a finite-dimensional Hopf algebra. Drinfeld defined a Hopf algebra $D(A)$ which contains copies of A and A°. As a coalgebra it is $A \otimes A^\circ$. However the multiplication is modified.

Since A is finite-dimensional the antipode of A is known to be invertible, which is required for the construction of the double.

The Drinfeld double is quasitriangular. The quasitriangularity of $U_q(\mathfrak{g})$ with q a root of unity can be deduced from this.

Weights

Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. If $\mathfrak{g}$ is the Lie algebra of a complex reductive Lie group G, then $\mathfrak{h}$ is the Lie algebra of a maximal torus T. It is abelian, and its elements are semisimple (diagonalizable) in any representation of G.

The group T is abelian, so its irreducible representations are one-dimensional. The group $X^*(T)$ of characters is called the weight lattice. Each such character (weight) λ induces a linear functional on $\mathfrak{h}$ so sometimes the weight lattice is identified with a subgroup of $\mathfrak{h}^*$.

Root space decomposition

Let Λ be the weight lattice. It may be identified with the group of rational characters of T, or as a group of linear functional on $\mathfrak{h}$. Since T and $\mathfrak{h}$ act on $\mathfrak{g}$ via the adjoint representation, it decomposes into weight eigenspaces. The nonzero characters of T that occur are called roots; the set of roots is denoted Φ. If $\alpha \in \Phi$ let $\mathfrak{X}_\alpha \subset \mathfrak{g}$ be the corresponding eigenspace. It is one-dimensional.

We have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{X}_\alpha.$$

We may decompose $\Phi = \Phi^+ \cup \Phi^-$ with Borel subalgebras

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{X}_\alpha, \qquad \mathfrak{b}_- = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^-} \mathfrak{X}_\alpha.$$

Simple roots

If α is a positive root that cannot be decomposed into other positive roots, then α is called simple. If $\mathfrak{g}$ is semisimple, then the number of simple roots equals the dimension of $\mathfrak{h}$; otherwise of $\mathfrak{h}$ can be slightly larger. Let α_i be the simple roots. Every positive root can be decomposed as a sum of α_i.

For example let $G = \mathrm{GL}(r)$. Then we may identify $\Lambda = \mathbb{Z}^r$. Thus if $\lambda = (\lambda_1, \cdots, \lambda_r)$ we identify λ with the weight (character of T)

$$\lambda \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} = \prod_i t_i^{\lambda_i}.$$

If $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{Z}^r$ the roots are $\mathbf{e}_i - \mathbf{e}_j$ where $i \neq j$. The root is positive if $i < j$ and simple if $j = i + 1$.

Cartan matrix

We choose a W-invariant inner product on the weight lattice Λ. With α_i be simple roots, let

$$a_{ij} = \frac{\langle 2\alpha_i, \beta_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

The matrix $A = (a_{ij})$ is called the Cartan matrix.

For example, if $\mathfrak{g} = \mathfrak{sl}_4$, the Cartan matrix is:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

If $\mathfrak{g}$ is semisimple the Cartan matrix contains enough information to reconstruct $\mathfrak{g}$.

Cartan matrix (continued)

The Cartan matrices that produce finite dimensional Lie algebras were classified by Cartan. In the 1970's Kac showed much theory generalizes to infinite dimensional Lie algebras obtained from more general Cartan matrices. For example

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

produces the affine Lie algebra $\widehat{\mathfrak{sl}}_4$ (See Lecture 6).

Kac-Moody Lie algebras occur naturally in mathematical physics, e.g. as current algebras. Kac-Moody Lie algebras also have nice quantized enveloping algebras.

Chevalley basis

Before Chevalley, finite groups of Lie type such as $\mathrm{SL}(n, \mathbb{F}_q)$ and the finite orthogonal and symplectic groups were constructed on a case-by-case basis. There was interest in these because many finite simple groups were obtained this way; for example the quotient of $\mathrm{SL}(n, \mathbb{F}_q)$ by its center is usually simple.

Chevalley showed that every complex semisimple Lie algebra $\mathfrak{g}$ had a basis with structure constants in $\mathbb{Z}$. Then the adjoint group becomes a group scheme over $\mathbb{Z}$. We can take its points over a finite field and obtain the groups of finite type over finite fields. This procedure had to be supplemented by Galois twisting (Steinberg, Suzuki, Tits, Ree). Together with the alternating groups, finite groups of Lie type account for all but 26 of the finite simple groups.

Serre presentation

If $\mathfrak{g}$ is a Lie algebra, the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the map $\mathrm{ad}(X)(Y) = [X, Y]$. It is a Lie algebra homomorphism.

Let A be a Cartan matrix of finite type. Let r be the rank, that is, the dimension of $\mathfrak{h}$. Then A is $r \times r$. The finite-dimensional semisimple Lie algebra $\mathfrak{g}$ has $3r$ generators H_i, E_i and F_i with relations

$$[H_i, H_j] = 0, \qquad [E_i, F_j] = \delta_{ij} H_i,$$

$$[H_i, E_j] = a_{ij} E_i, \qquad [H_i, F_j] = -a_{ij} F_j,$$

$$\mathrm{ad}(E_i)^{1-a_{ij}} E_j = \mathrm{ad}(F_i)^{1-a_{ij}} F_j = 0, \qquad (i \neq j).$$

The last relations are called the Serre relations. Note that the exponent $1 - a_{ij}$ is positive since $a_{ij} \leqslant 0$ when $i \neq j$.

Serre relations

For example, if $\mathfrak{g} = \mathfrak{sl}_4$,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

the Serre relation tells us $[E_i, E_j] = 0$ if $|i - j| > 1$ but

$$[E_i, [E_i, E_j]] = 0, \qquad j = i \pm 1.$$

To translate the Serre presentation to generators and relations for $U(\mathfrak{g})$ we replace $[A, B]$ by $AB - BA$. So this relation becomes

$$E_i^2 E_j - 2E_i E_j E_i + E_j E_i^2 = 0.$$

Generators and relations for $U(\mathfrak{g})$

To translate the Serre presentation to generators and relations for $U(\mathfrak{g})$ we replace $[A, B]$ by $AB - BA$. We obtain:

$$H_iH_j - H_jH_i = 0, \qquad E_iF_j - F_jE_i = \delta_{ij}H_i,$$

$$H_iE_j - E_jH_i = a_{ij}E_i, \qquad H_iF_j - F_jH_i = -a_{ij}F_i,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} E_i^{1-a_{ij}-k} E_j E_i^k,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} F_i^{1-a_{ij}-k} F_j F_i^k.$$

The Serre relation has been written using binomial coefficients.

The quantized enveloping algebra

To obtain the quantized enveloping algebra, we replace the generators H_i by group-like invertible generators K_i. These are assumed to commute.

The root system is called simply-laced if all roots have the same length. For simplicity we will assume this to be the case. We normalize the inner product on Λ so that $\langle \alpha, \alpha \rangle = 2$ if α is a root. If

$$K_i E_j K_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle} E_j, \qquad K_i F_j K_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

We still have to discuss the Serre relations.

The quantum Serre relations

For the Serre relations we replace the binomial coefficients by the corresponding Gaussian binomial coefficients.

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_q E_i^{1-a_{ij}-k} E_j E_i^k,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_q F_i^{1-a_{ij}-k} F_j F_i^k,$$

Here $i \neq j$ and

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \qquad [m]_{(q)} = \frac{q^m - q^{-m}}{q - q^{-1}},$$

$$[m]_q! = \prod_{k=1}^{m} [k]_q.$$

The quantum Serre relations (continued)

Actually we are assuming that the root system is simply-laced, so if $i \neq j$ then $a_{ij} = 0$ or -1. The quantum Serre relation relation says that if α_i and α_j are orthogonal then E_i and E_j commute, otherwise $a_{ij} = -1$ and

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$$

because

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = [2]_q = q + q^{-1}.$$

comultiplication

The quantized enveloping algebra is a Hopf algebra. The comultiplication is given on generators by:

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

If q is a root of unity, then $U_q(\mathfrak{g})$ has a finite-dimensional quotient that is quasitriangular. If q is not a root of unity, then $U_q(\mathfrak{g})$ is quasitriangular in a generalized sense.

In order to see why these Hopf algebras are quasitriangular we will take a different approach based on the Drinfeld double.