Lecture 1

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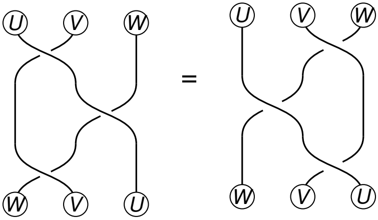


History

Quantum groups were invented in response to developments in mathematical physics. The initial motivation came from:

- Solvable lattice models in statistical mechanics
- Quantum integrable systems such as Heisenberg spin chains

Later important applications were found in Knot Theory and many other important areas. Potential applications exist in topological quantum computing.

My personal interest is in unexpected connections with representation theory of p-adic groups.

Origins in Physics

Solvable Lattice models are statistical mechanical systems, almost always 2 dimensional, that can be completely solved.

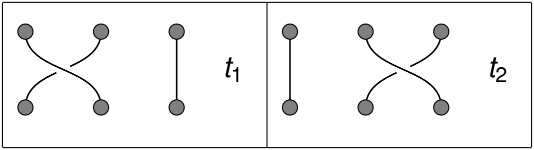
Historically the subject begins with Onsager's 1944 solution of the 2-dimensional Ising model. Another example leading directly to the invention of quantum groups is the six-vertex model.

Baxter invented a powerful tool for studying solvable lattice models, the Yang-Baxter equation.

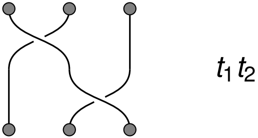
To give a feel for our subject, we will introduce this topic by explaining the Yang-Baxter equation and hinting at how it is applicable to Knot theory. Then we will consider how the YBE leads to quantum group territory.

Digression: the Artin Braid Group

The Braid group B_n models isotopy classes of n braids, which we will read from top to bottom. ▸ Braid Group (Wikipedia Link)
The braid group is generated by braids $t_1, \cdots, t_{n-1}$. If $n = 3$:



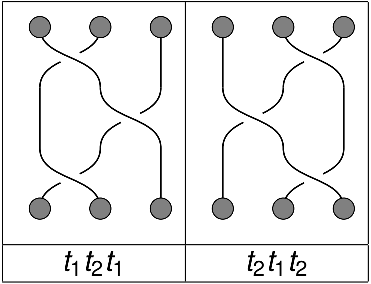
Braids are multiplied by concatenating.



The braid relations

The braid relations are:

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \text{ if } |i-j| > 1.$$

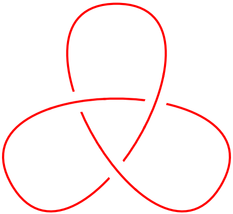


The braids are isotopic, hence we require the braid relation in the algebraic presentation of the braid group

$$B_n = \langle t_1, \cdots, t_{n-1} | \text{braid relations} \rangle .$$

The braid group and knot theory

Clearly the braid group captures some aspects of knot theory, but it is only a beginning.



In the braid group, strands can cross but can only move in one direction. Because of this a knot such as this trefoil knot cannot be modeled in the braid group. The braid group does capture some aspect of knot theory and it will be the first thing we see in quantum group theory.

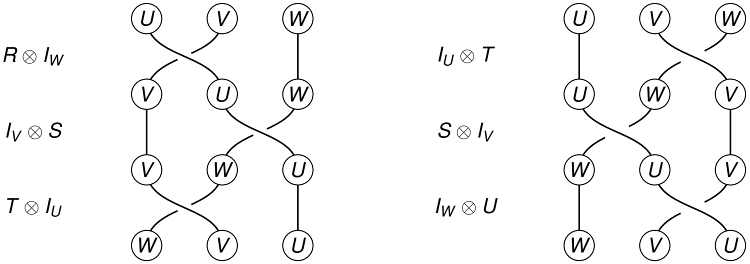
The Yang-Baxter equation (braid version)

Let U, V, W be vector spaces. We need

$$R \in \mathrm{End}(U \otimes V), \qquad S \in \mathrm{End}(U \otimes W), \qquad T \in \mathrm{End}(V \otimes W).$$

This version of the YBE is the identity

$$(T \otimes I_U)(I_V \otimes S)(R \otimes I_W) = (I_W \otimes U)(S \otimes I_V)(I_U \otimes T)\ .$$

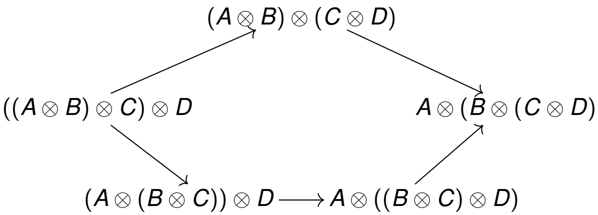


Monoidal Categories

A *monoidal category* is a category with a bifunctor $\otimes$ that is associative, that is we have natural isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

such that all identities such as the pentagon identity



are satisfied. We also require an identity object I with natural isomorphisms $A \otimes I \cong I \otimes A \cong A$. ▸ Monoidal Category (Wikipedia Link)

Examples of Monoidal Categories

The identity object must satisfy the following axiom.

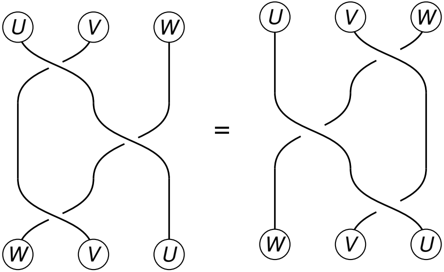
$$\begin{array}{ccc} (A \otimes I) \otimes C & \xrightarrow{\cong} & A \otimes (I \otimes C) \\ & {\scriptstyle \cong \otimes I_C} \searrow \quad \swarrow {\scriptstyle I_A \otimes \cong} & \\ & A \otimes C & \end{array}$$

We will encounter many examples of monoidal categories. For the moment there are two important ones.

- The category of sets. The composition functor is cartesian product $\times$, and the unit object is the set $I = \{1\}$ with one element.
- The category of vector spaces over a field K. The composition is tensor product, and K itself is the unit object.

Braided Monoidal Categories (Joyal and Street)

Since this lecture is introductory we will not give the definition of a braided monoidal category until later. Suffice it to say that it is a monoidal category with a natural morphism $c_{A,B} : A \otimes B \to B \otimes A$ for every pair of objects such satisfying certain axioms. These imply that if U, V, W are objects then the Yang-Baxter equation is satisfied:



Quantum Groups (Drinfeld, Jimbo)

Solutions to the Yang-Baxter equation have powerful applications. Conversely, such solutions have arisen in the process of solving practical problems. Braided monoidal categories are a rich source of solutions to the Yang-Baxter equation, and most often a braided category is found lurking behind such solutions.

Drinfeld formulated the notion of a quasitriangular Hopf algebra (QTHA) in 1986. The main fact is that the category of modules of a QTHA is braided. Hence:

$$\text{Quantum groups} \Rightarrow \text{Braided categories} \Rightarrow \text{applications}$$

where the applications include solvable lattice models, knot invariants such as the Jones polynomial, etc.

Deforming groups

In Drinfeld's view, a quantum group is a deformation of a group G, typically a Lie group. Thus the quantum group depends on a parameter q that can be thought of as $e^{\hbar}$, where $\hbar$ is Planck's constant. Thus in the "classical limit" $q \mapsto 1$ we recover the group G.

However groups are rigid and can't be deformed this way. The solution is to replace the group by another algebraic object whose representation theory is the same as that of G, and which lives in a category that does allow such deformations.

This is the category of Hopf algebras.

From Groups to Hopf Algebras

We would like to understand the notion of a group using maps exclusively. Then we will take the axioms which (in the category of sets) describe a group, and transfer them to the category of vector spaces. This will give the notion of a Hopf algebra.

We begin with the easier notion of a monoid. We will axiomatize a monoid in a way that uses maps instead of elements. Then transferring these axioms to the category of vector spaces will produce the notion of an algebra (if we do it in the simplest way) or a bialgebra (if we do it a better way).

As you know a monoid is a set M with an associative multiplication map $\mu : M \times M \to M$ and a unit element 1_M such that $1_M \cdot x = x \cdot 1_M = x$ where $a \cdot b$ is short for $\mu(a, b)$.

Slogans

- A monoid in the category of vector spaces is an algebra
- A monoid in the category of vector spaces is a bialgebra
- A group in the category of vector spaces is a Hopf algebra

The meaning of these is that if we abstract the notion of a monoid by writing the definition in terms of maps rather than elements, and transport the definition from the category of sets, we obtain a monoid. Monoid (Category Theory) (Wikipedia Link)

But if we repeat the same experiment including more useful structure (the diagonal) we obtain a bialgebra. And if we repeat it for groups, we obtain Hopf algebras.

Categorical notion of a monoid

The category of sets is monoidal with unit element $I = \{1\}$ (the set with one element) and monoidal operation $\times$. Let $\eta : I \to M$ map $1 \mapsto 1_M$ and $\mu : M \times M \to M$ be multiplication. A monoid can be defined in terms of these.

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times 1} & M \times M \\ \downarrow{\scriptstyle 1 \times \mu} & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M \end{array}$$

commutes. Also expressing $1 \cdot x = x \cdot 1 = x$:

$$\begin{array}{ccc} M \times M & & \\ {\scriptstyle I_M \times \eta}\uparrow & \searrow{\scriptstyle \mu} & \\ M \times I & \xrightarrow{\cong} & M \end{array} \qquad \begin{array}{ccc} M \times M & & \\ {\scriptstyle \eta \times I_M}\uparrow & \searrow{\scriptstyle \mu} & \\ I \times M & \xrightarrow{\cong} & M \end{array}$$

A basic monoid of vector spaces is an algebra (I)

Like the category of sets the category of vector spaces over the field K is monoidal category with unit object K and operation $\otimes$. Let A be a vector space with linear maps $\mu : A \otimes A \to A$ and $\eta : K \to M$. Assume commutative:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad \mu \quad} & A \end{array}$$

commutes. Also expressing $1 \cdot x = x \cdot 1 = x$:

$$\begin{array}{ccc} A \otimes A & & \\ \uparrow{\scriptstyle I_A \otimes \eta} & \searrow^{\mu} & \\ A \otimes I & \xrightarrow{\cong} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & & \\ \uparrow{\scriptstyle \eta \otimes I_A} & \searrow^{\mu} & \\ I \otimes A & \xrightarrow{\cong} & A \end{array}$$

A basic monoid of vector spaces is an algebra (II)

Define a multiplication on A by $a \cdot b = \mu(a \otimes b)$.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad \mu \quad} & A \end{array}$$

means $a(bc) = (ab)c$, so A is a ring with unit $1_A = \eta(1_K)$, and η embeds K in the center of A. So A is a K-algebra.

$$\begin{array}{ccc} A \otimes A & & \\ \uparrow{\scriptstyle I_A \otimes \eta} & \searrow^{\mu} & \\ A \otimes I & \xrightarrow{\cong} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & & \\ \uparrow{\scriptstyle \eta \otimes I_A} & \searrow^{\mu} & \\ I \otimes A & \xrightarrow{\cong} & A \end{array}$$

The diagonal

The categorical description of a monoid M can be improved with the diagonal map $\Delta : M \to M \times M$. This is needed to simulate any identity involving more than one copy of an element. Let $\varepsilon : M \to I$ be the constant map $x \to 1$.

$$
\begin{array}{ccc}
M & \xrightarrow{\quad\Delta\quad} & M \times M \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1_M \times \Delta} \\
M \times M & \xrightarrow{\Delta \times 1_M} & M \times M \times M
\end{array}
$$

$$
\begin{array}{ccc}
M \times M & & \\
{\scriptstyle I_M \times \varepsilon}\downarrow & \nwarrow{\scriptstyle \Delta} & \\
M \times I & \xrightarrow{\cong} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M \times M & & \\
{\scriptstyle \varepsilon \times I_M}\downarrow & \nwarrow{\scriptstyle \Delta} & \\
I \times M & \xrightarrow{\cong} & M
\end{array}
$$

Properties of the diagonal (I)

We took the basic properties of a monoid (associative law and unit properties), transferred them to an object in the category of vector spaces, and obtained an algebra.

But we want to add some properties of the diagonal map. Transferred to the category of vector spaces, these will give the notion of a bialgebra.

Let $\tau : M \to M$ be the map $\tau : (x, y) \to (y, x)$.

$$\begin{array}{ccccc} M \times M & \xrightarrow{\Delta \times \Delta} & M \times M \times M \times M & \xrightarrow{1_M \times \tau \times 1_M} & M \times M \times M \times M \\ \downarrow \mu & & & & \downarrow \mu \times \mu \\ M & & \xrightarrow{\qquad\qquad \Delta \qquad\qquad} & & M \times M \end{array}$$

commutes: both maps $(x, y) \to (xy, xy)$.

Properties of the diagonal (II)

Another couple of properties of the diagonal:

$$\begin{array}{ccc} M \times M & \xrightarrow{\varepsilon \times \varepsilon} & I \times I \\ \downarrow \mu & & \downarrow \cong \\ M & \xrightarrow{\varepsilon} & I \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{\eta} & M \\ \downarrow \cong & & \downarrow \Delta \\ I \times I & \xrightarrow{\eta \times \eta} & M \times M \end{array}$$

Coalgebras

We may dualize the notion of an algebra by reversing the arrows. That is, instead of a multiplication map $\mu : M \otimes M \to M$ we have a comultiplication $\Delta : M \to M \otimes M$ and instead of the unit a counit $\varepsilon : M \to I$. The axioms are dual:

$$\begin{array}{ccc}
M & \xrightarrow{\quad\Delta\quad} & M \otimes M \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle 1_M \otimes \Delta} \\
M \otimes M & \xrightarrow{\Delta \otimes 1_M} & M \otimes M \otimes M
\end{array}$$

$$\begin{array}{ccc}
M \otimes M & & \\
\Big\downarrow{\scriptstyle I_M \otimes \varepsilon} & \nwarrow{\scriptstyle \Delta} & \\
M \otimes I & \xrightarrow{\ \cong\ } & M
\end{array}
\qquad
\begin{array}{ccc}
M \otimes M & & \\
\Big\downarrow{\scriptstyle \varepsilon \otimes I_M} & \nwarrow{\scriptstyle \Delta} & \\
I \otimes M & \xrightarrow{\ \cong\ } & M
\end{array}$$

Bialgebras (I)

Let us collect all our properties and apply them to a K-vector space H. We have maps $\mu : H \otimes H \to H$ and $\Delta : H \to H \otimes H$ called multiplication and comultiplication, and maps $\eta : K \to H$, $\varepsilon : H \to K$ called unit and counit, subject to the following axioms.

Associativity and coassociativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\mu \otimes 1} & H \otimes H \\ \downarrow{\scriptstyle 1_H \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ H \otimes H & \xrightarrow{\mu} & H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1_H \otimes \Delta} \\ H \otimes H & \xrightarrow{\Delta \otimes 1_H} & H \otimes H \otimes H \end{array}$$

Bialgebras (II)

Unit:

$$
\begin{array}{ccc}
H\otimes H & & \\
{\scriptstyle 1_H\otimes\eta}\uparrow & \searrow^{\mu} & \\
H\otimes K & \xrightarrow{\cong} & H
\end{array}
\qquad\qquad
\begin{array}{ccc}
H\otimes H & & \\
{\scriptstyle \eta\otimes K_H}\uparrow & \searrow^{\mu} & \\
K\otimes H & \xrightarrow{\cong} & H
\end{array}
$$

Counit:

$$
\begin{array}{ccc}
H\otimes H & & \\
{\scriptstyle 1_H\otimes\varepsilon}\downarrow & \nwarrow^{\Delta} & \\
H\otimes I & \xrightarrow{\cong} & H
\end{array}
\qquad\qquad
\begin{array}{ccc}
H\otimes H & & \\
{\scriptstyle \varepsilon\otimes I_H}\downarrow & \nwarrow^{\Delta} & \\
I\otimes H & \xrightarrow{\cong} & H
\end{array}
$$

The associative and unit axioms make H an algebra; the coassociative and counit axioms make it a coalgebra.

Bialgebras (III)

The augmentation and coaugmentation axioms:

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\varepsilon \times \varepsilon} & I \times I \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \cong} \\
H & \xrightarrow{\quad \varepsilon \quad} & I
\end{array}
\qquad\qquad
\begin{array}{ccc}
I & \xrightarrow{\quad \eta \quad} & H \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \Delta} \\
I \times I & \xrightarrow{\eta \times \eta} & H \times H
\end{array}
$$

These say that the counit is an algebra homomorphism, and that the unit is a coalgebra homomorphism, respectively.

The Hopf axiom

A bialgebra has one more axiom. We will call this the Hopf axiom

$$\begin{array}{ccccc} H\otimes H & \xrightarrow{\Delta\otimes\Delta} & H\otimes H\otimes H\otimes H & \xrightarrow{1_H\otimes\tau\otimes 1_H} & H\otimes H\otimes H\otimes H \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu\otimes\mu} \\ H & & \xrightarrow{\quad\quad\Delta\quad\quad} & & H\otimes H \end{array}$$

Here $\tau : H \otimes H \to H \otimes H$ is the flip map $\tau(x \otimes y) = y \otimes x$.

With this, the axioms for a bialgebra are complete. It is notable that the axioms are unchanged if we reverse the direction of arrows. Most of the axioms are interchanged in pairs by such a changed, but the Hopf axiom is itself unchanged.

Significance of the Hopf axiom

If A and B are algebras, so is $A \otimes B$ with multiplication

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

If μ_A, μ_B and $\mu_{A\otimes B}$ are the multiplications in A, B and $A \otimes B$ this means

$$\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B).$$

Hence we write the Hopf axiom

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta\otimes\Delta} & (H \otimes H) \otimes (H \otimes H) \\ \downarrow{\scriptstyle \mu_H} & & \downarrow{\scriptstyle \mu_{H\otimes H}} \\ H & \xrightarrow{\quad\Delta\quad} & H \otimes H \end{array}$$

In other words, $\Delta : H \to H \otimes H$ is an algebra homomorphism.

The antipode

Remember the slogan: A group in the category of vector spaces is a Hopf algebra.

Let G be a group, and let $S : G \to G$ be the map $S(g) = g^{-1}$. We need to express the axiom $g \cdot g^{-1} = g^{-1} \cdot g = 1$ in terms of maps. We have at our disposal the diagonal map, the multiplication map, the unit and counit maps, and the solution to this problem will involve all of them.

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G \times G & \xrightarrow{1_G \times S} & G \times G \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\eta\quad} & & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G \times G & \xrightarrow{S \times 1_G} & G \times G \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\eta\quad} & & G \end{array}$$

Both directions are the map $g \mapsto 1_G$.

Hopf Algebras

So a Hopf algebra is a bialgebra H with a linear map $S : H \to H$ satisfying the additional axiom:

$$\begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{1_H \otimes S} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\quad \eta \quad\quad} & & H \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{S \otimes 1_H} & H \otimes H \\ \downarrow{\scriptstyle \varepsilon} & & & & \downarrow{\scriptstyle \mu} \\ K & & \xrightarrow{\quad\quad \eta \quad\quad} & & H \end{array}$$

This axiom is self dual, so reversing all arrows in the definition of a Hopf algebra does not change the definition.

Examples

Since we derived the notion of a Hopf algebra by abstracting the notion of a group, it is not surprising that Hopf algebras can be derived from a group by various procedures. There are two ways of obtaining a Hopf algebra from a finite group G.

The group algebra $K[G]$ has comultiplication $\Delta(g) = g \otimes g$ on a basis element $g \in G$, and it is a Hopf algebra.

Dually, the ring $\mathcal{O}(G)$ of functions on G is a Hopf algebra. The multiplication is pointwise (so this algebra is commutative). The comultiplication is obtained by identifying

$$\mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G),$$

so to define $\Delta(f)$ we need to describe a function on $G \times G$. This is the function $\Delta(f)(g, h) = f(gh)$.

Lie groups

If G is an affine algebraic group over $\mathbb{C}$, then the group $G(\mathbb{C})$ of complex points of G is a Lie group. Due to the theory of Chevalley we know that every semisimple Lie group such as SL_n is an algebraic group in this way.

There are Two Hopf algebras functorially associated with G. Let $\mathfrak{g}$ be the Lie algebra of G.

- The universal enveloping algebra $U(\mathfrak{g})$
- The affine algebra or coordinate ring $\mathcal{O}(G)$.

Both admit quantum deformations.

The enveloping algebra (I)

The Universal enveloping algebra $U = U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is characterized by the following universal property. First, it is an associative ring containing $\mathfrak{g}$ as a vector subspace, and if $X, Y \in \mathfrak{g}$ then

$$[X, Y] = XY - YX$$

where the bracket operation is in $\mathfrak{g}$, and the multiplications on the right are in $U(\mathfrak{g})$.

Universal property If A is any associative algebra and $\rho : \mathfrak{g} \to A$ is a linear map such that

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

then ρ extends to an algebra homomorphism $U \to A$.

The enveloping algebra (II)

Elements of $\mathfrak{g}$ can be identified with left invariant vector fields on $G(\mathbb{C})$. These are differential operators. Indeed if $X \in \mathfrak{g}$, if f is a smooth function on $G(\mathbb{C})$ we may define Xf to be the function obtained by differentiating f along this vector field. So the enveloping algebra may be defined to be the ring of left invariant differential operators generated by $\mathfrak{g}$.

The enveloping algebra may be also identified with the convolution ring of distributions on G concentrated at the identity.

It has a comultiplication, counit and antipode making it a Hopf algebra. If $X \in \mathfrak{g}$ then

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \epsilon(X) = 0, \qquad S(X) = -X.$$

The enveloping algebra of SL_2

The Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ consists of 2×2 complex matrices of trace zero. As a vector space, it is 3 dimensional generated by:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with bracket operations

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y.$$

So the enveloping algebra is the associative algebra generated by three elements X, Y, H subject to the relations

$$XY - YX = H, \qquad HX - XH = 2X, \qquad HY - YH = -2Y.$$

Caution: The multiplication here is not matrix multiplication, but multiplication in the enveloping algebra.

Function algebras

The other type of Hopf algebra associated with algebraic groups is the ring $\mathcal{O} = \mathcal{O}(G(\mathbb{C}))$ of rational (polynomial) functions. Unlike the enveloping algebra this is a commutative algebra, and the multiplication is encoded in the comultiplication.

If V is any affine algebraic variety and $\mathcal{O}(V)$ is its affine algebra we may identify

$$\mathcal{O}(V \times V) \cong \mathcal{O}(V) \otimes \mathcal{O}(V)$$

and so the multiplication morphism $G \times G \to G$ corresponds to an algebra homomorphism $\Delta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$. The inverse map is realized as an antipode and $\mathcal{O}$ is a Hopf algebra.

Dual pairing

Let us contrast $U = U(G)$ with $\mathcal{O} = \mathcal{O}(G)$.

The universal enveloping algebra is cocommutative, i.e. $\tau \circ \Delta = \Delta$ if $\tau : U \otimes U \to U \otimes U$ is the flip map $x \otimes y \to y \otimes x$. The multiplication encodes the multiplication of the group.

The function algebra is commutative, and the comultiplication encodes the multiplication of the group.

If we identify $U(G)$ as the algebra of distributions on G that are concentrated at the identity, there is a pairing $U(G) \times \mathcal{O}(G) \to \mathbb{C}$, namely we may apply a distribution to a function to obtain a complex number. We often encounter Hopf algebras in duality this way.

Quantum Groups

Both $U(G)$ and $\mathcal{O}(G)$ have deformations that involve a parameter q, and these are the subject of our topic.

The specialization $q \to 1$ should recover the classical non-deformed theory, while the specialization $q \to 0$ leads to the theory of crystal bases.

Even the case $G = \mathrm{SL}(2)$ is enough for significant applications such as the Jones polynomial in knot theory, or the six-vertex model in statistical mechanics.