BRAIDED CATEGORIES

A BRAIDED CATEGORY (JOYAL AND STREET 1986) IS A MONOCAIRAL CATEGORY $C$; IF $A, B \in C$ THERE IS DEFINED A COMMUTATIVITY CONSTRAINT

$C_A: A \otimes B \rightarrow B \otimes A$

THAT IS NATURAL. IF $\rho: A \rightarrow A'$
$\sigma: B \rightarrow B'$ THERE IS A COMMUTATIVE DIAGRAM

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{(\rho \otimes \sigma)} & A' \otimes B' \\
\downarrow C_{A,B} & & \downarrow C'_{A',B'} \\
B \otimes A & \xleftarrow{\sigma \otimes \rho} & B' \otimes A'
\end{array}
\]

REPRESENT $\rho: A \rightarrow A'$ AS A DOT ON A LINE:

\[
\begin{array}{ccc}
& \bullet & \\
| & \downarrow \rho & \\
\bullet & \rightarrow & \bullet
\end{array}
\]

\[
\begin{array}{ccc}
& \bullet & \\
| & \uparrow \rho' & \\
\bullet & \leftarrow & \bullet
\end{array}
\]
TWO AXIOMS.

\[(A \odot B) \odot C\]

\[\alpha_{A,B,C}\]

\[(B \odot A) \odot C\]

\[\alpha_{B,A,C}\]

\[B \odot (A \odot C)\]

\[\odot_{B,A,C}\]

\[B \odot (C \odot A)\]

"HEXAGON EQUATION"

IN A STRICT MONOIDAL CATEGORY ELIMINATE ASSOCIATORS
Also \[ A \bowtie B \bowtie C \xrightarrow{\text{CBA}} C \bowtie A \bowtie B \]
\[ A \bowtie C \bowtie B \xrightarrow{\text{C}_A \bowtie \bowtie \text{C}_B} C \bowtie A \bowtie B \]

First Axiom:

\[ A \bowtie B \bowtie C \xrightarrow{\text{C}_A \bowtie \bowtie \text{C}_B} C \bowtie A \bowtie B \]
\[ B \bowtie A \bowtie C \xrightarrow{\text{I}_B \bowtie \bowtie \text{C}_A} C \bowtie A \bowtie B \]

I will show this implies Yang Baxter Equation.

\[ \left\{ \begin{array}{c}
    C : A \bowtie B \rightarrow B \bowtie A \\
    C : B \bowtie A \rightarrow A \bowtie B
\end{array} \right. \] not assumed inverses
Yang Baxter Equation

To reduce this we need both braided axioms and naturality
Use naturality to move the last morphism first.

Unravel this to get the other side.

YBE is true in a braided category.
Example of a braided category:

Category of Braids

$B_n$: braid group

two parallel planes in $\mathbb{R}^3$

$w$ marked points in each plane.

A braid is an isotopy class of monotone smooth curves between the 2 planes connecting dots in some permutation.

Multiplication comes from stacking 2 braids.

$w = 3$

$w = 4$
IF \( n \) POINTS \( P_1, \ldots, P_n \) (top) \( Q_1, \ldots, Q_n \) (bot.)

THE \( m_i \) \( (i = 1, \ldots, n-1) \) generate \( B_n \).

BRAID RELATION: \( b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \)

IF \( |i-j| > 1 \) \( m_i m_j = m_j m_i \)
Braid relation

\[ B_n = \langle \mu_1, \ldots, \mu_{n-1} \mid \text{braid relations} \rangle \]
Given a braid we have a permutation $b$

$\sigma_i(b) = p_i$ is connected to $q_{<0}$

$\sigma_i$ is a hom $B_n \to S_n$ symmetric group.

In Lie theory it is proved $S_n$ (type $A_{n-1}$ Weyl group) is a Coxeter group. With presentation

$S_n = \langle A_1, \ldots, A_n | \text{braid relations} \rangle$

Braid category $B$ has 1 element for each $n = 0, 1, 2, \ldots$

$\text{Hom}(n, n) = \emptyset$ unless $n = m$.

$\text{Hom}(n, n) = B_n$.

Composition of morphisms $= \text{multiplication of braids}$. 
$\mathcal{B}$ is monoidal, \( \mathfrak{m} \circ \mathfrak{n} = \mathfrak{n} + \mathfrak{m} \).

If \( f : \mathfrak{m} \rightarrow \mathfrak{m} \), \( g : \mathfrak{n} \rightarrow \mathfrak{n} \), then \( f \circ g : \mathfrak{m} \circ \mathfrak{n} \rightarrow \mathfrak{n} + \mathfrak{m} \).

This is just the juxtaposition of the braids.

\( \mathfrak{n} = 2 \), \( \mathfrak{m} = 3 \).

\[ \mathfrak{m}^2 : 2 \rightarrow 2 \quad \mathfrak{n}, \mathfrak{m}_2 : \mathcal{B} \rightarrow 3 \]

\[ \mathfrak{m}^2 \circ \mathfrak{n} \circ \mathfrak{m}_2 : S \rightarrow S \]

This category \( \mathcal{B} \) is a braided category.

(think about this.)
\[ C_{2,3} : S \rightarrow S \]

**This is a braided category where**

\[ C_{A,B} \neq C_{B,A} \]

**Categories of modules over quasi-triangular Hopf algebras**

(see Majid Ch. 5) are braided.

\[ U_q(SL_2) \text{ and Jones polynomial.} \]

**Hopf algebras**
An algebra can be formulated as a vector space with linear maps

\[ \lambda : A \otimes A \rightarrow A \otimes A \]

associative

\[ A \otimes A \otimes A \xrightarrow{\lambda_3} A \otimes A \]

commutes

\[ \begin{array}{ccc}
1 & A & \otimes \lambda \\
\downarrow & \downarrow & \downarrow \\
A & A & \otimes A
\end{array} \]

unit \( \eta : F \rightarrow A \) linear map

\[ \begin{array}{ccc}
A & \xrightarrow{\eta} & A \otimes F \\
\| & \downarrow & \downarrow \circ \eta \\
A & \xleftarrow{\eta} & A \otimes A \\
\end{array} \]

We can dualize these axioms and arrive at a coalgebra \( C \).

\[ \Delta : C \rightarrow C \otimes C \]

cofactor

\[ \varepsilon : C \rightarrow F \]

\( \nu y = \nu(x \circ y) \)
\[ C \xrightarrow{\Delta} C \otimes C \]
\[ \cong \]
\[ C \xrightarrow{\Delta} C \otimes C \]
\[ C \xrightarrow{\cong} C \otimes C \]
\[ \cdot \]

**Notation for Coalgebras & Hopf alg. Sweedler (Majid style)**

\[ \Delta x = \sum x_i \otimes x_j \quad \text{for some } x_i, x_j \in C \]

Instead write \[ \Delta x = \sum x_i \otimes x_{(i)} \]

(Conop is quickly)

\[ (\Delta \otimes I) \Delta x = (1 \otimes \Delta) \Delta x \]

\[ (\Delta \otimes I) (x_{(1)} \otimes x_{(2)}) = x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)} \]

So Coassociativity in Sweedler notation

\[ x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)} \]
Knowing this write both sides as
\[ x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\Delta \otimes 1) \Delta x = (1 \otimes \Delta) \Delta x = \Delta^3 x. \]

Axioms for a Hopf Alg:

Both an Alg & CoAlg.

Comultiplication \( \Delta : H \rightarrow H \otimes H \) is algebra Hom. Equivalently, \( \mu : H \otimes H \rightarrow H \) is a coalgebra Hom.

\[
\begin{array}{c}
H \otimes H \\
\downarrow \Delta \\
H \\
\end{array} \xrightarrow{\Delta \otimes \Delta} \begin{array}{c}
H \otimes H \otimes H \\
\downarrow \mu \\
H \otimes H \\
\end{array}
\]

Reverse arrows obtain same diagram.

We should think about right arrow.

Introduce \( \tau : H \otimes H \rightarrow H \otimes H \)
\[ \tau(x \otimes y) = yax \]
This map is the composition

\[
\begin{array}{ccc}
\top & \downarrow & \text{H@H@H@H} \\
\text{H@H} & \downarrow & \text{H@H@H@H} \\
\text{H@H@H} & \downarrow & \text{H@H@H@H} \\
\text{H@H} & \downarrow & \text{H@H@H@H} \\
\end{array}
\]

Reverse arrows

\[
\begin{array}{ccc}
\text{H@H} & \rightarrow & \text{H@H@H@H} \\
\text{H@H@H@H} & \rightarrow & \text{H@H@H@H} \\
\end{array}
\]
Sweedler discussion to be continued