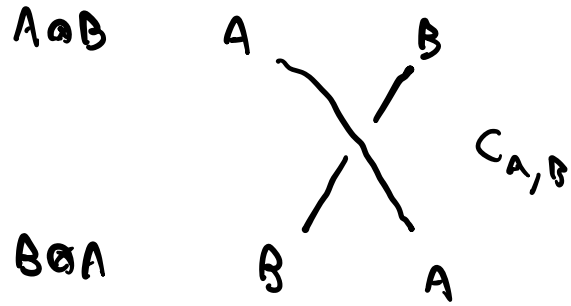
BRAIDED CATEGORIES

A BRAIDED CATEGORY (JOYAL AND STREET 1986) IS A MONOIDAL CATEGORY $\mathcal{C}$: IF $A, B \in \mathcal{C}$ THERE IS DEFINED A COMMUTATIVITY CONSTRAINT

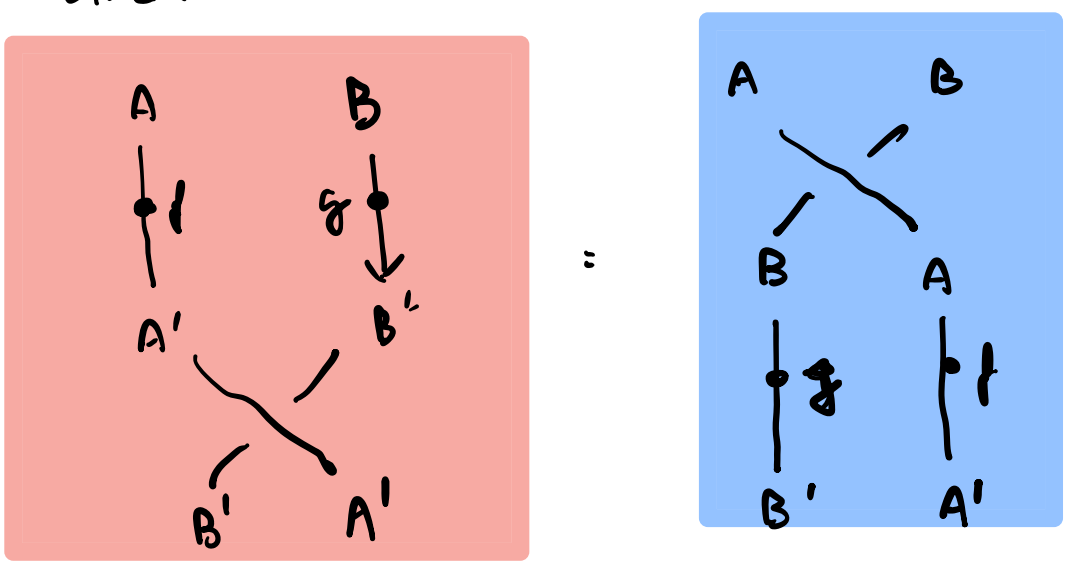
$$c_{A}: A \otimes B \longrightarrow B \otimes A$$

THAT IS NATURAL. IF $f: A \to A'$ $g: B \to B'$ THERE IS A COMMUTIVE DIAGRAM

$$\begin{tikzcd}
A \otimes B \arrow[r, "f \otimes g"] \arrow[d, "c_{A,B}"'] & A' \otimes B' \arrow[d, "c_{A',B'}"] \\
B \otimes A \arrow[r, "g \otimes f"] & B' \otimes A'
\end{tikzcd}$$



REPRESENT $f: A \to A'$ AS A DOT ON A LINE:



TWO AXIOMS.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[dr, "\alpha_{A,B,C}"] \arrow[d, "c_{A,B} \otimes 1_C"'] & \\
(B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] & A \otimes (B \otimes C) \arrow[d, "c_{A,B \otimes C}"] \\
B \otimes (A \otimes C) \arrow[dr, "1_B \otimes c_{A,C}"] & (B \otimes C) \otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
& B \otimes (C \otimes A)
\end{tikzcd}$$

"HEXAGON EQUATION"

IN A STRICT MONOIDAL CATEGORY ELIMINATE ASSOCIATORS

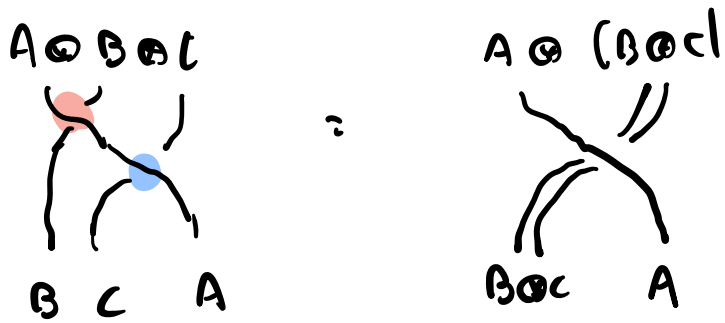
$$\begin{tikzcd}
A \otimes B \otimes C \arrow[d, "c_{A,B} \otimes 1_C"] \arrow[dr, "c_{A,B \otimes C}"] & \\
B \otimes A \otimes C & B \otimes C \otimes A \arrow[ul, "1_B \otimes c_{A,C}"']
\end{tikzcd}$$

ALSO

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[d, "1_A\otimes c_{B,C}"] \arrow[dr, "c_{A\otimes B,C}"] & \\
A\otimes C\otimes B \arrow[r, "c_{A,C}\otimes 1_B"'] & C\otimes A\otimes B
\end{tikzcd}$$

FIRST AXIOM:

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[d, "c_{A,B}\otimes 1_C"] \arrow[dr, "c_{A,B\otimes C}"] & \\
B\otimes A\otimes C \arrow[r, "1_B\otimes c_{A,C}"'] & B\otimes C\otimes A
\end{tikzcd}$$



I WILL SHOW THIS IMPLIES YANG BAXTER EQUATION.

$$\left\{\begin{array}{l} c_{A,B}: A\otimes B \to B\otimes A \\ c_{B,A}: B\otimes A \to A\otimes C \end{array}\right\} \text{ NOT ASSUMED INVERSES}$$

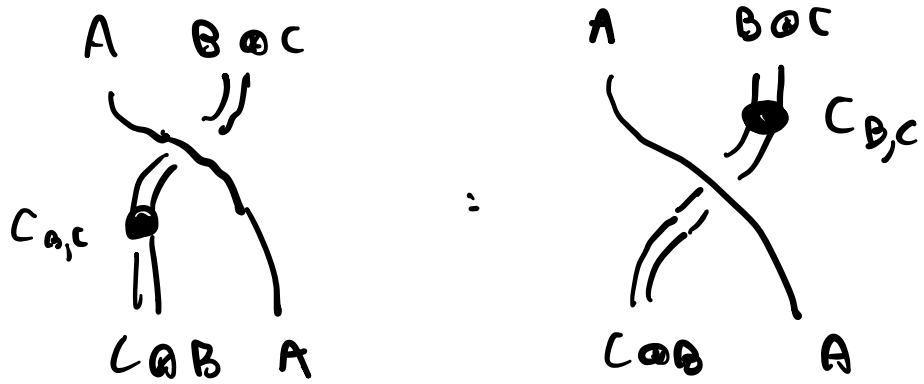
YANG BAXTER EQUATION

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[d, "c \otimes 1_c"'] \arrow[dr] & \\
B \otimes A \otimes C \arrow[d, "1_B \otimes c"] & A \otimes C \otimes B \arrow[d] \\
B \otimes C \otimes A \arrow[dr, "c \otimes 1_A"'] & C \otimes A \otimes B \arrow[d] \\
& C \otimes B \otimes A
\end{tikzcd}$$

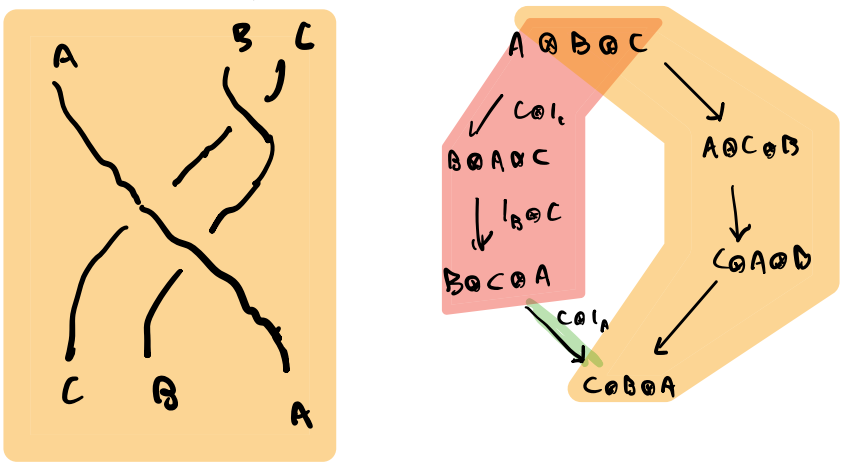
TO REDUCE THIS WE NEED BOTH BRAIDED AXIOMS AND NATURALITY

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[d] \\
B \otimes A \otimes C \arrow[d] \\
B \otimes C \otimes A \arrow[d, "c \otimes 1_A"] \\
C \otimes B \otimes A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \otimes B \otimes C \arrow[d, "c_{A, B \otimes C}"] \\
B \otimes C \otimes A \arrow[d, "c \otimes 1_A"] \\
C \otimes B \otimes A
\end{tikzcd}$$

USE NATURALITY TO MOVE THE LAST MORPHISM FIRST



UNRAVEL THIS TO GET THE OTHER SIDE



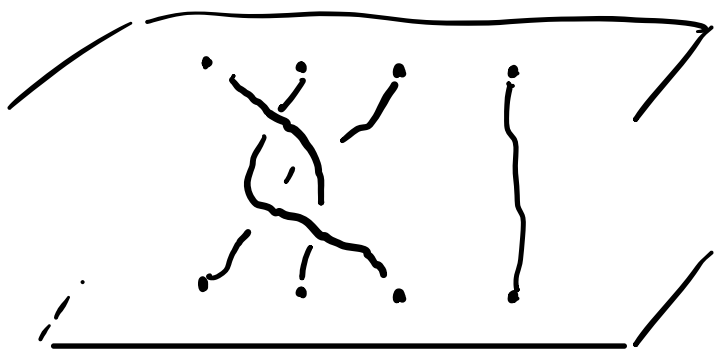
YBE IS TRUE IN A BRAIDED CATEGORY.

EXAMPLE OF A BRAIDED CATEGORY:

CATEGORY OF BRAIDS.

$\mathbb{B}_n$: BRAID GROUP

TWO PARALLEL PLANES IN $\mathbb{R}^3$



$n = 4$

n MARKED POINTS IN EACH PLANE.

A BRAID IS AN ISOTOPY CLASS OF MONOTONIC SMOOTH CURVES BETWEEN THE 2 PLANES CONNECTING DOTS IN SOME PERMUTATION

MULTIPLICATION COMES FROM STACKING 2 BRAIDS

$n = 3$

$\mathbb{N}_1$

σ_2

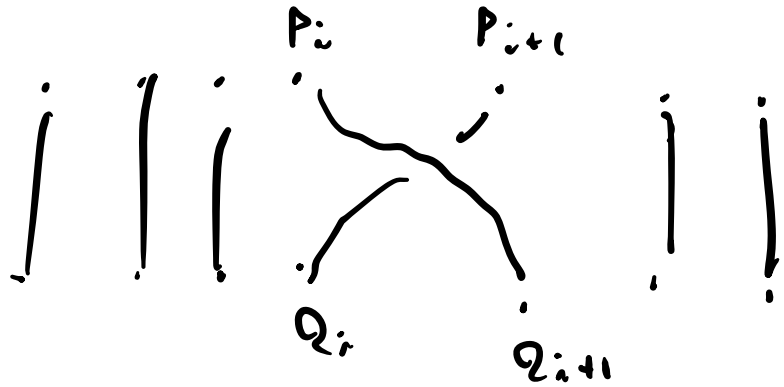
$\sigma_2\sigma_1$:

$\sigma_2\sigma_1$

IF n POINTS $P_1, \dots, P_n$ (TOP)

$Q_1, \dots, Q_n$ (BOT.)

σ_i

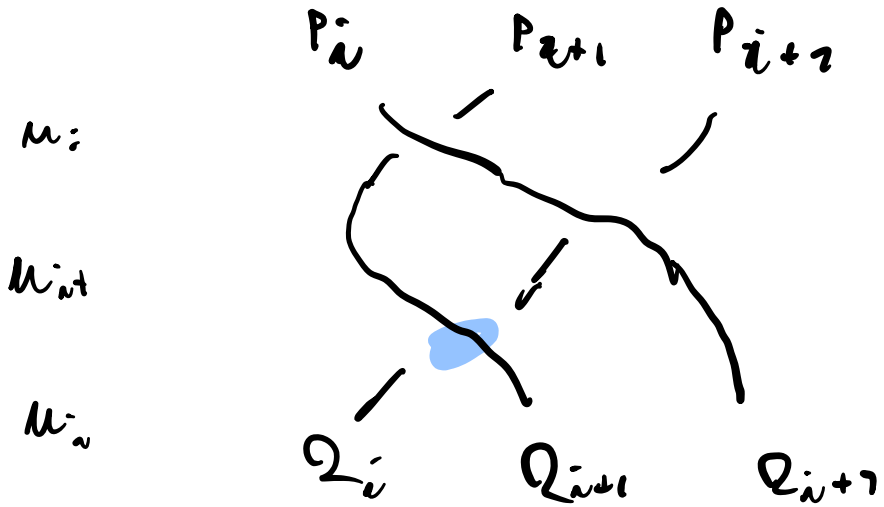


THE σ_i $(i = 1, \dots, n-1)$ GENERATE B_n.

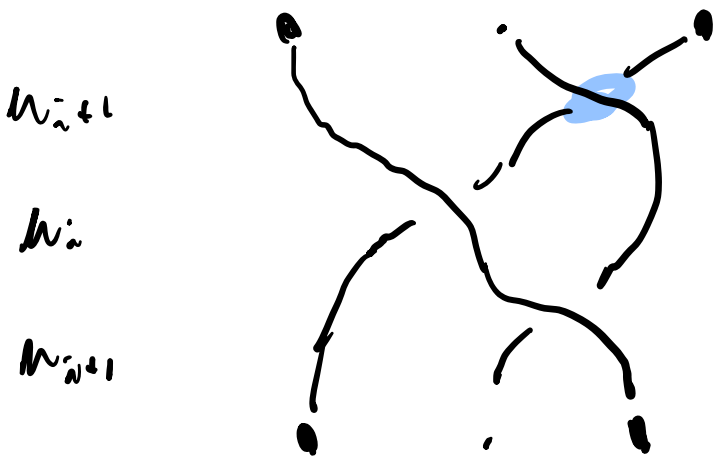
BRAID RELATION: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

IF $|i-j| > 1$ $\sigma_i\sigma_j = \sigma_j\sigma_i$

BRAID RELATION



IS ISOTOPIC TO



$\sigma(u_i u_{i+1} u_i)$
$\shortparallel$
$(i, i+2)$

$$B_n = \langle u_1, \dots, u_{n-1} \mid \text{BRAID RELATIONS} \rangle$$

GIVEN A BRAID b WE HAVE A PERMUTATION

$\sigma(b)$ $\quad P_i$ IS CONNECTED TO $Q_{\sigma(i)}$

σ IS A HOM $\mathcal{B}_n \to S_n$ SYMMETRIC GROUP.

IN LIE THEORY IT IS PROVED

S_n (TYPE A_{n-1} WEYL GROUP)

IS A COXETER GROUP. WITH PRESENTATION

$$S_n = \langle s_1, \dots, s_n \mid \text{BRAID RELATIONS},\ s_i^2 = 1 \rangle$$

BRAID CATEGORY $\mathcal{B}$ HAS 1 ELEMENT FOR EACH $n = 0, 1, 2, \dots$

$$\mathrm{Hom}(n, m) = \emptyset \quad \text{UNLESS } n = m.$$

$$\mathrm{Hom}(n, n) = \mathcal{B}_n.$$

COMPOSITION OF MORPHISMS = MULTIPLICATION OF BRAIDS.

$\mathcal{B}$ IS MONOIDAL. $\quad n \otimes m = n+m$.

IF $f : n \to n$, $\quad g : m \to m$

THEN $f \otimes g : \underset{\substack{\| \\ n+m}}{n \otimes m} \longrightarrow n+m$

IS JUST THE JUXTAPOSITION OF THE BRAIDS.

$n = 2 \qquad m = 3$

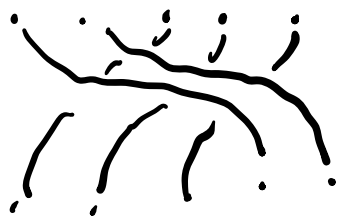
$\sigma_1^2 : 2 \to 2 \qquad \sigma_1 \sigma_2 : 3 \to 3$

$\sigma_1^2 \otimes \sigma_1 \sigma_2 : 5 \to 5$

THIS CATEGORY $\mathcal{B}$ IS A BRAIDED CATEGORY.

(THINK ABOUT THIS.)

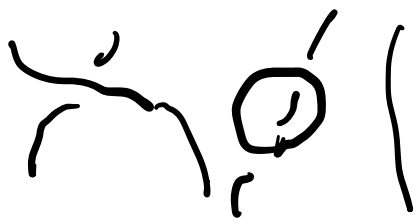
$C_{2,3} : 5 \to 5$



THIS IS A BRAIDED CATEGORY WHERE

$$C_{A,B} \neq C_{B,A}^{-1}.$$

CATEGORIES OF MODULES OVER QUASITRIANGULAR HOPF ALGEBRAS (SEE MAJID CH. 5) ARE BRAIDED.



TANGLE CATEGORY

$U_q(sl_2) \rightsquigarrow$ JONES POLYNOMIAL.

HOPF ALGEBRAS

AN ALGEBRA CAN BE FORMULATED

A V.S. A WITH LINEAR MAPS

$$\mu : A \otimes A \longrightarrow A \otimes A$$

ASSOCIATIVE

COMMUTES

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

UNIT $\eta : F \to A$ LINEAR MAP

$$\begin{tikzcd} A \arrow[r, phantom, "\cong"] \arrow[d, equal] & A \otimes F \arrow[d, "1 \otimes \eta"] \\ A & A \otimes A \arrow[l, "\mu"] \end{tikzcd}$$

$\otimes : \otimes_F$

MIRROR ALSO

WE CAN DUALIZE THESE AXIOMS ARRIVE AT A COALGEBRA C.

$\Delta : C \to C \otimes C$ COMULTIPLICATION

$\varepsilon : C \longrightarrow F$ COUNIT.

$xy = \mu(x \otimes y)$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes 1"] \\ C \otimes C \arrow[r, "1 \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

COASSOCIATIVITY

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \otimes F \\ C \arrow[r, "\Delta"] & C \otimes C \arrow[u, "1 \otimes \varepsilon"'] \end{tikzcd}$$

NOTATION FOR COALGEBRAS & HOPF ALG.
SWEEDLER (MAJID STYLE)

$$\Delta x = \sum x_i \otimes x_j \quad \text{FOR SOME } x_i, x_j \in C$$

INSTEAD WRITE $\Delta x = \sum x_{(1)} \otimes x_{(2)}$

(DROP $\sum$ QUICKLY)

$$(\Delta \otimes 1)\Delta x = (1 \otimes \Delta)\Delta x$$

$$(\Delta \otimes 1)(x_{(1)} \otimes x_{(2)}) = x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)}$$

SO COASSOCIATIVITY IN SWEEDLER NOTATION

$$x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)}$$

KNOWING THIS WRITE BOTH SIDES AS

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\Delta \otimes 1)\Delta x$$
$$= (1 \otimes \Delta)\Delta x = \Delta^{(3)} x.$$

AXIOMS FOR A HOPF ALG;

BOTH AN ALG & COALG.

COMULTIPLICATION $\Delta : H \to H \otimes H$ IS ALGEBRA HOM. EQUIVALENTLY; $\mu : H \otimes H \to H$ IS A COALGEBRA HOM.

$$\begin{tikzcd}
H \otimes H \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "\mu"] & H \otimes H \otimes H \otimes H \arrow[d, "a \otimes b \otimes c \otimes d \mapsto ac \otimes bd."] \\
H \arrow[r, "\Delta"] & H \otimes H
\end{tikzcd}$$

REVERSE ARROWS OBTAIN SAME DIAGRAM.

WE SHOULD THINK ABOUT RIGHT ARROW.

INTRODUCE $\tau : H \otimes H \to H \otimes H$

$$\tau(x \otimes y) = y \otimes x$$

$$\begin{array}{ccc} H\otimes H\otimes H\otimes H & & a\otimes b\otimes c\otimes d \\ \downarrow & & \downarrow \\ H\otimes H & & ac\otimes bd. \end{array}$$

THIS MAP IS THE COMPOSITION

$$\begin{array}{ccc} H\otimes H\otimes H\otimes H & & a\otimes b\otimes c\otimes d \\ \downarrow{\scriptstyle 1\otimes\tau\otimes 1} & & \\ H\otimes H\otimes H\otimes H & & a\otimes c\otimes b\otimes d \\ \downarrow{\scriptstyle \mu\otimes\mu} & & \\ H\otimes H & & ac\otimes bd \end{array}$$

REVERSE ARROWS

$$H\otimes H \xrightarrow[\Delta\otimes\Delta]{} H\otimes H\otimes H\otimes H \xrightarrow[1\otimes\tau\otimes 1]{} H\otimes H\otimes H\otimes H$$

$$x \otimes y \qquad x_{(1)} \otimes x_{(2)} \otimes y_{(1)} \otimes y_{(2)}$$

$$x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$$

$$\begin{tikzcd}
H \otimes H & H \otimes H \otimes H \otimes H \arrow[l, "\mu \otimes \mu"'] \\
H \arrow[u, "\Delta"] & H \otimes H \arrow[l, "\mu"'] \arrow[u]
\end{tikzcd}$$

SWEEDLER DISCUSSION TO BE CONTINUED