Modular Tensor Categories

MTC are simplest kinds of interesting ribbon categories.

Module category of the Drinfeld double of a finite group.

Categories $C(q_{12})$ of a Lie algebra $\mathfrak{g}$.

Many constructions:

Module category of $U_q(\mathfrak{g})$ where

$q$ is a root of unity related to $q_{12}$.

"Semi-simplify" a nice category emerges whose simple objects are in bijection with the dominant weights of $\mathfrak{g}$ in level $k$ above.

\[ (\lambda, \alpha_i^+) \geq 0 \quad \alpha_i^+ = \text{simple roots} \]
LEVEL $h$ AT COVE $i$ HIGHEST ROOT $\alpha_i$.

$$\langle \lambda, \alpha_i \rangle \leq 0.$$ 

See Baranov-Kirillov Ch. 3.

**Definition:** A NTC is a ribbon category that is an abelian category, in which there are a finite number of simple objects, and every object is a direct sum of these. The $S$-matrix is invertible.
Two versions of diagram calculus. Unlike BK, Turaev, I will read diagrams from top to bottom. Denote coev and eval as before.

\[ \text{coev}_v : I \to V \otimes V^* \]

\[ \text{ev} : V^* \otimes V \to I. \]

Modification (similar to Turaev, BK): label the edges by simple object. Assign an orientation.
If the simple objects are $V_i$ (i.e., I).

\[
\begin{array}{c}
\text{i} \\
\text{MEANS} \\
\text{v}\_i \\
\end{array}
\]

\[
\begin{array}{c}
\text{v}\_i^* \\
\text{MEANS} \\
\text{v}\_i \\
\end{array}
\]

Modify caps by including twist when needed.

\[
\text{coev : I } \rightarrow \text{ } V_i @ V_i^*
\]

\[
\begin{array}{c}
\text{i} \\
\text{MEANS} \\
\text{v}\_i \\
\end{array}
\]

\[
\begin{array}{c}
\text{v}\_i^* \\
\text{WOULD} \\
\text{MEAN} \\
\text{v}\_i \\
\end{array}
\]

Which is wrong so
\[ I \xrightarrow{\text{coev}} V^* \otimes V \xrightarrow{\text{coev}} V^* \otimes V \xrightarrow{\theta^* \otimes 1} V^* \otimes V \]

\( \ast \) = old style coev.

\( \ast \) = modified definition.

\[ \text{ev} : V^* \otimes V \rightarrow I \]
WITH THIS CONVENTION THE TWO FOLLOWING DIAGRAMS BOTH MAKE SENSE.

MEAN

MEAN S
These diagrams represent endomorphisms of $I$, i.e., scalars. The first one is $d_{\lambda} = "Quantum Dimension"$ of $V_{\lambda}$, second is $d_{\lambda^*} = \dim(V_{\lambda^*})$. They are equal and we'll consider a generalization of this fact.

Including a ' $\sim$ ' make dimension mult.

$$\dim(V_{\lambda} \otimes V_{\lambda'}) = \dim(V_{\lambda}) \dim(V_{\lambda'}).$$

Trace: If $f: V \rightarrow V$ is any endomorphism, trace $tr(f)$ can be defined in 2 ways: $V = V_{\lambda}$.

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]
These 2 definitions produce same answer.

First diagram unfolds to

Second

Naturality.
As a special case of the equality of the two definitions we take \( f = I_u \).

\[
\dim(v) = \ldots
\]

\[
\dim(v^*) = \ldots
\]

So \(\dim(v) = \dim(v^*)\).

The S-matrix if \( \bar{w}, \tilde{\delta} \in I \)

\[
\tilde{S}_{\bar{w}, \tilde{\delta}} = \ldots
\]

By the equality of two definitions of trace.
THE MATRIX $\hat{A}_{ij}$ IS ASSUMED INVERTIBLE.
A couple of other matrices need:

$$t = \left( \delta_{ij} \theta_i \right)_{\text{diagonal}}$$

$$C = \left( \delta_{ij}, c_i \right)$$

$$C^2 = 1$$

\textbf{Theorem 3.1.7.} Define the matrices \( \tilde{s} = (\tilde{s}_{ij}), t = (t_{ij}) \) and \( c = (c_{ij}) \) ("charge conjugation matrix") by (3.1.1) and

(3.1.8) \hspace{1cm} t_{ij} = \delta_{ij} \theta_i,
(3.1.9) \hspace{1cm} c_{ij} = \delta_{ij} c_i.

Then we have:

(3.1.10) \hspace{1cm} (\tilde{s} t)^3 = p^+ \tilde{s}^2,
(3.1.11) \hspace{1cm} (\tilde{s} t^{-1})^3 = p^- \tilde{s}^2 c,
(3.1.12) \hspace{1cm} ct = t c, \quad c \tilde{s} = \tilde{s} c, \quad c^2 = 1,

where \( p^\pm \) are defined by (3.1.7). Moreover, when \( \tilde{s} \) is invertible, we have

(3.1.13) \hspace{1cm} \tilde{s}^2 = p^+ p^- c.

This gives us a rep'n of \( SL_2(q) \) on the category. (i.e. on the free abelian group gen'd by the simple objects.)
$\text{SL}_2(\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad S^2 = -I$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (ST)^3 = I$$

$$ST = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (ST)^2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$(ST)^3 = S^2$$  \text{This will play the role of $C$}.$$

ADJUSTING $Ω$ OR SCALAR:

$$Δ = \frac{1}{\sqrt{Ω}} \sim Δ$$

gives a rep' of $\text{SL}_2(\mathbb{Z})$.

RELATED TO THE MTC PRODUCES A TOPOLOGICAL QFT WHICH ACTS ON
The mapping class group of any surface:

\[ \text{MCG}(S \times S') = \text{monodromy classes of diffeomorphisms of } T \]

\[ T = \mathbb{R}^2 / \mathbb{Z}^2. \]

Any diffeomorphism is homotopic to a linear one induced by an elt of \( SL_2(\mathbb{Z}) \).

**Lemma:** Fix \( \mathbf{w} \)

\[ \sum_{i,j} d_{ij}(\theta) = \rho^+ \cdot \theta^{-1} \]

Since \( V_\mathbf{w} \) is simple, \( \theta_{V_\mathbf{w}} : V_\mathbf{w} \to \tilde{V}_\mathbf{w} \) is a scalar, like \( d_{ij} \), where \( \rho^+ = \sum_{\mathbf{w}} d_{w}^2 \theta_{\mathbf{w}}. \)
\[ \sum_{i} Z d_{j} \left( \frac{i}{\theta} \right)^{2} = \Theta^{-1} \]

where \( \Theta^{-1} = \sum_{n} \alpha_{n} \).

Automatic this is true for some \( \rho^{+}, \rho^{-} \).

Problem is to compute. We take trace:

\[ \sum_{i} a_{i} \left( \frac{i}{\theta} \right)^{2} = \rho^{+} \Phi^{1} \]

As \( z \) is a constant I can move to left and take trace:

\[ \sum_{i} \theta_{i} \phi_{i} \left( \frac{i}{\theta} \right)^{2} = v_{i} \theta_{i} \text{ (LHS)} \]

\[ = \rho^{+} \text{ dim (V;)} \]
LHS: \[ \sum_{i} \sum_{j} \]
\[ V_n \otimes V_j = \sum_{h} N_{ij}^{hk} V_h \]

\[ N_{ij}^{hk} = \text{"Fusion Coefficients"} \]

\[ = \sum_{j} d_{j} N_{i,j}^{j} \theta_{n} \theta_{k} \]

\[ = \sum_{j} d_{j} N_{i,j}^{j} \theta_{n} \theta_{d_{h}} = \sum_{i} d_{i} N_{i,j}^{j} \theta_{e} \theta_{d_{i}} \]

\[ \Rightarrow \quad \text{dim} d_{n} = \sum_{i} d_{i} N_{i,m}^{i} \quad d_{i} - d_{j} \]

Comes from the multiplicativity of the quantum dimension.

\[ \text{LHS} = \text{dim}(V_n \otimes V_{k}) = \sum_{i} N_{i,j}^{i} \text{dim}(V_{j}^{*}) \]

\[ N_{i,j}^{i} \times N_{i,m}^{i} \]

\[ \text{LHS} = \text{dim} \quad V_{n} \quad \Rightarrow \quad V_{n} \otimes V_{j} = \text{dim}(1, V_{n} \otimes V_{j} \otimes V_{k}) \]

\[ \text{RHS} = \text{dim} \quad V_{h} \]
\[ = \sum_{i} \delta_{i} \chi_{i} \sum_{h} d_{i} \theta_{h} \sum_{h} \]

\[ = \sum_{i} \sum_{h} d_{i} \delta_{i} \theta_{h} \sum_{h} \]

\[ = \sum_{h} d_{i} \delta_{i} \theta_{h} : d_{i} \theta_{h} ^{+} \]

Comparing with \[ \theta_{i}^{+} \quad \theta_{i} \]

shows \[ \theta_{i}^{+} = \sum_{h} \frac{d_{h}^{2}}{\hbar} \theta_{h} \]

See also Lecture 14 of Old Notes.