Modular Tensor Categories

MTC ARE SIMPLEST KINDS OF INTERESTING RIBBON CATEGORIES.

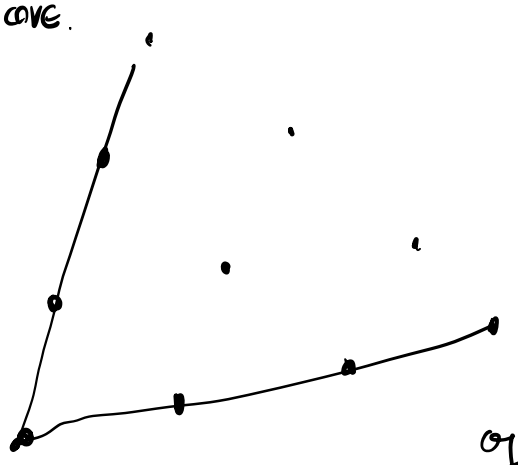
MODULE CATEGORY OF THE DRINFELD DOUBLE OF A FINITE GROUP.

CATEGORIES $\underline{\mathcal{C}(\mathfrak{g}, k)}$ $\mathfrak{g}$ A LIE ALG. k = "LEVEL"

MANY CONSTRUCTIONS:

MODULE CATEGORY OF $U_q(\mathfrak{g})$ WHERE q IS A ROOT OF UNITY RELATED TO k.

"SEMISIMPLIFY" A NICE CATEGORY EMERGES WHOSE SIMPLE OBJECTS ARE IN BIJECTION WITH THE DOMINANT WEIGHTS OF $\mathfrak{g}$ IN LEVEL k ALCOVE.



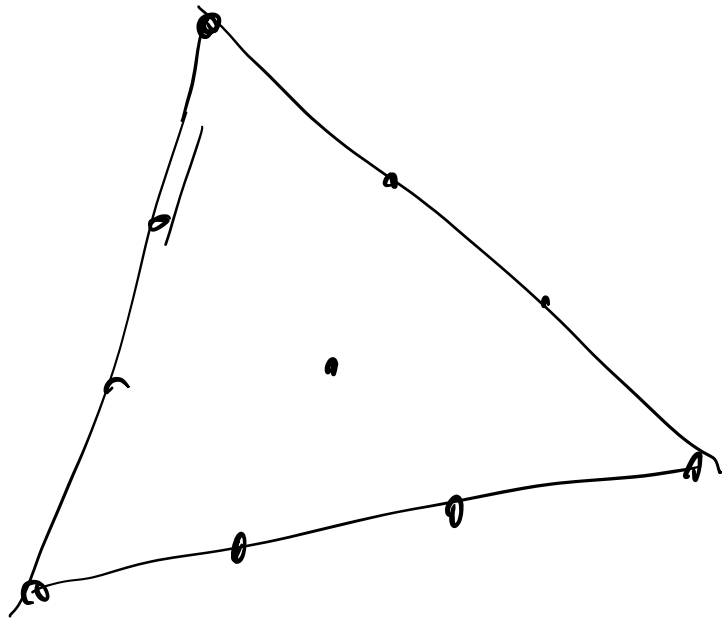
λ DOMINANT

$\mathfrak{g}$ $\mathfrak{sl}_3$

$\langle \lambda, \alpha_i^\vee \rangle \geq 0$ α_i = SIMPLE ROOTS

LEVEL k ALCOVE; HIGHEST ROOT α_n.

$$\langle \lambda, \alpha_m^\vee \rangle \leq k.$$



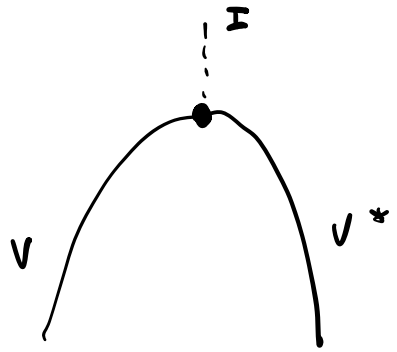
SEE BAKALOV - KIRILLOV CH. 3.

DEFINITION: A MTC IS A RIBBON CATEGORY THAT IS AN ABELIAN CATEGORY. IN WHICH THERE ARE A FINITE NUMBER OF SIMPLE OBJECTS. AND EVERY OBJECT IS A DIRECT SUM OF THESE. THE S-MATRIX IS INVERTIBLE.

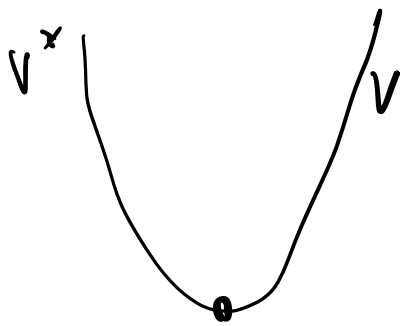
TWO VERSIONS OF DIAGRAM CALCULUS.

UNLIKE BK, TURAEV, I WILL READ DIAGRAMS FROM TOP TO BOTTOM.

DENOTE COEV AND EVAL AS BEFORE.



$\mathrm{coev}_V : I \to V \otimes V^*$



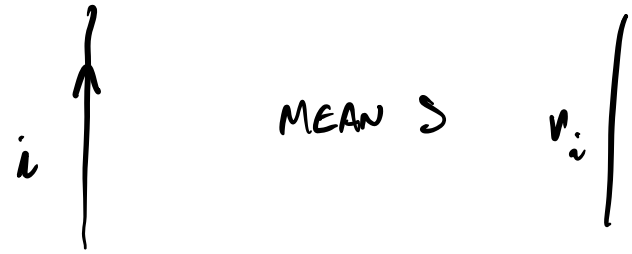
$\mathrm{ev} : V^* \otimes V \to I.$

MODIFICATION (SIMILAR TO TURAEV, BK):

LABEL THE EDGES BY SIMPLE (USUALLY) OBJECT.

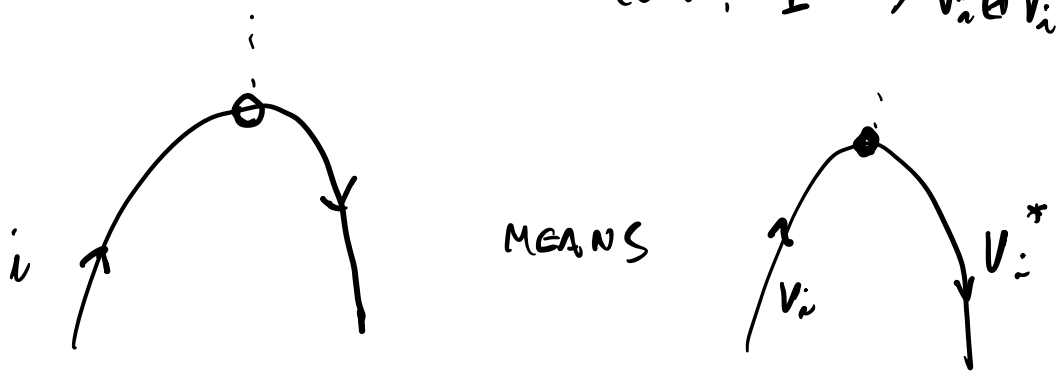
ASSIGN AN ORIENTATION.

IF THE SIMPLE OBJECTS ARE V_i $(i \in I)$.

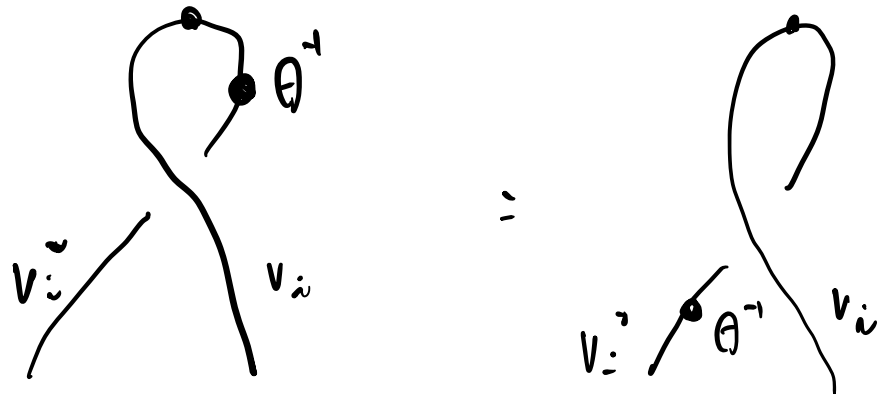


MODIFY CAPS BY INCLUDING TWIST WHEN NEEDED:

$\mathrm{coev}: I \to V_i \otimes V_i^*$



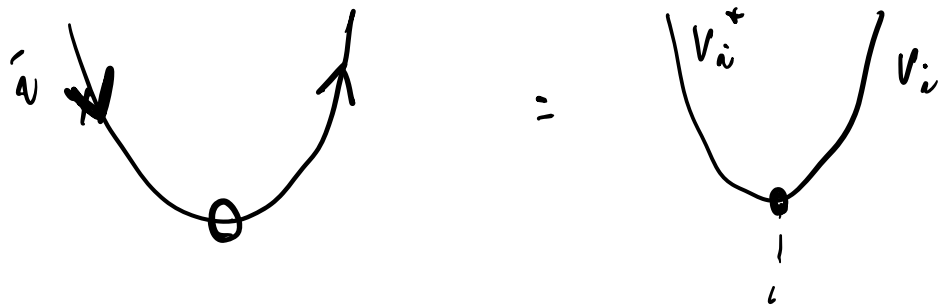
WHICH IS WRONG SO



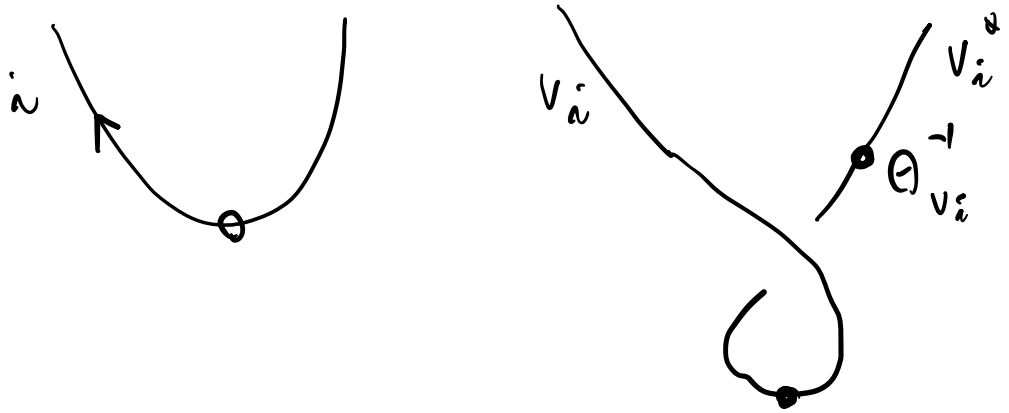
$$I \xrightarrow{\text{coev}} V_i \otimes V_i^* \xrightarrow{c_{V_i, V_i^*}} V_i^* \otimes V_i \xrightarrow{\theta^{-1} \otimes 1} V_i^* \otimes V_i$$

● = OLD STYLE COEV.

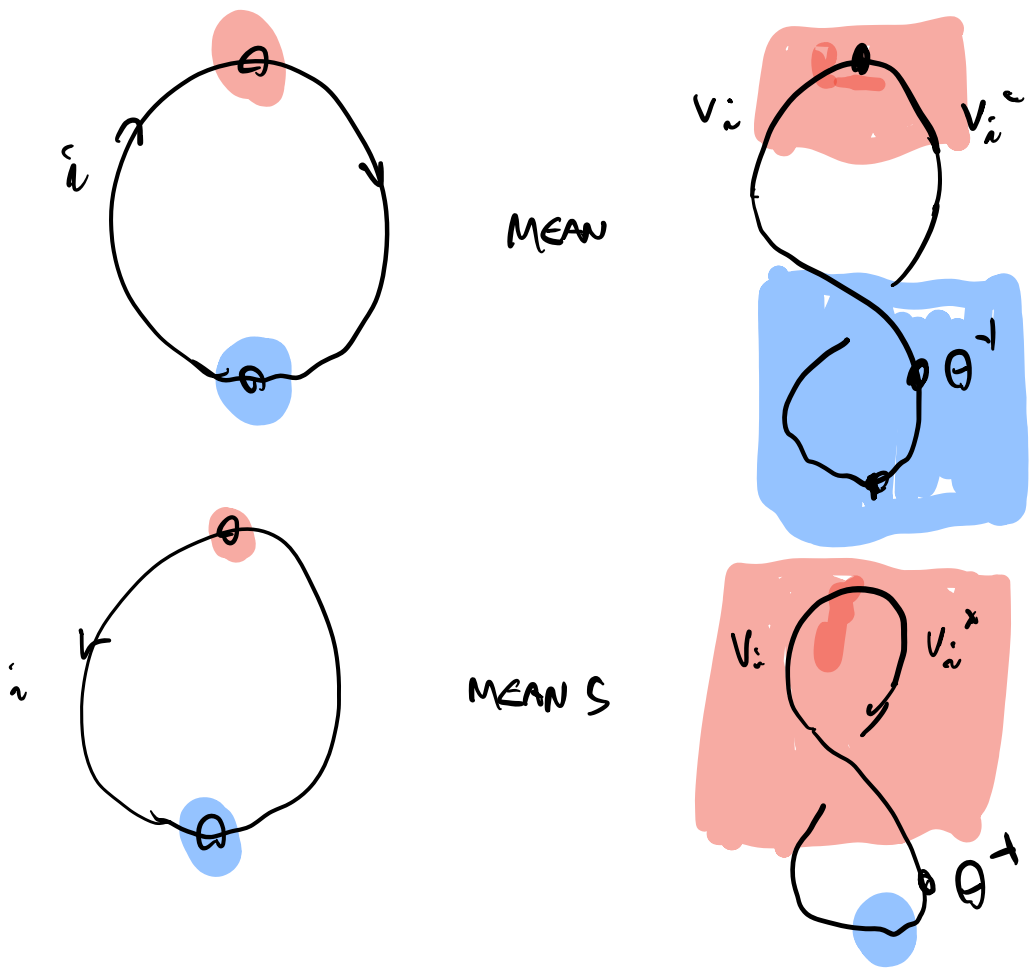
○ = MODIFIED DEFINITION.



$$\mathrm{ev}_i : V_i^* \otimes V_i \to I$$



WITH THIS CONVENTION THE TWO FOLLOWING DIAGRAMS GOTTA MAKE SENSE;



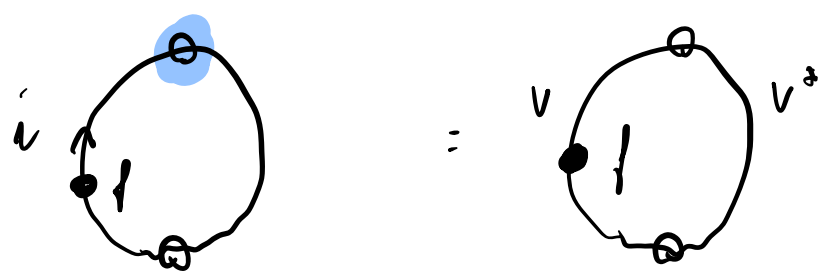
THESE DIAGRAMS REPRESENT ENDOMORPHISMS OF I, I.E. SCALARS. THE FIRST ONE IS d_i = "QUANTUM DIMENSION" OF V_i SECOND IS $d_{i^*} = \mathrm{DIM}(V_i^*)$

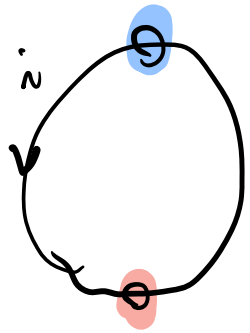
THEY ARE EQUAL AND WE'LL CONSIDER A GENERALIZATION OF THIS FACT.

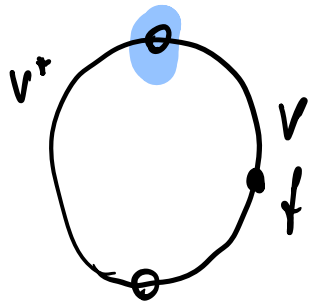
INCLUDING θ^{-1} MAKE DIMENSION MULT.

$$\mathrm{DIM}(V_i \otimes V_j) = \mathrm{DIM}(V_i)\,\mathrm{DIM}(V_j).$$

TRACE: IF $f: V \to V$ IS ANY ENDOMORPHISM TRACE $\mathrm{TR}(f)$ CAN BE DEFINED IN 2 WAYS; $V = V_i$

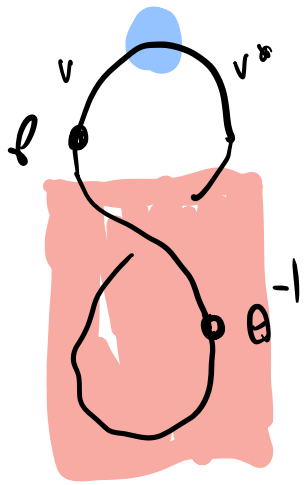




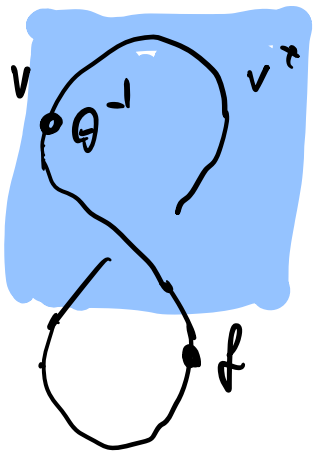


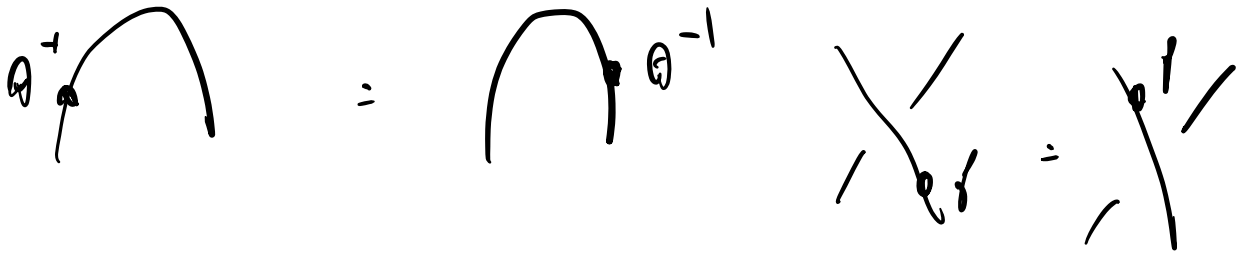
THESE 2 DEFINITIONS PRODUCE SAME ANSWER.

FIRST DIAGRAM UNFOLDS TO



SECOND

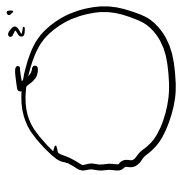




NATURALITY.

AS A SPECIAL CASE OF THE EQUALITY OF THE TWO DEFINITIONS WE TAKE $f = 1_V$.

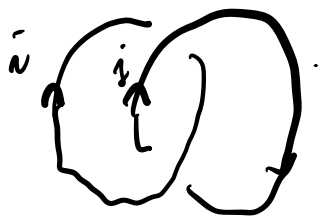
$\mathrm{DIM}(V) =$ 

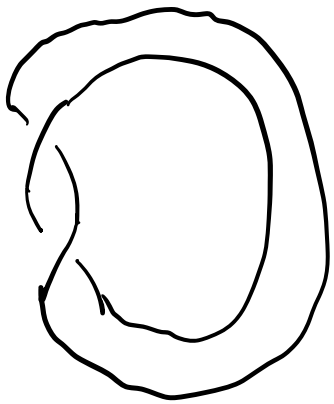
$\mathrm{DIM}(V^*) =$ 

$=$ 

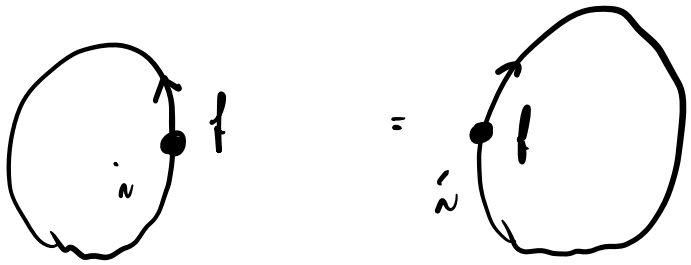
SO $\mathrm{DIM}(V) = \mathrm{DIM}(V^*)$.

THE S-MATRIX IF $i, j \in I$

$\tilde{S}_{i,j} =$ 

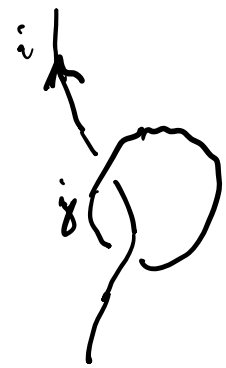
$=$ 

BY THE EQUALITY OF TWO DEFINITIONS OF TRACE.



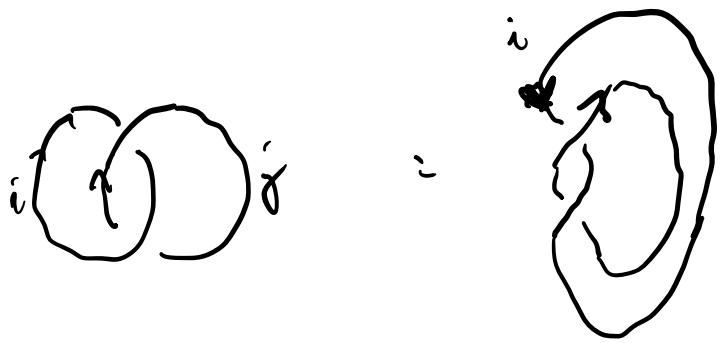
$f \in \mathrm{END}(V_i)$

I'M APPLY THIS WITH f BEING THIS MORPHISM



$\in \mathrm{END}(V_i)$

$\tilde{\Delta}_{ij}$



$$= \mathrm{tr}\left(C_{V_\gamma, V_i^*} \circ C_{V_i^*, V_\gamma} \right).$$

THE MATRIX $\hat{\Delta}_{ij}$ IS ASSUMED INVERTIBLE.

A COUPLE OF OTHER MATRICES NEED:

$$t = (\theta_i \delta_{ij}) \quad \text{DIAGONAL}$$

$$c = (\delta_{i,i^*}) \qquad c_{ij} = 1 \text{ IF } V_i = V_j^*$$

0 OTHERWISE.

$$c^2 = 1$$

THEOREM 3.1.7. *Define the matrices* $\tilde{s} = (\tilde{s}_{ij})$, $t = (t_{ij})$ *and* $c = (c_{ij})$ (*"charge conjugation matrix"*) *by* (3.1.1) *and*

$$t_{ij} = \delta_{ij}\theta_i, \tag{3.1.8}$$

$$c_{ij} = \delta_{ij^*}. \tag{3.1.9}$$

Then we have:

$$(\tilde{s}t)^3 = p^+\tilde{s}^2, \tag{3.1.10}$$

$$(\tilde{s}t^{-1})^3 = p^-\tilde{s}^2 c, \tag{3.1.11}$$

$$ct = tc, \quad c\tilde{s} = \tilde{s}c, \quad c^2 = 1, \tag{3.1.12}$$

where $p^\pm$ *are defined by* (3.1.7). *Moreover, when* $\tilde{s}$ *is invertible, we have*

$$\tilde{s}^2 = p^+p^-c. \tag{3.1.13}$$

THIS GIVES US A REP'N OF $SL_2(\mathbb{Z})$ ON THE CATEGORY. (I.E. ON THE FREE ABELIAN GROUP GEN'D BY THE SIMPLE OBJECTS.)

$SL_2(\mathbb{Z})$ IS GEN'D BY

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \qquad S^2 = -I$$

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \qquad (ST)^3 = I.$$

$$ST = \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix} \qquad (ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(ST)^3 = \begin{pmatrix} -I & \\ & -I \end{pmatrix}$$

$(ST)^3 = S^2$ THIS WILL PLAY THE ROLE OF C.

ADJUSTING $\tilde{\Delta}$ BY SCALAR:

$$\Delta = \frac{1}{\sqrt{\theta^+\theta^-}}\tilde{\Delta}$$

GIVES A REP'N OF $SL_2(\mathbb{Z})$.

RELATED TO THE MTC PRODUCES A TOPOLOGICAL QFT WHICH ACTS ON

THE MAPPING CLASS GROUP OF ANY SURFACE:

$$MCG(\underbrace{S^1 \times S^1}_{T}) = \text{HOMOTOPY CLASSES OF DIFFEOMORPHISMS OF } T$$

$T = \mathbb{R}^2 / \mathbb{Z}^2$.

ANY DIFFEOMORPHISM IS HOMOTOPIC TO A LINEAR ONE INDUCED BY AN ELT OF $SL_2(\mathbb{Z})$.

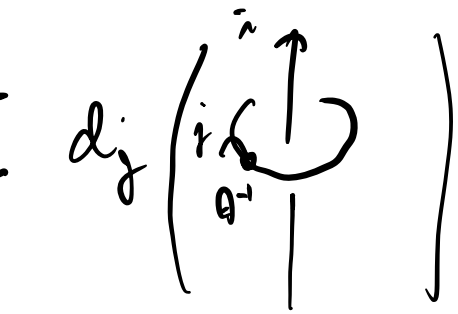
LEMMA: FIX i

$$\sum_{j} d_j \left(\begin{array}{c} i \\ j \\ \theta \end{array} \right) = \begin{array}{c} i \\ \rho^+ \quad \theta^{-1} \end{array}$$

SINCE V_i IS SIMPLE $\theta_{V_i} : V_i \to V_i$ IS A SCALAR. LIKE d_i

WHERE $\rho^+ = \sum_k d_k^2 \theta_k$.

SIMILARLY

$$\sum_j d_j$$ 

$$= p^-$$ 

WHERE $p_- = \sum d_h^2 \theta_h^{-1}$.

AUTOMATIC THIS IS TRUE FOR SOME p_+, p_-.

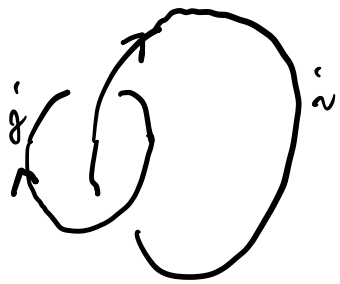
PROBLEM IS TO COMPUTE. WE TAKE TRACE:

$$\sum_j d_j$$ 

$$= p^+$$ 

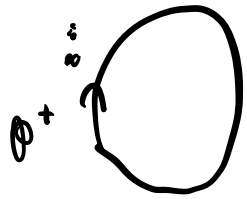
θ_i IS A CONSTANT I CAN MOVE TO LEFT:

AND TAKE TRACE:

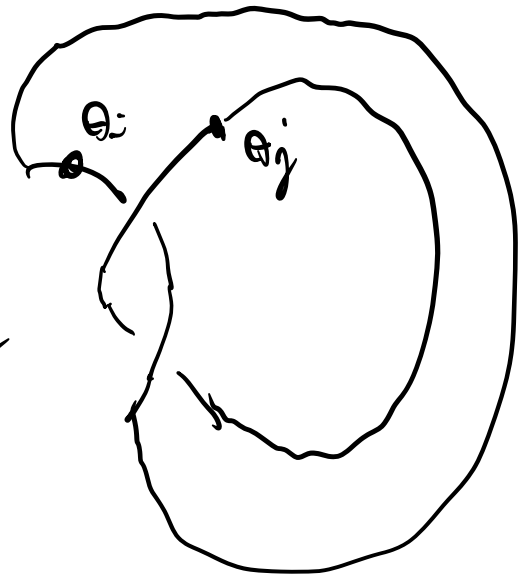
$$\sum_j d_j \theta_i \theta_j$$ 

$$= \operatorname{tr} \theta_i (\text{LHS}).$$

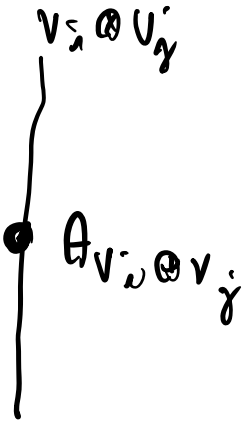
$$= p^+ \operatorname{Dim}(V_i)$$

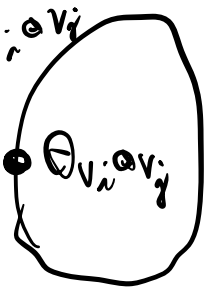
RHS: 

$= p^* d_i$.

LHS: $\sum_j d_j$ 

REMEMBER 

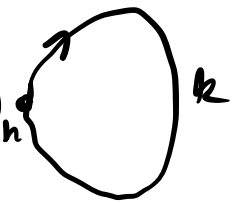
$=$ 

LHS: $\sum_j d_j$ 

$$V_i \otimes V_j = \sum_k N_{ij}^{k} V_k \qquad N_{ij}^{k} =$$

"FUSION COEFFICIENTS"

$$= \sum_j d_j N_{ij}^{k} \theta_k$$



$$= \sum_j d_j N_{ij}^{k} \theta_k d_k = \sum_j d_i N_{ik^*}^{j^*} \theta_k d_k$$

$$\rightarrow \quad d_i d_k = \sum_j d_j N_{ik^*}^{j^*} \qquad d_j = d_{j^*}$$

COMES FROM THE MULTIPLICATIVITY OF THE QUANTUM DIMENSION.

$$\text{LHS} = \text{DIM}(V_i \otimes V_k) = \sum_j N_{ik}^{j^*} \text{DIM}(V_j^*)$$

$$N_{ij}^{k} = N_{ik^*}^{j^*}$$

$$\text{LHS} = \text{DIM } V_k \text{ IN } V_i \otimes V_j = \text{DIM}(1, V_i \otimes V_j \otimes V_k^*$$

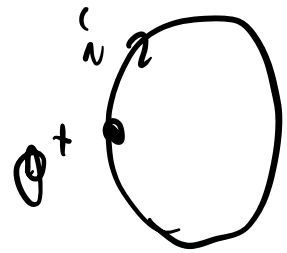
$$\text{RHS} = \text{DIM } V_k^*$$

$$= \sum_{j,k} d_j N_{ij}^{k} \theta_k d_k$$

$$= \sum_{j,k} N_{ik^*}^{j^*} d_j d_k \theta_k$$

$$= \sum_{k} d_i d_k^2 \theta_k = d_i \phi_+$$

COMPARING WITH



SHOWS $\phi_+ = \sum_k d_k^2 \theta_k$.

SEE ALSO LECTURE 14 OF OLD NOTES.