Drinfeld Double Modular Tensor Categories

We created $D(A)^*$ by Drinfeld twisting $A^* \otimes A^{\text{op}}$. Thus as an algebra $D(A)^* = A^* \otimes A^{\text{op}}$ but the comultiplication is

$$\Delta_F(x) = F\Delta(x)F^{-1}$$

where

$$F = (1_{A^*} \otimes S^{-1}e_i) \otimes (e^i \otimes 1_A),$$

$$F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A).$$

Then we define $D(A)$ to be the dual of $D(A)^*$. As a coalgebra it is $A \otimes A^\circ$, where $A^\circ = (A^{\text{op}})^* = (A^*)^{\text{cop}}$. Since the comultiplication of $D(A)^*$ differs from the comultiplication in $A^* \otimes A^{\text{op}}$, the multiplication in $D(A)$ is correspondingly modified from $A \otimes A^{\text{op}}$.

\[
\begin{align*}
A & \text{ Hopf alg.} \\
A^{\text{op}} &= A \text{ with reversed unit} \\
A^{\text{cop}} &= A^{\text{op}} \\
A^\circ &= (A^{\text{op}})^* = (A^*)^{\text{cop}}
\end{align*}
\]

The Drinfeld double is $A \otimes A^\circ$ with modified multiplication construct $D(A)^* = A^* \otimes A^{\text{op}}$ with modified comultiplication: Drinfeld twisting.
IF $H$ IS A HOPF ALG AND $F \in H \otimes H$ SATISFIES

$$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$$

OBTAIN SUCH AN ELEMENT WITH $H \cdot A^* \otimes A^{op}$ BY TAKING $F = 1 \otimes S^{-1}e_n \otimes e_i^* \otimes 1$.

$e_i^*$, $e_n^*$ DUAL BASES OF $A_i$, $A^n$.

$$A^* \otimes A^{op} \xrightarrow{\text{twist}} \varphi \Delta \Delta_{\Lambda} \Lambda_{\Delta} \Lambda \Lambda_{\Lambda}$$

THIS HOPF ALG IS $D(A)^*$. DEFINE $D(A)^*$.

DUAL OF THIS $= A \oplus A^o$ $A^o : (A^{op})^*$

CLAIM THIS CONTAINS COPIES OF $A_i$, $A^o$.

SO WE HAVE TO CHECK:

Denote $x \in A_i$, $\lambda \in A^o$ $x \downarrow \lambda$ = IMAGE OF $x \otimes \lambda$ IN THIS $D(A)$.

**THEOREM**: $x \rightarrow x \downarrow 1$ IS A HOM $A \rightarrow D(A)$

$\lambda \rightarrow 1 \downarrow \lambda$ IS A HOM $A^o \rightarrow D(A)$.

SEE LECTURE 12 FOR DETAILS.
THEOREM: \( D(A) \) IS QUASI TRIANGULAR.

With \( R \)-matrix \( R = (e^v \mathbb{R} 1_{A^+}) \mathbb{R} (1_{A^+} \mathbb{R} e^w) \)

See Lecture 12 on Kassel for details.

\( \otimes e^v \otimes e^w \otimes 1 \) in \( D(A)^* = A^+ \otimes A^0 \)

This canonical element was used to produce the Dinfeld twist. The other canonical one is used to produce the \( R \)-matrix.

The procedure doesn't actually require finite-dim Hopf with some care one can adapt it to a pair of Hopf algs. with a dual pairing. You can easily construct \( U_q(\mathfrak{gl}^+) \), \( U_q(\mathfrak{gl}^-) \)

\[ \begin{pmatrix} \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \end{pmatrix} \in \mathfrak{gl}(W) \]

\[ \begin{pmatrix} \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \end{pmatrix} \]

There is a dual pairing so this leads to \( U_q(\mathfrak{gl}(W)) \) and its quasi triangularity.
MODULAR RIBBON CATEGORIES

These are ribbon categories with a finite number of simple objects. They arise in different ways:

I. Quantum groups $U_q(G)$ with $q$ a root of unity.

II. Drinfeld double of a finite group, similar to I but coming from different sources.

I'. Certain WZW conformal field theory

I''. Representations of a fixed level of affine Lie algebras.

I'''. Monodromy of KZ differential equations.

Despite the fact these ribbon categories have different sources they are easy to work with.

Reference: Book of BAKALOV and KIRILLOV.
(Available from Kirillov's web page.)

Abelian categories are like category of modules over a ring places where you can do homological algebra.

Additive category: Hom spaces are abelian groups, and composition

\[ \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C) \]

are bilinear.

Abelian category C add enough axioms to (for example) state and prove snake lemma.

A morphism \( \phi : A \rightarrow B \) is a monomorphism if \( f \circ g = f \circ g' \) (\( f, g', C \rightarrow A \))

\[ \Rightarrow g = g' \]

(f.g. injective Homs in module category).

Epimorphisms are dual.

You require in the category every morphism \( \phi : A \rightarrow B \) factors as
There is a zero object $0$ which is an initial object in $C$.

Kernel $\ker f_A : A \to B$ is an object $K$ with a morphism $i : K \to A$ such that $f \circ i = 0$ in $\text{Hom}(K, B)$.

And any morphisms $f : L \to A$ with $f \circ y = 0$ factors uniquely through it.

Cokernels are defined dually.

Assume every monomorphism is kernel of its cokernel and dually.
Also assume if \( A, B \) are objects there is a coproduct \( A \oplus B \) which is also a product.

Then there is enough structure to do homological algebra.

The category is **semisimple** if every S.E.S. splits.

\[
0 \rightarrow M' \xrightarrow{b} M \xrightarrow{g} M'' \rightarrow 0
\]

\( g \cdot \alpha = 1_{M''} \) this forces

\[
M = M' \oplus M''.
\]

An object \( X \) if \( Y \hookrightarrow X \) is a monomorphism it is zero or an isomorphism.

I want every object to be a direct sum of simple objects.
TENSOR CATEGORIES,

AN ABELIAN CATEGORY WITH 0 OBJECT AND DIRECT SUMS \( \oplus \). ASSUME A MONOIDAL STRUCTURE WITH \( \otimes \) NATURAL WHICH IMPLIES

\[
(A \oplus B) \otimes C \equiv (A \otimes C) \oplus (B \otimes C).
\]

\( A \oplus B \) HAS MORPHISMS

\[
\begin{array}{c}
A \xrightarrow{\phi} A \oplus B \xleftarrow{\psi} B
\end{array}
\]

\[
\begin{array}{c}
\phi \circ j = 0 & p \circ \iota = 1_A & q \circ \iota = 1_B
\end{array}
\]

\[
\phi \circ j \circ \iota = 1_{A \oplus B}.
\]

APPLYING ANY ADDITIVE FUNCTION \( Y \) SUCH AS

\( A \to A \oplus C \) (\( C \) FIXED)

PRODUCES SIMILAR MAPS IN

\( Y(C \oplus B) \)

PROVING \( Y(C \oplus B) = Y(C) \oplus Y(B) \)
A MTC is a semisimple ribbon category with a finite number of simple objects \( i, j, k, \ldots \) such that a certain \( S \)-matrix is invertible.

Digression: in a ribbon category, a right dual is also a left dual.

**Rigidity**:\[ \text{coev}: F \rightarrow V \otimes V^* \]

\[ V^* \otimes V \rightarrow F \]

\[ V^* \otimes V \rightarrow F \rightarrow V^* \]

\[ V^* \otimes V \rightarrow F \rightarrow V^* \]

\[ V \otimes V \rightarrow V \]

\[ V \otimes V \rightarrow V \]

\[ V \otimes V \rightarrow V \]

\[ V \otimes V \rightarrow V \]

\[ V \otimes V \rightarrow V \]
Using ribbon element we can produce similar morphisms $\text{coev}_v : F \to V^* \otimes V$,

$\text{ev}_v : V^* \otimes V \to F$

Define them thus:

These make $V^*$ into a left dual.
In the ribbon $CA$
The $S$-matrix in the modular $T$, $C$. Let $i$, $j$ be objects.

\[ \begin{array}{cc}
V_i & V_j \\
V_j & V_i
\end{array} \]

This is a morphism $F \rightarrow F$ hence an element of the ring $\text{End}(F)$. In the examples $\text{End}(F)$ will be a field.

\[ \delta_{ij} : \text{this scalar.} \]

The $S$-matrix $\tilde{S} = (\delta_{ij})$

indexed by pairs of simple objects.

**Theorem:** There is an action of $\text{SL}_2(\mathbb{Z})$ on the simple objects of the category.
$SL_2(\mathbb{R}) = \langle (1, -1), (1, 0), (,-1, 1) \rangle$

$s = T$

$ST = (1, -1) \quad (ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

$(ST)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$s^2 = I$ central $\quad (-I)^2 = I$

$(ST)^3 = -I$

This is a presentation.

$SL_2(\mathbb{R}) \subset \mathbb{R}^2 = \{ z = x + iy \mid y > 0 \}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \rightarrow \frac{az + b}{cz + d}$

\[ FUNDAMENTAL\ \DOMAIN \]
$S: \mathbb{Z} \rightarrow -\frac{1}{2}$

$T: (\mathbb{Z}^+) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^+$

- $T$ acts minimally on $\mathbb{Z}$. 

"Fixed point"