Drinfeld Double
Modular Tensor Categories

We created $D(A)^*$ by Drinfeld twisting $A^* \otimes A^{\mathrm{op}}$. Thus as an algebra $D(A)^* = A^* \otimes A^{\mathrm{op}}$ but the comultiplication is

$$\Delta_F(x) = F\Delta(x)F^{-1}$$

where

$$F = (1_{A^*} \otimes S^{-1}e_i) \otimes (e^i \otimes 1_A),$$
$$F^{-1} = (1_{A^*} \otimes e_i) \otimes (e^i \otimes 1_A).$$

Then we define $D(A)$ to be the dual of $D(A)^*$. As a coalgebra it is $A \otimes A^\circ$, where $A^\circ = (A^{\mathrm{op}})^* = (A^*)^{\mathrm{cop}}$. Since the comultiplication of $D(A)^*$ differs from the comultiplication in $A^* \otimes A^{\mathrm{op}}$, the multiplication in $D(A)$ is correspondingly modified from $A \otimes A^{\mathrm{op}}$.

A HOPF ALG. A^{op} = A WITH REVERSED MULT

A^{cop} = " COMULT

$A^\circ := (A^{op})^* = (A^*)^{cop}$

DOUBLE IS $A \otimes A^\circ$ WITH MODIFIED MULTIPLICATION

CONSTRUCT $D(A)^* = A^* \otimes A^{op}$ WITH MODIFIED COMULTIPLICATION : DRINFELD TWISTING,

IF H IS A HOPF ALG AND $F \in H \otimes H$

SATISFIES $$F_{12}(\Delta \otimes 1)(F) = F_{23}(1 \otimes \Delta)(F)$$

OBTAIN SUCH AN ELEMENT WITH $H = A^* \otimes A^{op}$ BY

TAKING $F = 1 \otimes S^{-1} e_i \otimes e_i^* \otimes 1$

e_i, e_i^* DUAL BASES OF A, A^*.

$$A^* \otimes A^{op} \xrightarrow{\text{TWIST}} \text{REPLACE } \Delta \text{ BY}$$

$$\Delta_F(x) = F \Delta(x) F^{-1}.$$

THIS HOPF ALG IS $D(A)^*$. DEFINE $D(A)$ = DUAL OF THIS = $A \otimes A^0$ $\qquad A^0 = (A^{op})^*$

CLAIM THIS CONTAINS COPIES OF A, A^0

SO WE HAVE TO CHECK:

DENOTE $x \in A$, $\lambda \in A^0$ $\qquad x \boxtimes \lambda$ = IMAGE OF $x \otimes \lambda$ IN THIS $D(A)$.

THEOREM: $x \to x \boxtimes 1$ IS A HOM $A \to D(A)$

$\lambda \to 1 \boxtimes \lambda$ IS A HOM $A^0 \to D(A)$.

SEE LECTURE 12 FOR DETAILS.

THEOREM: $D(A)$ IS QUASITRIANGULAR.

WITH R-MATRIX $R = (e_i \boxtimes 1_{A^\circ}) \boxtimes (1_A \boxtimes e_i^*)$

SEE LECTURE 12 OR KASSEL FOR DETAILS.

$$1 \otimes e_i \otimes e_i^* \otimes 1 \quad \text{IN} \quad D(A)^* = A^* \otimes A^{op}$$

THIS CANONICAL ELEMENT WAS USED TO PRODUCE THE DRINFELD TWIST. THE OTHER CANONICAL ELT. IS USED TO PRODUCE THE R-MATRIX.

THE PROCEDURE DOESN'T ACTUALLY REQUIRE FINITE-DIM'L HOPF WITH SOME CARE ONE CAN ADAPT IT TO A PAIR OF HOPF ALG. WITH A DUAL PAIRING.

YOU CAN EASILY CONSTRUCT $U_q(\mathfrak{b}^+)$, $U_q(\mathfrak{b}^-)$

$$\mathfrak{b}^+ = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \subset \mathfrak{gl}(n)$$

$$\mathfrak{b}^- = \begin{pmatrix} * & & \\ & \ddots & \\ * & & * \end{pmatrix}$$

THERE IS A DUAL PAIRING SO THIS LEADS TO $U_q(\mathfrak{gl}(n))$ AND ITS QUASITRIANGULARITY.

MODULAR TENSOR CATEGORIES

THESE ARE RIBBON CATEGORIES WITH A FINITE NUMBER OF SIMPLE OBJECTS.

THEY ARISE IN DIFFERENT WAYS:

I. QUANTUM GROUPS $U_q(\mathfrak{g})$ q A ROOT OF UNITY.

II. DRINFELD DOUBLE OF A FINITE GROUP.

SIMILAR TO I BUT COMING FROM DIFFERENT SOURCES:

I'. CERTAIN WZW CONFORMAL FIELD THEORY

I'' REPRESENTATIONS OF A FIXED LEVEL OF AFFINE LIE ALGEBRAS.

I''' MONODROMY OF KZ DIFFERENTIAL EQUATIONS.

DESPITE THE FACT THESE RIBBON CATEGORIES HAVE DIFFERENT SOURCES THEY ARE EASY TO WORK WITH.

REFERENCE: BOOK OF BAKALOV AND KIRILLOV.

(AVAILABLE FROM KINLICHON'S WEB PAGE.)

ABELIAN CATEGORIES ARE LIKE CATEGORY OF MODULES OVER A RING PLACES WHERE YOU CAN DO HOMOLOGICAL ALGEBRA.

ADDITIVE CATEGORY: HOM SPACES ARE ABELIAN GROUPS, AND COMPOSITION

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$$

ARE BILINEAR.

ABELIAN CATEGORY $\mathcal{C}$ ADD ENOUGH AXIOMS TO (FOR EXAMPLE) STATE AND PROVE SNAKE LEMMA.

A MORPHISM $f: A \to B$ IS A MONOMORPHISM IF $f \circ g = f \circ g'$ $(g, g': C \to A)$ $\Rightarrow g = g'$.

(E.G. INJECTIVE HOMS IN MODULE CATEGORY).

EPIMORPHISMS ARE DUAL.

YOU REQUIRE IN THE CATEGORY EVERY MORPHISM $f: A \to B$ FACTORS AS

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{EPI}"'] & & B \\ & I \arrow[ur, "\text{MONO}"'] & \end{tikzcd}$$

THERE IS A ZERO OBJECT 0 WHICH IS AN INITIAL OBJECT IN $\mathcal{C}$

KERNEL $f: A \to B$ IS AN OBJECT K WITH A MORPHISM

$$i: K \to A$$

SUCH THAT $f \circ i = 0$ IN $\mathrm{HOM}(K, B)$

AND ANY MORPHISMS $\gamma: L \to A$ WITH $f \circ \gamma = 0$ FACTORS UNIQUELY THROUGH IT.

$$\begin{tikzcd} L \arrow[r] \arrow[dr, dashed] & A \arrow[r, "f"] & B \\ & K \arrow[u] & \end{tikzcd}$$

COKERNELS ARE DEFINED DUALLY.

ASSUME EVERY MONOMORPHISM IS KERNEL OF ITS COKERNEL AND DUALLY

ALSO ASSUME IF A, B ARE OBJECTS THERE IS A COPRODUCT $A \oplus B$ WHICH IS ALSO A PRODUCT.

THEN THERE IS ENOUGH STRUCTURE TO DO HOMOLOGICAL ALGEBRA.

THE CATEGORY IS SEMISIMPLE IF EVERY S.E.S. SPLITS.

$$0 \to M' \xrightarrow{f} M \underset{h}{\overset{g}{\rightleftarrows}} M'' \to 0$$

$gh = 1_{M''}$ THIS FORCES

$$M = M' \oplus M''.$$

AN OBJECT X IF $Y \to X$ IS A MONOMORPHISM IT IS ZERO OR AN ISOMORPHISM.

I WANT EVERY OBJECT TO BE A DIRECT SUM OF SIMPLE OBJECTS.

TENSOR CATEGORIES:

AN ABELIAN CATEGORY WITH 0 OBJECT AND DIRECT SUMS $\oplus$. ASSUME A MONOIDAL STRUCTURE WITH $\otimes$ NATURAL WHICH IMPLIES

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C).$$

$A \oplus B$ HAS MORPHISMS

$$\begin{tikzcd} A \arrow[r, shift left, "i"] & A \oplus B \arrow[l, shift left, "p"] \arrow[r, shift right, "q"'] & B \arrow[l, shift right, "j"'] \end{tikzcd}$$

$$qi = 0 \qquad pj = 0 \qquad pi = 1_A$$
$$qj = 1_B$$

$$ip + jq = 1_{A \oplus B}.$$

APPLYING ANY ADDITIVE FUNCTOR $\mathcal{F}$ SUCH AS

$A \rightsquigarrow A \otimes C$ (C FIXED)

PRODUCES SIMILAR MAPS IN

$$\mathcal{F}(A \oplus B)$$

PROVING $\mathcal{F}(A \oplus B) = \mathcal{F}(A) \oplus \mathcal{F}(B)$

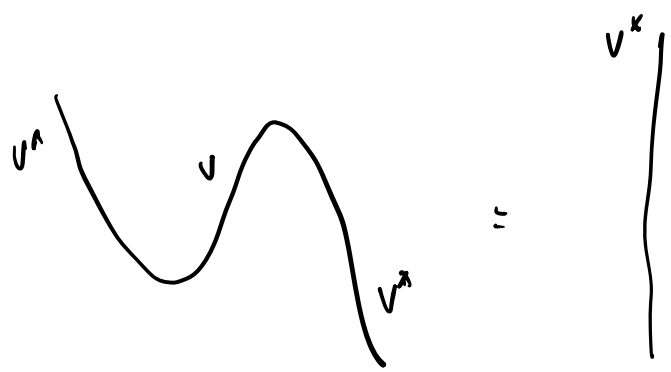
A MTC IS A SEMISIMPLE RIBBON CATEGORY WITH A FINITE NUMBER OF SIMPLE OBJECTS $\{i, j, k, \dots\}$ SUCH THAT A CERTAIN S-MATRIX IS INVERTIBLE.

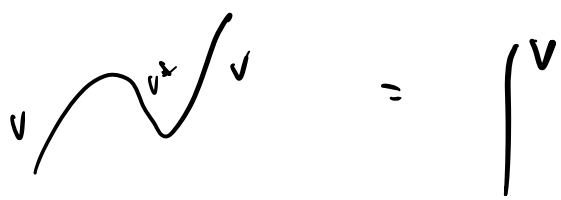
DIGRESSION: IN A RIBBON CATEGORY A RIGHT DUAL IS ALSO A LEFT DUAL.

RIGIDITY: $\mathrm{coev}: F \to V \otimes V^*$

IDENTITY OBJECT

$V^* \otimes V \to F$

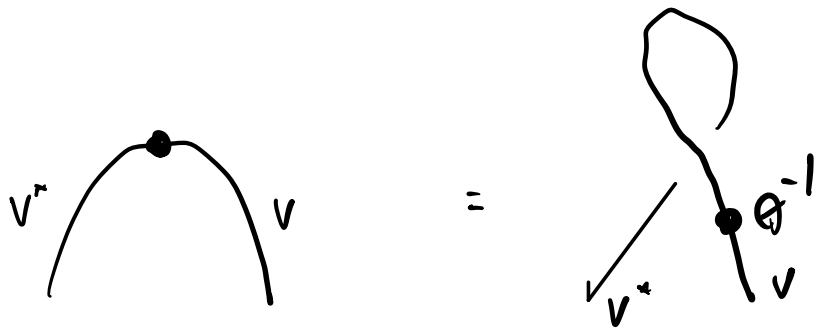


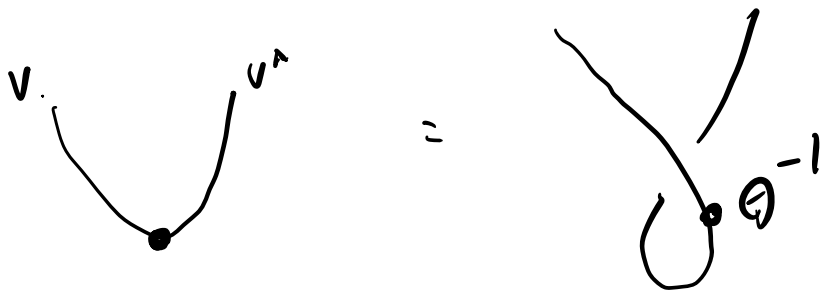


USING RIBBON ELEMENT WE CAN PRODUCE SIMILAR MORPHISMS $\mathrm{coev}'_{V^*} : F \to V^* \otimes V$

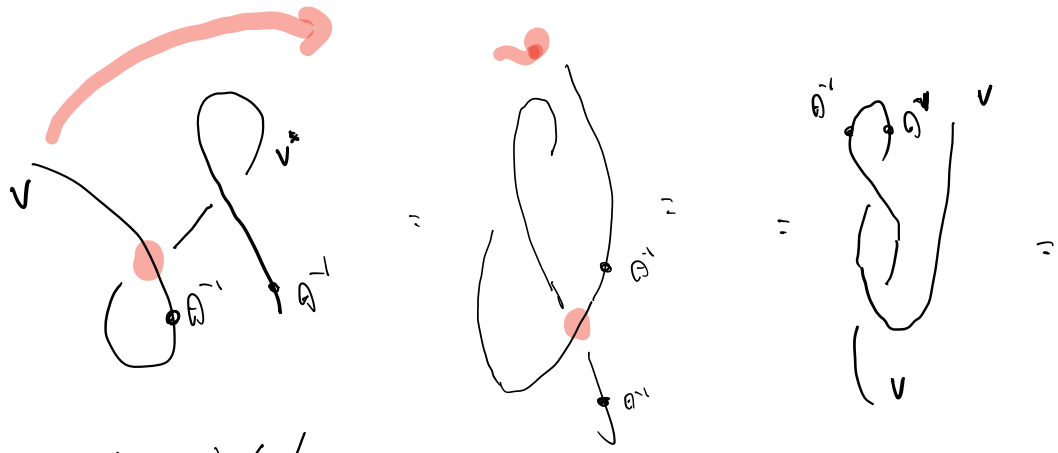
$\mathrm{ev}'_V : V^* \otimes V \to F$

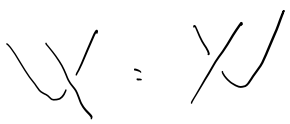
DEFINE THEM THUS:



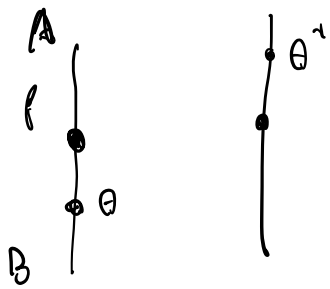


THESE MAKE V^* INTO A LEFT DUAL.





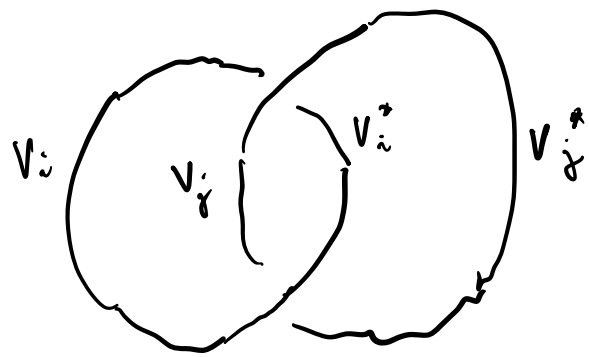
$\theta_F = 1_F$.



IN THE RIBBON CA

THE S-MATRIX IN THE MODULAR T.C.

LET i, j BE OBJECTS



THIS IS A MORPHISM $F \to F$ HENCE AN ELEMENT OF THIS RING $\mathrm{END}(F)$.

IN THE EXAMPLES $\mathrm{END}(F)$ WILL BE A FIELD.

$\tilde{\Delta}_{i,j}$ = THIS SCALAR.

THE S-MATRIX $\tilde{S} = (\tilde{\Delta}_{i,j})$

INDEXED BY PAIRS OF SIMPLE OBJECTS.

<u>THEOREM</u>: THERE IS AN ACTION OF $SL_2(\mathbb{Z})$ ON THE SIMPLE OBJECTS OF THE CATEGORY.

$$SL_2(\mathbb{Z}) = \left\langle \underset{S}{\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}, \underset{T}{\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\rangle$$

$$ST = \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix} \qquad (ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(ST)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

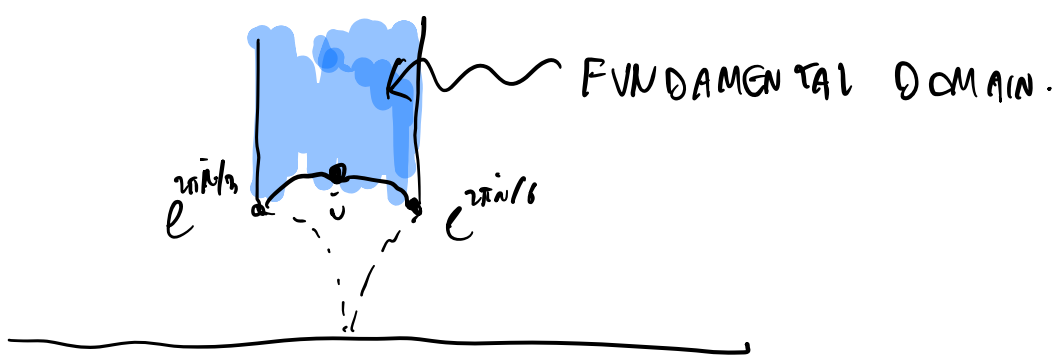
$S^2 = -I$ CENTRAL $\qquad (-I)^2 = 1$

$(ST)^3 = -I$

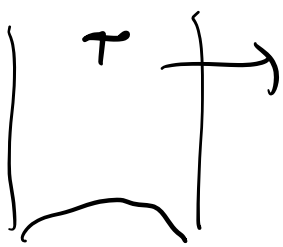
THIS IS A PRESENTATION.

$SL_2(\mathbb{R}) \circlearrowright \mathcal{H} = \{ z = x + iy \mid y > 0 \}$

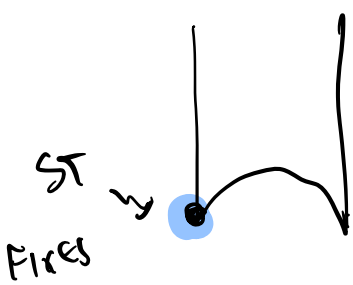
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \dfrac{az+b}{cz+d}$



$$S : z \to -\frac{1}{z}$$



$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} : z \to z+1$$



$-I$ ACTS TRIVIALLY ON $\mathcal{H}$.