Drinfeld Double (continued)
Jones Polynomial

RIBBON HOPF ALGEBRA

$\downarrow$

RIBBON CATEGORY

$\downarrow$ TURAEV-RESH.

KNOT INVARIANTS

FOLLOWING FOR ABOVE: KASSEL, ROSSO, TURAEV (1997)
HARD TO FIND.

DRINFELD DOUBLE:

A = FINITE DIM'L HOPF ALGEBRA

$$H = A^* \otimes A^{op}$$

$$H \otimes H = A^* \otimes A^{op} \otimes A^* \otimes A^{op}$$

CONTAINS 2 COPIES OF THE CANONICAL ELT

$$T = \sum_{i=1}^{\dim H} e_i \otimes e_i^* \quad \text{IN} \quad A^{opp} \otimes A^* .$$

$$T^{-1} = S^{-1}(e_i) \otimes e_i^*$$ PROVED THURSDAY.

THE 2 COPIES ARE:

$$1 \otimes e_i \otimes e_i^* \otimes 1 \rightsquigarrow \text{DRINFELD TWIST } D(A)^*.$$

AND $$e_i^* \otimes 1 \otimes 1 \otimes e_i \rightsquigarrow \text{QUASITRIANGULAR STRUCTURE IN } D(A).$$

YOU CAN GENERALIZE CONSTRUCTION TO WORK WITH $A^* \otimes B^{op}$, A, B ARE HOPF ALGEBRAS (NOT NEC. FINITE-DIM'L) IN DUALITY.

THIS LEADS TO $U_q(\mathfrak{g})$ AND ITS QUASITRIANGULARITY. (IF q IS A ROOT OF UNITY $U_q(\mathfrak{g})$ HAS A FINITE DIM'L QUOTIENT THAT IS QUASITRIANGULAR. OTHERWISE IT IS STILL TRUE $U_q(\mathfrak{g})$ IS "MORALLY QUASITRIANGULAR".

$$A, B = \underset{\substack{\uparrow \\ \text{UPPER } \Delta \\ \text{ELTS OF} \\ \mathfrak{g} = \mathfrak{gl}(n)}}{U_q(\mathfrak{b}_+)}, \qquad \underset{\substack{\uparrow \\ \text{LOWER } \Delta.}}{U_q(\mathfrak{b}_-)}.$$

MORE IMMEDIATELY IF G IS A FINITE GROUP

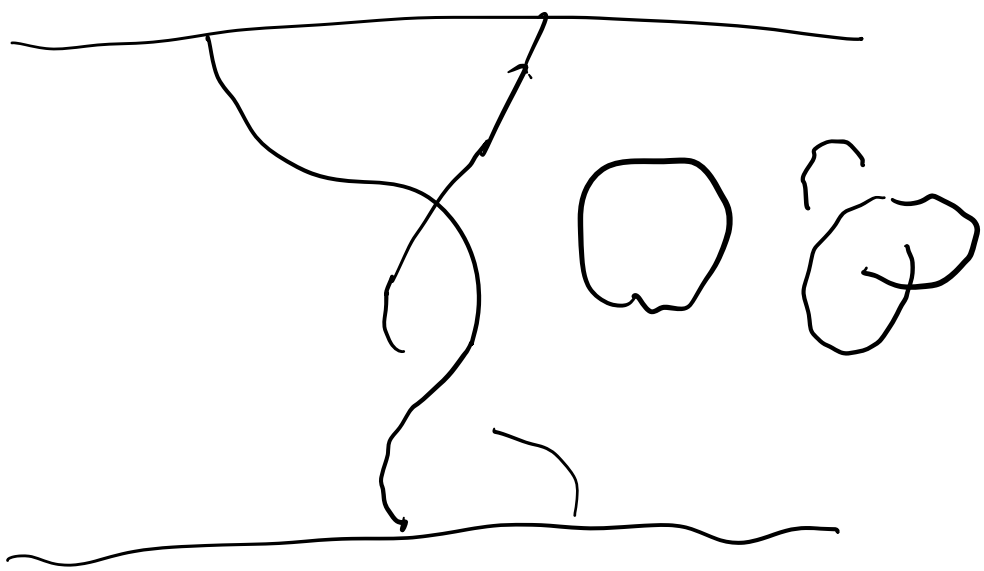
$D(A)$ $\quad A = \mathbb{C}[G]$ PRODUCES KNOT INVARIANTS RELATED TO

$$\mathrm{Hom}(G, \pi_1(S_3 - \text{KNOT}))$$

E.G. SUFFICIENT TO SHOW TREFOIL KNOT AND ITS MIRROR IMAGE ARE DISTINCT.

TODAY:

CATEGORY OF TANGLES



$\downarrow$ FUNCTOR

GIVEN RIBBON CATEGORY

GIVEN A RIBBON CATEGORY $V \cong V^{**}$ CANONICALLY.

CONSTRUCTION OF $D(A)^*$ = TWIST OF $A^* \otimes A^{op} = H$.

DRINFELD TWISTING: IF $F \in H \otimes H$ INVERTIBLE TRY TO MODIFY COMULTIPLICATION OF H BY

$$\Delta_F(x) = F\Delta(x)F^{-1}$$

FOR THIS TO BE COASSOCIATIVE PROVED CONDITION IS

$$\boxed{\underbrace{F_{12}(\Delta\otimes 1)(F)}_{\in\, H\otimes H\otimes H} = F_{23}(1\otimes\Delta)(F)}$$

PROVED THURSDAY. ALSO NEED $\underbrace{(\varepsilon\otimes 1)F}_{=\,(1\otimes\varepsilon)F} = 1_{H\otimes H}$

$$F^{-1} = 1 \otimes e_i \otimes e_i^* \otimes 1 \in A^* \otimes A^{op} \otimes A^* \otimes A^{op}$$

$$F = 1 \otimes S^{-1}(e_i) \otimes e_i^* \otimes 1$$ (PROVED USING CONVOLUTION THEORY).

WANT TO CHECK $F_{12}(\Delta \otimes 1)(F) = \cdots$

$T = e_i \otimes e_i^* \in A^{op} \otimes A^*$ SATISFIES

$$(\Delta \otimes 1)T = T_{13}T_{23} \quad \text{IN } A^{op} \otimes A^{op} \otimes A^*$$

$$(1 \otimes \Delta)T = T_{13}T_{12} \quad \text{IN } A^{op} \otimes A^* \otimes A^*$$

PROVED LAST TUESDAY. THIS IMPLIES

$$(\Delta \otimes 1)(F^{-1}) = (F^{-1})_{13}(F^{-1})_{23}$$

$$(\Delta \otimes 1)(F^{-1}) =$$

$$(\Delta \otimes 1)(1 \otimes e_i \otimes e_i^* \otimes 1)$$

$$= (1 \otimes (e_i)_{(1)} \otimes 1 \otimes (e_i)_{(2)} \otimes e_i^* \otimes 1)$$

$$\in H \otimes H \otimes H$$

IF C, D ARE HOPF ALGEBRAS

$$\Delta_{C \otimes D}(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$$

$$= (1 \otimes e_i \otimes 1 \otimes 1 \otimes e_i^* \otimes 1)$$
$$(1 \otimes 1 \otimes 1 \otimes e_j \otimes e_j^* \otimes 1) = (F^{-1})_{13} (F^{-1})_{23}$$

$$(\Delta \otimes 1) T = (\Delta \otimes 1)(\sum e_i \otimes e_i^*) =$$
$$T_{13} T_{23} = (e_i \otimes 1 \otimes 1 \otimes e_i^*)$$
$$(1 \otimes 1 \otimes e_j \otimes e_j^*)$$

$$T = \sum e_i \otimes e_i^* \qquad A^{op} \otimes A^* \otimes A^{op} \otimes A^*$$
$$\in$$
$$A^{op} \otimes A^*$$

$$(\Delta \otimes 1)(F^{-1}) = (F^{-1})_{13} (F^{-1})_{23}$$

TAKE INVERSES

$$(\Delta \otimes 1)(F) = F_{23} F_{13}.$$

SIMILARLY

$$(1 \otimes \Delta)(F) = F_{12} F_{13}.$$

$$F_{12}(\Delta \otimes 1)(F) = F_{12} F_{23} F_{13}$$
$$= F_{23} F_{12} F_{13} = F_{23}(1 \otimes \Delta)(F).$$

BUT F_{12} AND F_{23} COMMUTE:

$$F_{12} = 1 \otimes S^{-1}(e_i) \otimes e_i^* \otimes 1 \otimes 1 \otimes 1$$

$$F_{23} = 1 \otimes 1 \otimes 1 \otimes S^{-1}(e_i) \otimes e_i^* \otimes 1$$

SO IT IS POSSIBLE TO TWIST.

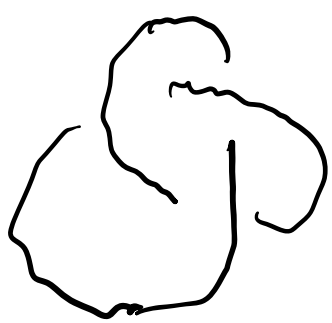
SEE LECTURE 11 FOR WRITTEN OUT PROOF.

SO $$D(A)^* = A^* \otimes A^{op}$$ WITH

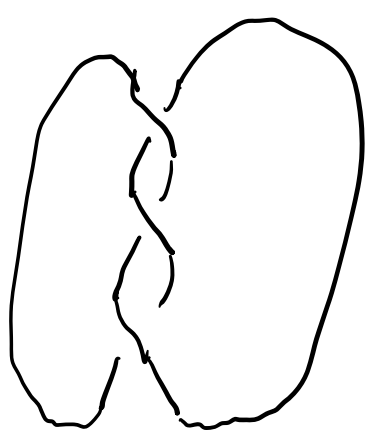
COMULTIPLICATION MODIFIED.

APPROACH GOING BACK TO TAIT.

PROJECT KNOT ONTO PLANE ALSO



TREFOIL KNOT



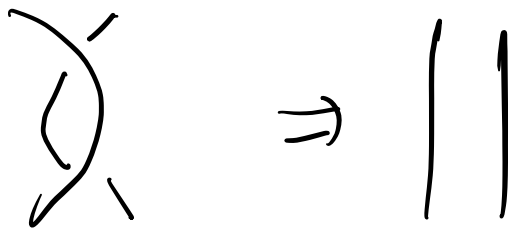
TREFOIL KNOT.

KNOT DIAGRAMS IN PLANE ARE SIMPLE CLOSED CURVES WITH CROSSINGS MARKED AS OVER OR UNDER

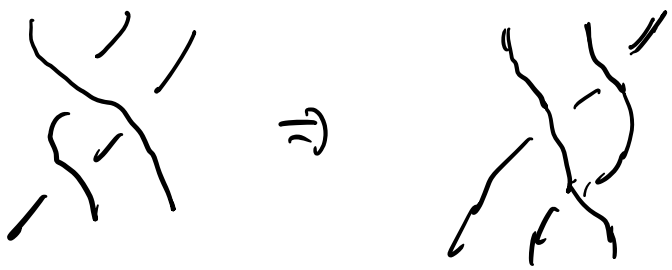
LINK OR KNOT DIAGRAMS ARE EQUIVALENT $\Leftrightarrow$ EQUIVALENT BY REIDEMEISTER MOVES

R1

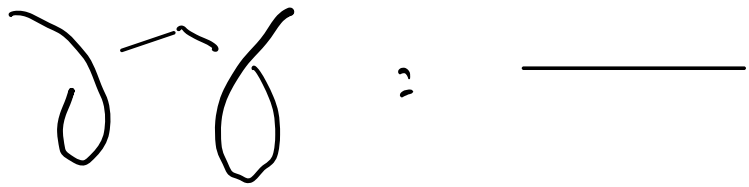
R2



R3



R1 IS PROBLEMATIC. WEAKER:

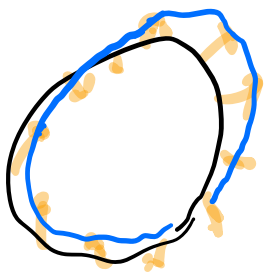
R0 

OVER AND UNDER CROSSINGS CAN BE CANCELLED.

TWO WAYS OF DECORATING A KNOT OR LINK.

(1) ORIENTING; ASSIGN AN ORIENTATION TO EACH CONNECTED COMPONENT. A KNOT HAS 2 ORIENTATIONS

(2) FRAMING; ASSIGN TO EACH CONNECTED COMPONENT A NORMAL VECTOR FIELD WHICH UP TO ISOTOPY IS DETERMINED BY AN INTEGER.



OBTAIN A RIBBON BY DEFORMING THE KNOT IN DIRECTION OF V.F.

THE LINKING NUMBER OF THE ORIGINAL AND

DEFORMED CIRCLES IS AN INTEGER

$\frac{1}{2}$(# OF OVERCROSSINGS − # OF UNDER.)

FOR FRAMED LINKS, R1 FAILS BUT R0 REMAINS OK!

TWO KNOTS ARE EQUIV. $\Leftrightarrow$ KNOT DIAGRAMS W.R.T. PLANE PROJECTIONS DIFFER BY APPLICATION OF R1, R2, R3.

FRAMED KNOTS (OR LINKS): USE R0, R2, R3.

AN EXAMPLE OF AN INVARIANT OF FRAMED LINKS IS KAUFMANN BRACKET.

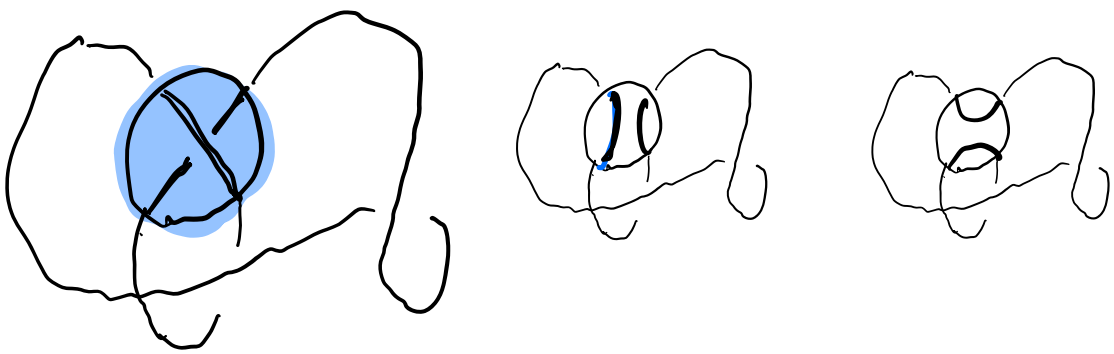
EXAMPLE OF INVARIANT OF ORIENTED LINKS IS JONES POLYNIMIAL.

FIX A NUMBER $a \in \mathbb{C}^\times$, $d = -a^2 - a^{-2}$

KAUFFMAN BRACKET: $\langle L \rangle$ L = LINK DIAGRAM.

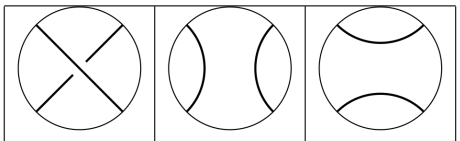
DEFINED BY SPECIFYING A SKEIN RELATION.

SUPPOSE 3 KNOTS OR LINKS ARE IDENTICAL EXCEPT FOR A SMALL REGION.



L L' L''

$$\langle L \rangle = a\langle L' \rangle + a^{-1}\langle L'' \rangle$$



L L' L''

SEE LECTURE 8.

The figures agree outside the circled portion. We define:

$$\langle L \rangle = a\langle L' \rangle + a^{-1}\langle L'' \rangle.$$

WE ALSO WANT TO SPECIFY IF L IS A DISJOINT UNION OF n UNLINKED CIRCLES

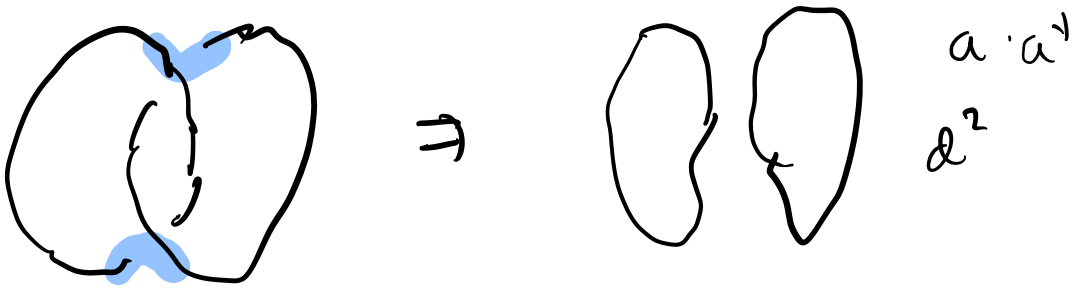
$$\langle L \rangle = d^n \qquad d = -a^2 - a^{-2}.$$

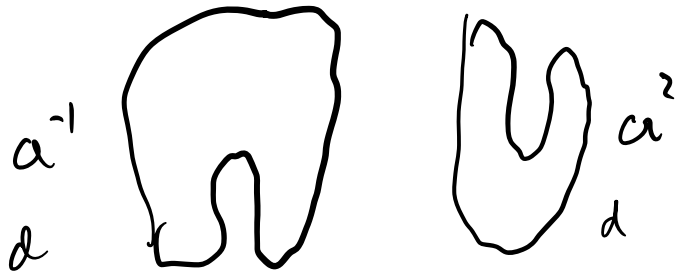
HOW DO WE KNOW WE CAN DO THIS?

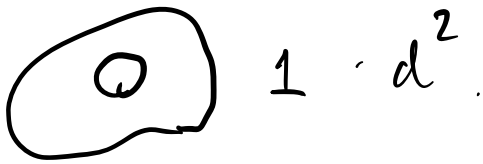
THE LINK DIAGRAM WILL HAVE m CROSSINGS

FOR EACH OF THESE WE CAN CHOOSE TO REPLACE BY EITHER

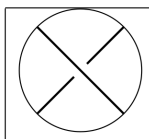
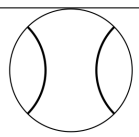
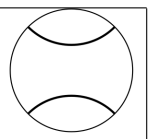
AND THERE ARE 2^m DIFFERENT DIAGRAMS OBTAINABLE BY THESE CHOICES. EACH CHOICE MULTIPLIES BY EITHER a OR a^{-1}. AND IT DOESN'T MATTER THE ORDER IN WHICH WE DO THEM. EACH CHOICE RESULT IN SOME NUMBER k OF UNLINKED CIRCLES.







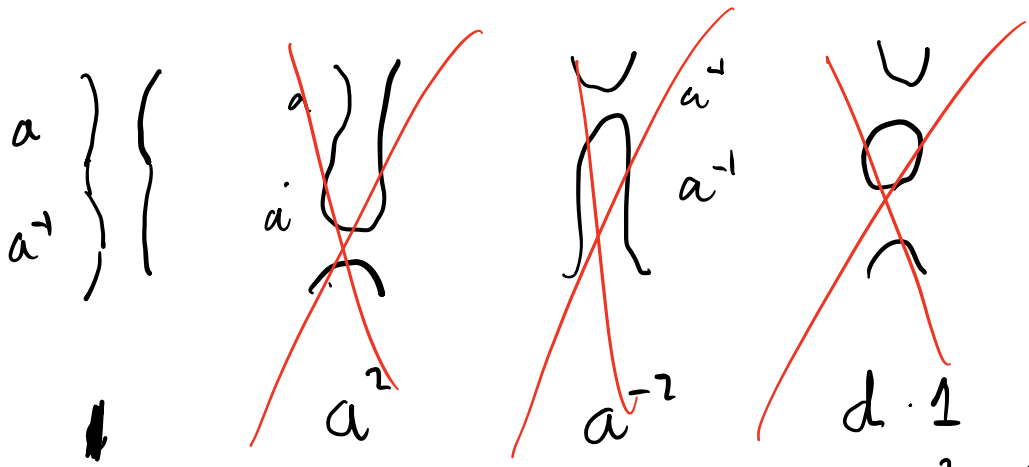
SO DEFINING KAUFFMAN BRACKET IS NOT A PROBLEM BUT I NEED TO CHECK IT IS UNCHANGED BY R0, R2, R3.

	a	a^{-1}
		
L	L'	L''

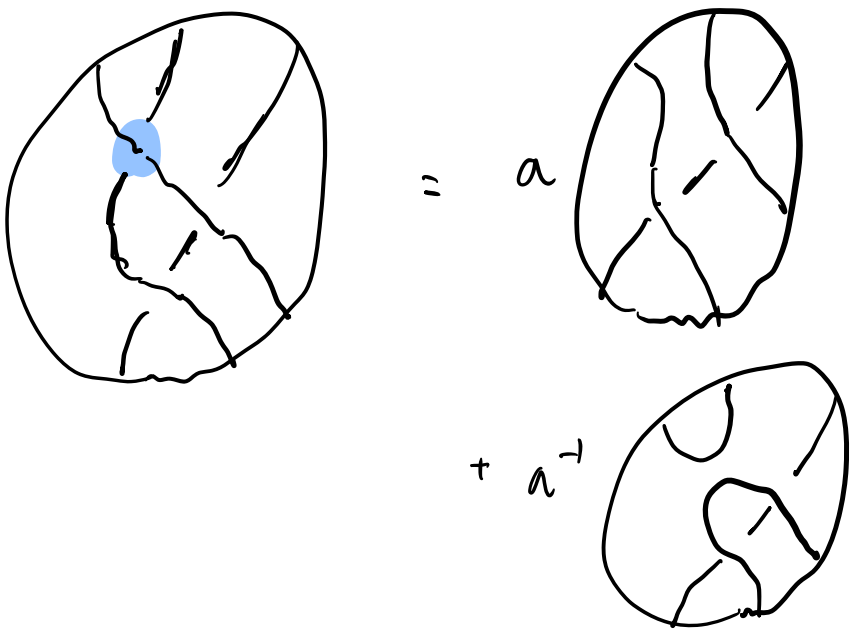
The figures agree outside the circled portion. We define:

$$\langle L \rangle = a\langle L' \rangle + a^{-1}\langle L'' \rangle.$$

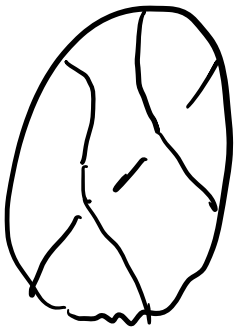
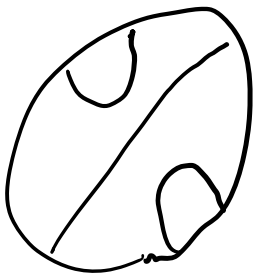
R2

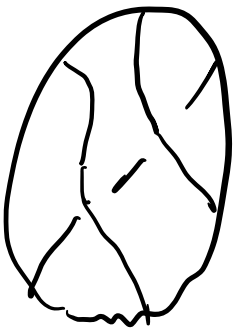
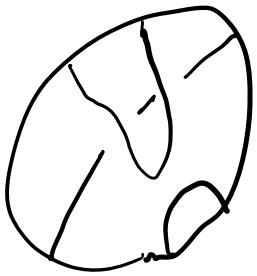


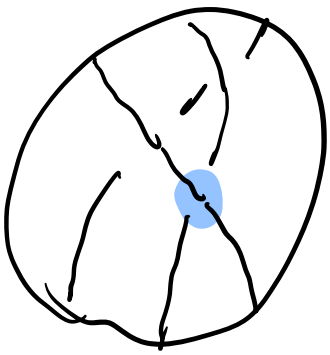
$= -a^2 - a^{-2}.$



USING R2 INVARIANCE THIS EQUALS

$=$ a  $+$ a^{-1} 

$=$ a  $+$ a^{-1} 

$=$ 

$\langle L \rangle$ GIVES US AN INVARIANT OF FRAMED LINKS. FRAMING NEEDED SINCE HAVE R2, BUT NOT R1.