Ribbon Categories

MAJID LEMMA 5.2

1. $(\epsilon \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes \epsilon)\mathcal{R} = 1$.

2. $(H, \mathcal{R}_{21}^{-1})$ *is also a quasitriangular bialgebra* ($\mathcal{R}_{21}^{-1} = \tau(\mathcal{R}^{-1})$ *is called the 'conjugate' quasitriangular structure).*

3. $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ *holds in* $H \otimes H \otimes H$ *(the Yang–Baxter equation).*

QTHA MEANS: IF $h \in H$

(1) $\tau \Delta h = R (\Delta h) R^{-1}$ IN $H \otimes H$

(2) $(\Delta \otimes 1) R = R_{13} R_{23}$

(3) $(1 \otimes \Delta) R = R_{13} R_{12}$

PROOF OF LEMMA:

$(\varepsilon \otimes 1) R = 1$ IN $H \otimes H$

$(\varepsilon \otimes 1 \otimes 1)(\Delta \otimes 1) R = (\varepsilon \otimes 1 \otimes 1) R_{13} \cancel{R_{23}}$

$(\varepsilon \otimes 1 \otimes 1) R^{(1)}{}_{(1)} \otimes R^{(1)}{}_{(2)} \otimes R^{(2)}$ $\qquad \varepsilon(R^{(1)}{}_{(1)}) \in F$

$1 \otimes \varepsilon(R^{(1)}{}_{(1)}) R^{(1)}{}_{(2)} \otimes R^{(2)}$

COUNIT PROPERTY: $\varepsilon(a_{(1)}) a_{(2)} = a$.

$= 1 \otimes R^{(1)} \otimes R^{(2)} = \cancel{R_{23}}$

BUT R_{23} IS INVERTIBLE

$$(\varepsilon \otimes 1 \otimes 1) R_{13} = 1 \quad \text{IN} \quad H \otimes H \otimes H$$

$$((\varepsilon \otimes 1) R)_{13}$$

THIS IMPLIES $(\varepsilon \otimes 1) R = 1$ IN $H \otimes H$.

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

PROOF: $(1 \otimes \tau \Delta) R =$

$$(1 \otimes \tau)(1 \otimes \Delta) R = (1 \otimes \tau) R_{13} R_{12}$$

$$= R_{12} R_{13}$$

ALTERNATIVELY REMEMBER

$$\tau \Delta h = R \Delta h R^{-1}$$

$$(1 \otimes \tau \Delta R) = R^{(1)} \otimes \tau \Delta R^{(2)}$$

$$= R^{(1)} \otimes (R \Delta R^{(2)} R^{-1})$$

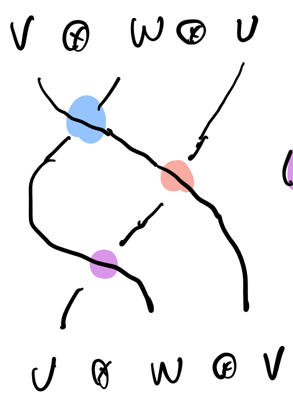
$$R_{23} ((1 \otimes \Delta) R) R_{23}^{-1}$$

$$= \boxed{R_{23} R_{13} R_{12} R_{23}^{-1}}$$

SO $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$.

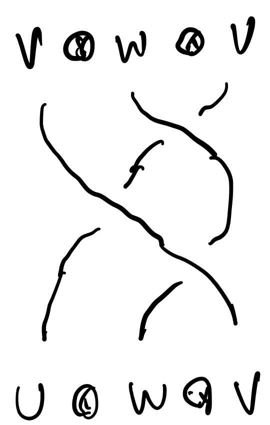
WHY DOES $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ DESERVE TO BE CALLED YANG-BAXTER EQ?

IN A BRAIDED CATEGORY, YBE IS THE EQUIVALENCE OF



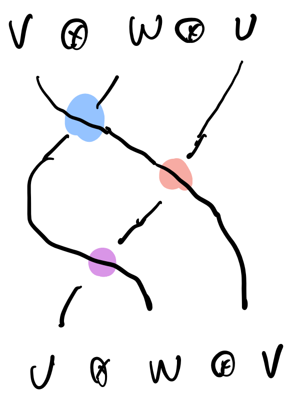
$$(c_{w,u} \otimes 1)(1 \otimes c_{v,u})(c_{v,w} \otimes 1_u)$$

$$\parallel$$



$$(1 \otimes c_{v,w})(c_{v,u} \otimes 1)(1 \otimes c_{w,u})$$

TO RELATE THIS TO $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ REMEMBER $c_{U,V} : U\otimes V \to V \otimes U$ IS IMPLEMENTED BY MULT BY $R \in H\otimes H$ FOLLOWED BY THE FLIP. THE ACTUAL YBE SHOULD BE WRITTEN



$$(\tau R\otimes 1)(1\otimes \tau R)(\tau R\otimes 1)$$
$$= (1\otimes \tau R)(\tau R\otimes 1)(1\otimes \tau R)$$

"BRAID RELATION"

LET'S MOVE THE τ TO ONE SIDE.

LHS: $(\tau\otimes 1)R_{12}(1\otimes\tau)R_{23}(\tau\otimes 1)R_{12}$

$= (\tau\otimes 1)(1\otimes\tau)R_{13}R_{23}(\tau\otimes 1)R_{12}$

$= (\tau\otimes 1)(1\otimes\tau)(\tau\otimes 1)R_{23}R_{13}R_{12}$

RHS: (SIMILAR) $(1\otimes\tau)(\tau\otimes 1)(1\otimes\tau)R_{12}R_{13}R_{23}$

IN S_3 $\quad (12)(23)(12) = (23)(12)(23)$

MEANING $(1\otimes T)(T\otimes 1)(1\otimes T) = (T\otimes 1)(1\otimes T)(T\otimes 1)$.

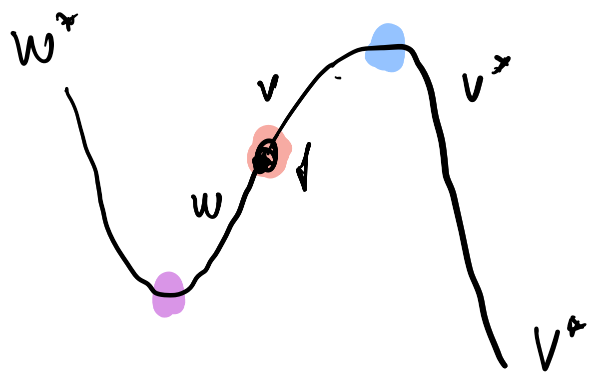
TWO PROOFS OF YBE IN MODULE CATEGORY OF A QTHA.

IN A RIGID CATEGORY DUAL IS A CONTRAVARIANT FUNCTOR,

$$f : V \to W$$

I CAN DEFINE $f^* : W^* \to V^*$

USING PROPERTIES OF EVAL AND COEVAL.



$$W^* \xrightarrow{1\otimes \mathrm{coev}_V} W^*\otimes V\otimes V^* \xrightarrow{1\otimes f\otimes 1} W^*\otimes W\otimes V^* \xrightarrow{\mathrm{ev}_W\otimes 1_{V^*}} V^*$$

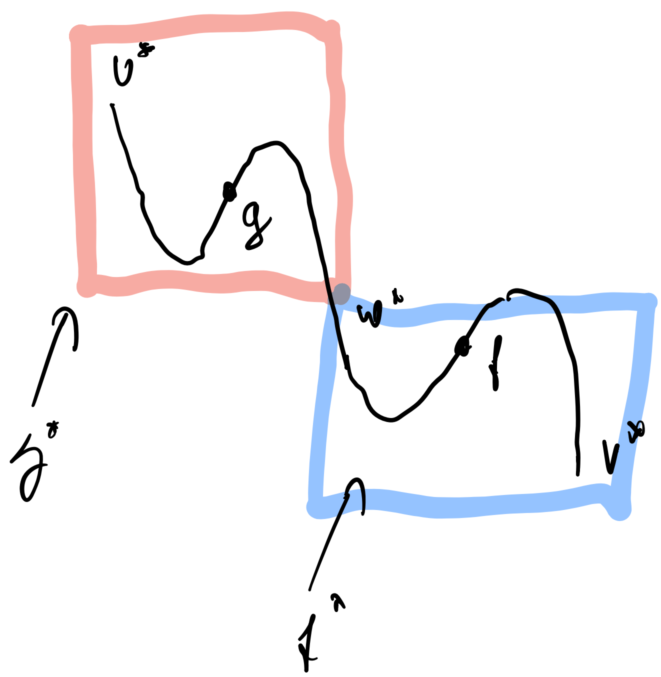
STRAIGHTFORWARD TO CHECK IF

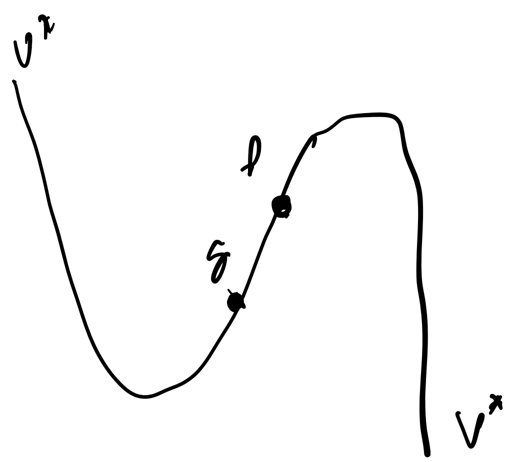
$$V \xrightarrow{f} W \xrightarrow{g} U$$

$$(gf)^* = f^* g^*$$

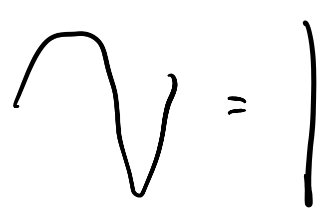
PROOF:

$f^* g^* =$





USING



$= (gf)^*$

ALSO $(V \otimes W)^* = W^* \otimes V^*$.

DEFINE SUITABLE EV AND COEV FOR

$V \otimes W$

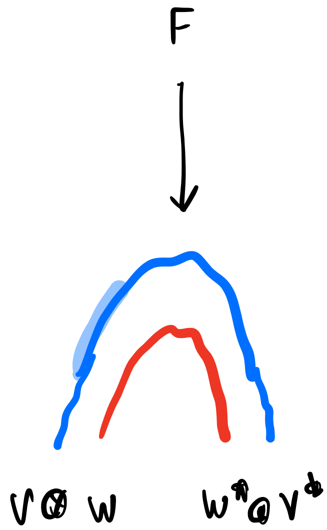
$$\mathrm{coev} : F \to (V \otimes W) \otimes (W^* \otimes V^*)$$

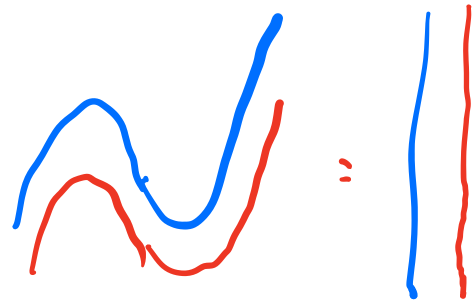
PURPORTED DUAL.

$$\mathrm{ev} : (W^* \otimes V^*) \otimes (V \otimes W) \to F$$

WHICH HAVE STRAIGHTENING PROPERTIES.

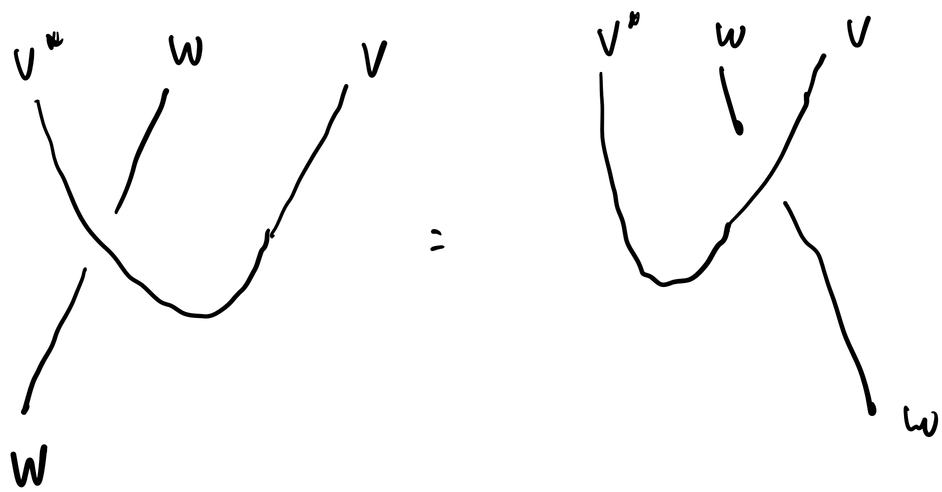
$$\begin{tikzcd}
F \arrow[d, "\mathrm{coev}_V"] \arrow[dd, bend right=60, "\mathrm{coev}_{V \otimes W}"'] & \\
V \otimes V^* \cong V \otimes F \otimes V^* \arrow[d, "1 \otimes \mathrm{coev}_W"] & \\
V \otimes W \otimes W^* \otimes V^* &
\end{tikzcd}$$





FOLLOWS FROM STRAIGHTENING FOR V, W, SEPARATELY.

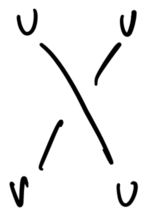
NEXT IN A BRAIDED RIGID CATEGORY



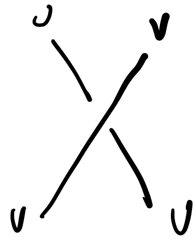
$$V^* \otimes W \otimes V \xrightarrow{c_{V^*,W} \otimes 1_V} W \otimes V^* \otimes V \xrightarrow{1 \otimes ev_V} W$$

LEFT PICTURE

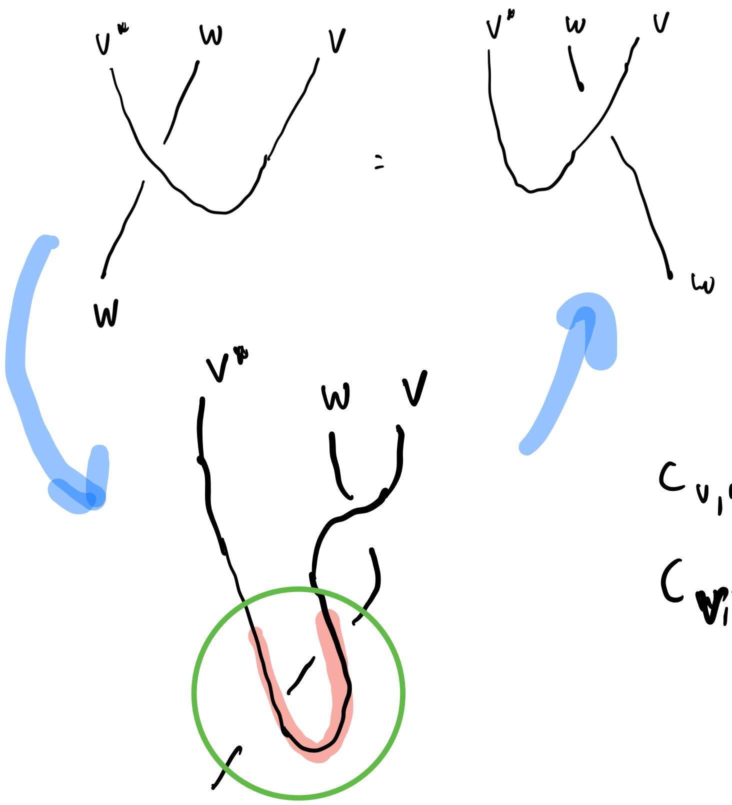
$$V^* \otimes W \otimes V \xrightarrow{1 \otimes c_{V,W}^{-1}} V^* \otimes V \otimes W \xrightarrow{ev_V \otimes 1_W} W$$



$c_{V,V}$



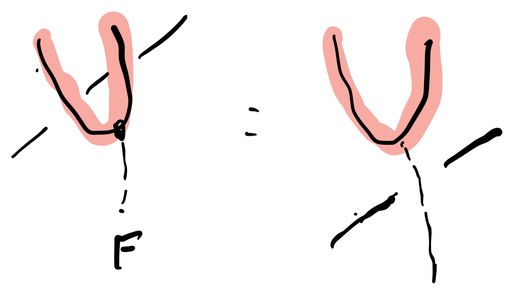
$c_{V,V}^{-1}$



$c_{V,W}^{-1}$

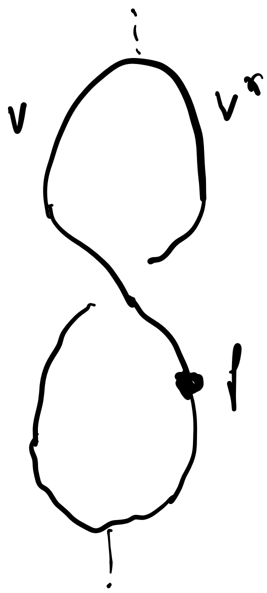
$c_{V,W}$

USE NATURALITY:



RIBBON CATEGORY.

LIKE TO DEFINE $\mathrm{TR}(f)$ $\quad f: V \to V$



THIS GIVES A MORPHISM

$F \to F$

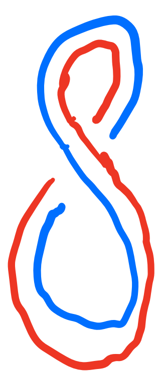
I.E. A SCALAR WHICH WE CAN DEFINE TO BE THE TRACE BUT

$\mathrm{TR}(f \otimes g)$ $\quad f \in \mathrm{END}(V)$

$\parallel$ $\quad g \in \mathrm{END}(W)$

$\mathrm{TR}(f)\,\mathrm{TR}(g)$

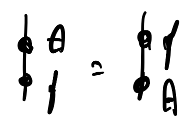
BECAUSE



TWO CIRCLES ARE LINKED

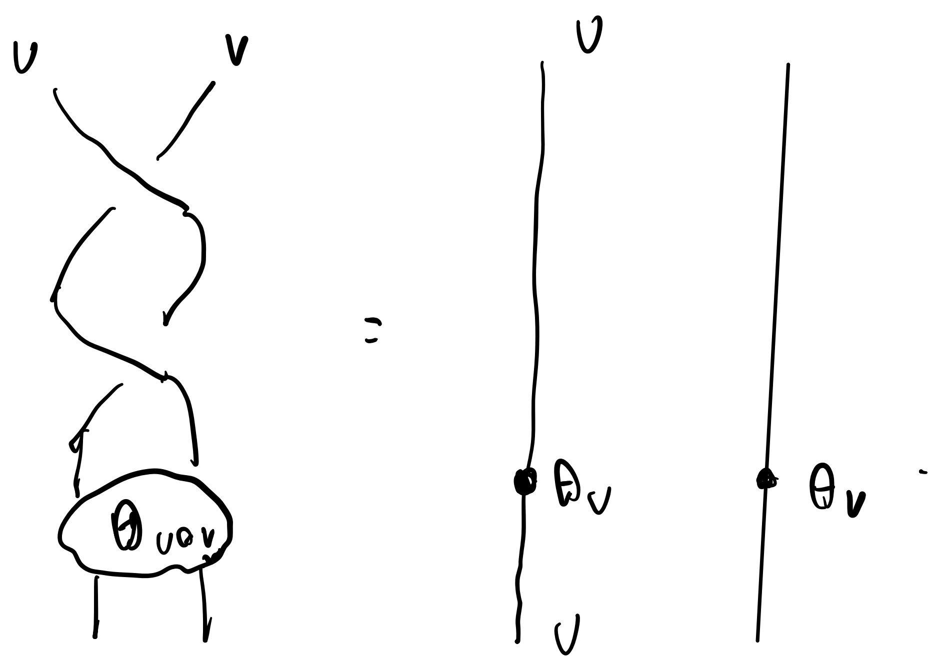
IN THE RIBBON CATEGORY YOU CAN FIX THIS.

DEFINITION: A RIBBON CATEGORY IS ONE WHERE EVERY OBJECT V HAS A TWIST

$\theta_V : V \to V$ NATURAL 

SUCH THAT

$$\theta_{U\otimes V}(c_{V,U} \circ c_{U,V}) = \theta_U \otimes \theta_V$$



ALSO ASSUME $\theta_{V^*} = (\theta_V)^*$

Definition 12.2 *A braided category is called* ribbon *(or 'tortile') if the natural transformation* $\mathfrak{v} \circ \mathfrak{u}$ *has a square root natural isomorphism* $\nu : \mathrm{id} \to \mathrm{id}$ *(*id *the identity functor) characterised by a collection of functorial isomorphisms obeying*

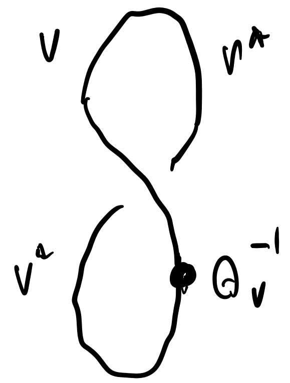
$$\nu_V^2 = \mathfrak{v}_V \circ \mathfrak{u}_V, \qquad \nu_{V \otimes W} = \Psi^{-1}_{V,W} \circ \Psi^{-1}_{W,V} \circ (\nu_V \otimes \nu_W),$$

$$\nu_{\underline{1}} = \mathrm{id}, \qquad \nu_{V^*} = (\nu_V)^*.$$

These conditions are not independent (for example, one can conclude the first from the latter three). In this case, one can restore multiplicativity by using a modified notion $\underline{\dim}'$ of dimension, as shown in Figure 12.3(b).

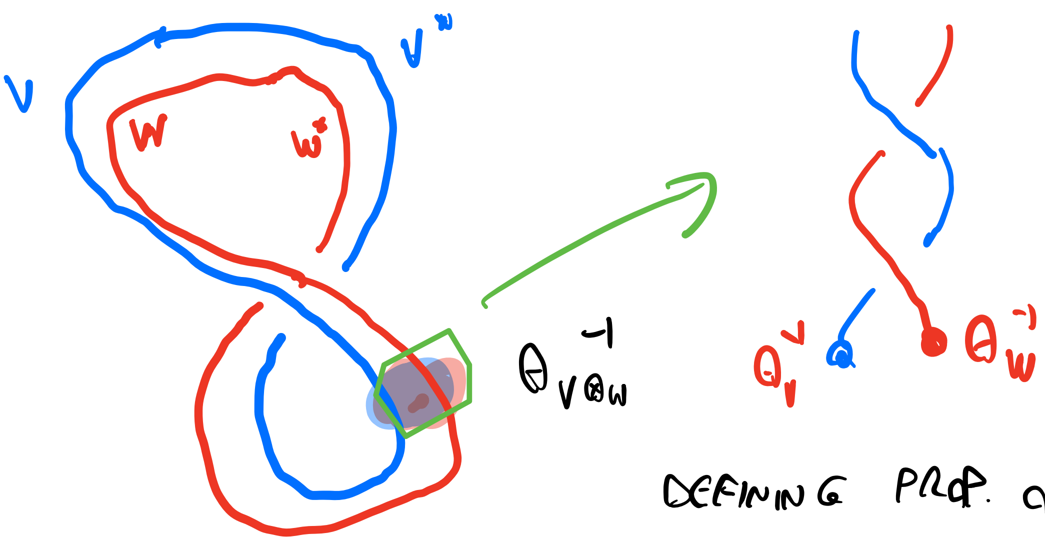
HOW THE RIBBON AXIOM FIXES THE DIFFICULTY WITH DIMENSION AND TRACE

DEFINE THE DIMENSION OF THE OBJECT V "CORRECTLY"

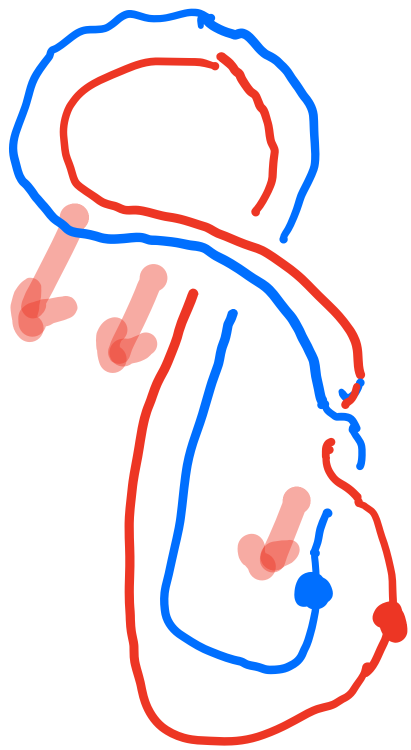


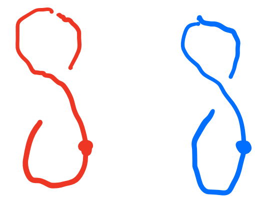
$= \dim(V)$

PROOF THAT $\dim(V \otimes W) = \dim(V) \otimes \dim(W)$



DEFINING PROP. OF θ.





USING RIBBON PROPERTY WE HAVE A GOOD DEF OF "QUANTUM" TRACES AND DIMENSIONS