Ribbon Categories

Majid Lemma 5.2

1. \((\varepsilon \otimes \text{id})R = (\text{id} \otimes \varepsilon)R = 1\).

2. \((H, R_{21}^{-1})\) is also a quasitriangular bialgebra \((R_{21}^{-1} = \tau(R^{-1})\) is called the ‘conjugate’ quasitriangular structure).

3. \(R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\) holds in \(H \otimes H \otimes H\) (the Yang–Baxter equation).

RTHA means: if \(A \in H\)

\begin{align*}
1. \quad \Delta A & = R(\Delta A) R^{-1} \quad \text{in} \quad H \otimes H \\
2. \quad (\Delta \otimes 1)R & = R_{13}R_{23} \\
3. \quad (1 \otimes \Delta)R & = R_{13}R_{12}
\end{align*}

Proof of Lemma:

\((\varepsilon \otimes \varepsilon)R = 1 \quad \text{in} \quad H \otimes H\)

\((\varepsilon \otimes \varepsilon)(\Delta \otimes \varepsilon)R = (\varepsilon \otimes \varepsilon)R_{13}R_{23}\)

\begin{align*}
(\varepsilon \otimes \varepsilon)R & = 1 \otimes \varepsilon \left( R^{(1)} \right) \otimes R^{(2)} \otimes R^{(3)} \\
& \quad \varepsilon \left( \Omega^{(1)} \right) \cdot F \\
& \quad 1 \otimes \varepsilon \left( \Omega^{(1)} \right) R^{(1)} \otimes R^{(2)} \\
& \quad \varepsilon \left( \alpha_{(1)} \right) \alpha_{(2)} : a
\end{align*}

\begin{align*}
\varepsilon & = 1 \otimes R^{(1)} \otimes R^{(2)} = R_{13}^{-1}
\end{align*}
But $R_{22}$ is invertible

$$(E \otimes (E \otimes I)) R_{13} = I \quad \text{in } H \otimes H \otimes H$$

$$(E \otimes I) R_{13}$$

This implies $(E \otimes I) R = I$ in $H \otimes H$.

$R_{12} R_{12} R_{23} = R_{23} R_{12} R_{12}$

**Proof:**

$(1 \otimes T \Delta) R = (1 \otimes T) R_{13} R_{12}$

$= R_{12} R_{13}$

**Alternatively, remember**

$T \Delta R = R \Lambda H R^{-1}$

$(1 \otimes T \Delta R) = R'' \otimes T \Delta R''$

$= R'' \otimes (R \Lambda H R^{-1})$

$R_{23} (1 \otimes A) R | R_{23}^{-1}$
= \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \mathcal{R}_{23}^{-1}

\begin{align*}
\text{so } \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} &= \mathcal{R}_{33} \mathcal{R}_{13} \mathcal{R}_{12}.
\end{align*}

Why does \( \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \) deserve to be called Yang–Baxter eq?

In a braided category, \( \mathcal{R} \) be is the equivalence of

\[
\begin{array}{cc}
V \otimes W \otimes V & (C_{w,v} \otimes 1) (C_{v,w} \otimes 1) (C_{v,w} \otimes 1) \\
U \otimes W \otimes V &
\end{array}
\]
to relate this to $q_{12} q_{13} q_{23} = r_{23} r_{13} r_{12}$

Remember $c_{u,v}$: $u \boxtimes v \rightarrow v \boxtimes u$ is implemented by $n_{u,v}$ by $r \in [0,1]$. Followed by the flip. The actual yba should be written.

\[ (TR @ 1) (1 @ TQ) (CT @ 1) = (1 @ TQ) (TR @ 1) (1 @ TR) \]

"BRAIO Relation".

\[ \text{LHS:} \quad (T @ 01) R_{12} (1 @ 02) R_{23} (T @ 01) R_{12} \]
\[ \quad = (T @ 01) (1 @ 02) R_{13} R_{23} (T @ 01) R_{12} \]
\[ \quad = (T @ 01) (1 @ 0T) (T @ 01) R_{23} R_{13} R_{12} \]

\[ \text{RHS:} \quad \text{(SIMILAR)} \quad (1 @ 02) (T @ 01) (1 @ 0T) R_{23} R_{13} R_{23} \]
In $S_3$, $(12)(13)(12) = (13)(12)(23)$ meaning $(10^{1})(20^{1})(10^{1}) = (20^{1})(10^{1})(10^{1})$.

Two proofs of YBE in module category of a QHA.

In a rigid category dual is a contravariant functor,

\[ f : V \to W \]

I can define $f^* : W^* \to V^*$ using properties of EVAL and COEVAL.
STRAIGHTFORWARD to CHECK IF

\[ V \xrightarrow{f} W \xrightarrow{g} V \]

\[ (g \circ f)^{*} = \lambda \ast f^* \]

**Proof:**

\[ \lambda \ast f^* \]

\[ \lambda \ast q^* \]

\[ \lambda \ast q^* \]

USING $G$

\[ V = 1 \]

\[ = (g \circ f)^* \]
Also \((V \otimes W)^* = W^* \otimes V^*\).

Defining suitable \(EV\) and \(C_{EV}\) for \(V \otimes W\)

\[
C_{EV} : F \rightarrow (V \otimes W) \otimes (W^* \otimes V^*)
\]

\[
EV : (W^* \otimes V^*) \otimes (V \otimes W) \rightarrow F
\]

which have straightening properties.

\[
\begin{array}{c}
V \otimes V^* \subseteq V \otimes F \otimes V^*
\end{array}
\]
\[ F \]
\[ \downarrow \]
\[ V \otimes W \quad W^* \otimes V^* \]

Follows from straightening from \( V \otimes W \), separately.

**Next in a braided rigid category**

\[ U^* \quad W \quad V \]
\[ \quad \quad \]
\[ V^* \quad W \quad V \]
\[ W \quad \quad W \]

\[ V^* \otimes W \otimes V \xrightarrow{C_{V^*,V} \otimes 1_V} \quad W \otimes V^* \otimes V \xrightarrow{1 \otimes C_{V,W}} \quad W \]

**Left picture**

\[ V^* \otimes W \otimes V \xrightarrow{1 \otimes C_{V,W}^{-1}} \quad U^* \otimes V \otimes W \xrightarrow{C_{V,W} \otimes 1_W} \quad W \]
Use naturality:

\[ \text{Diagram here.} \]
RIBBON CATEGORY

LIKE TO DEFINE $\text{tr}(f) : V \to V$

\[ \text{tr}(f \circ g) = \text{tr}(g) \cdot \text{tr}(f) \]

BECAUSE

THIS GIVES A MORPHISM

$F \to F$

I.E. A SCALAR WHICH WE CAN DEFINE TO BE THE TRACE BUT

$\lambda \in \text{End}(V)$

$\lambda \in \text{End}(W)$

BECAUSE

TWO CYLINDERS ARE LINKED
In the ribbon category you can fix this.

Definition: A ribbon category is one where every object $V$ has a twist

$$\theta_V : V \to V \text{ natural}$$

such that

$$Q_{u,v}(C_{v,u} \circ C_{u,v}) = \theta_u \circ \theta_v$$

Also assume $\theta_v^* = (\theta_v)^*$. 
Definition 12.2 A braided category is called ribbon (or 'tortile') if the natural transformation $\nu \circ \mu$ has a square root natural isomorphism $\nu : \text{id} \rightarrow \text{id}$ (id the identity functor) characterised by a collection of functorial isomorphisms obeying

$$\nu_V^2 = \nu_V \circ \mu_V, \quad \nu_V \otimes W = \Psi_{V,W}^{-1} \circ \Psi_{W,V}^{-1} \circ (\nu_V \otimes \nu_W),$$

$$\nu_\perp = \text{id}, \quad \nu_{\nu^*} = (\nu_V)^*.$$

These conditions are not independent (for example, one can conclude the first from the latter three). In this case, one can restore multiplicativity by using a modified notion $\text{dim}'$ of dimension, as shown in Figure 12.3(b).
Define the dimension of the object $V$.

Proof that $\dim(V \oplus W) = \dim(V) \oplus \dim(W)$.

Defining prop. of $\theta$. 
Using ribbon property, we have a good OGF of "quantum" traces and dimensions.