Point Set Topology

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Our textbook emphasizes metric spaces. However metric spaces are special cases of a more fundamental class of "spaces," namely *topological spaces*, and these are more fundamental than metric spaces. So we will introduce topological spaces before we introduce metric spaces before returning to the book.

Let X be a set. A *topology* is a set $\mathfrak{X}$ of subsets of X with the following properties.

Axiom 1 *Both X and the empty set $\varnothing$ are in $\mathfrak{X}$.*

Axiom 2 *An arbitrary intersection of elements of $\mathfrak{X}$ is in $\mathfrak{X}$. Thus if $\mathcal{U} \subset \mathfrak{X}$ then*

$$\bigcup_{U \in \mathcal{U}} U \in \mathfrak{X}.$$

Axiom 3 *A finite intersection of elements of $\mathfrak{X}$ is in $\mathfrak{X}$. Thus if $U_1, \cdots, U_n \in \mathfrak{X}$, then*

$$\bigcap_{i=1}^{b} U_i \in \mathfrak{X}.$$

The pair $(X, \mathfrak{X})$ is called a *topological space*. If the following additional axiom is satisfied, the space is called a *Hausdorff topological space*:

Hausdorff Axiom. *If $x, y \in X$ are distinct, there exist disjoint U_x and U_y in $\mathfrak{X}$ such that $x \in U_x$ and $y \in U_y$.*

Basic definitions

If $(X, \mathfrak{X})$ is a topological space, an element of $\mathfrak{X}$ is called an *open set*. If U is open, then its complement $X - U$ is called a *closed set*.

Every subset S of X contains a largest open subset, called its *interior*. Indeed, the interior S° may be defined to be be the union of all open subsets contained in S; then S° is open, and any open subset X contained in S is contained in S°. A subset U of X is open if and only if $U = U^\circ$. Thus $(U^\circ)^\circ = U^\circ$.

Similarly, every subset S is contained in a unique smallest closed set $\bar{S}$, called its *closure*. We may define $\bar{S}$ to be the intersection of all closed subsets of X that contain S. A subset V of X is closed if and only if $V = \bar{V}$.

If $x \in X$ then a *neighborhood* of x is any subset N of X such that $x \in N$, and such that N contains an open subset U of X with $x \in U$ and $U \subseteq N$. The neighborhood N may or may not be open. An open neighborhood of x is just an open set containing x.

If X and Y are sets, and $f : X \longrightarrow Y$ is a map, and $U \subseteq Y$ is any subset, then by definition

$$f^{-1}(U) = \{x \in X | f(x) \in U\}.$$

If $(X, \mathfrak{X})$ and $(Y, \mathfrak{Y})$ are topological spaces, then a map $f : X \longrightarrow Y$ is called *continuous* if $f^{-1}(U) \in \mathfrak{X}$ for all $U \in \mathfrak{Y}$. In words, f is continuous if the inverse image of an open set is open.

It is easy to see that a composition of continuous maps is continuous. That is, if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then so is $g \circ f : X \longrightarrow Z$.

Suppose that $(X, \mathfrak{X})$ is a topological space, and S is any subset. Then S may be given a topology, known as the *subspace topology*, as follows. Let $\mathfrak{S}$ be the set of subsets of S of the form $U \cap S$ where $U \in \mathfrak{X}$. It is straightforward to prove that this is a topology.

Let X and Y be topological spaces. Let $X \times Y$ be the Cartesian product of the sets X and Y. We will define a topology on $X \times Y$, called the *product topology*. A subset U of $X \times Y$ is *open in the product topology* if for every $(x, y) \in U$ (so $x \in X, y \in Y$) there exist open neighborhoods V and W of x and y, respectively, such that $V \times W \subseteq U$. Equivalently, a subset of $X \times Y$ is open if and only if it is a union of sets of the form $V \times W$ with V and W open subsets of X and Y, respectively.

Proposition 1 *Define $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ by $p(x, y) = x$ and $q(x, y) = y$. Then p and q are continuous maps. Suppose Z is any topological space and $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ are continuous maps. Define $F : Z \longrightarrow X \times Y$ by*

$$F(z) = (f(z), g(z)).$$

Then F is a continuous map.

Proof If V is an open set in X, then $p^{-1}(V) = V \times Y$ is open in $X \times Y$, so p is continuous, and similarly q. To prove that F is continuous, we must prove that if U is an open subset of $X \times Y$, then $F^{-1}(U)$ is an open subset of Z. Since every open subset of $X \times Y$ is a union of subsets of the form $V \times W$ with $V \subseteq X$ and $W \subseteq Y$ open, we may assume that $U = V \times W$ is of this form. Then $F^{-1}(U) = f^{-1}(V) \cap g^{-1}(W)$ is an intersection of two open sets, hence open. $\square$

Compactness

If X is a topological space, an *open cover* is set $\mathcal{U}$ of open subsets such that

$$X = \bigcup_{U \in \mathcal{U}} U.$$

If $\mathcal{V}$ is another open cover such that $\mathcal{V} \subseteq \mathcal{U}$ then $\mathcal{V}$ is called a *subcover*. A topological space X is called *quasicompact* if every open cover has a finite subcover. A quasicompact Hausdorff space is called *compact*. Quasicompact spaces that are not Hausdorff arise in algebraic geometry, with the Zariski topology.

The Hausdorff assumption for compact spaces is not important for us for two reasons. On the one hand, the book emphasizes metric spaces, which are automatically compact. On the other hand, the writing assignment will concern compact spaces that are not assumed to be metric spaces. However the Hausdorff assumption will play no role in the proofs.

The Heine-Borel theorem says that a closed interval $[a, b]$ in $\mathbb{R}$ is compact. Let me state without proof three further important facts about compact spaces. These are important enough that I think you should remember them. You may find it intructive to prove them yourself.

Proposition 2 *Let X be a compact topological space, and let Y be a closed subspace. Then Y is compact with the subspace topology.*

Proposition 3 *Let X and Y be compact topological spaces. Then $X \times Y$ is compact.*

Proposition 4 *Let X and Y be topological spaces with X compact. Let $f : X \longrightarrow Y$ be a continuous map. Then $f(X)$ is compact.*

Sequential compactness

If X is a topological space, then X is called *sequentially compact* if every sequence of elements of X has a convergent subsequence. Thus the Bolzano-Weierstrass asserts that a closed unit interval $[a, b]$ is sequentially compact.

It is proved in the book that if X is a metric space, then X is compact if and only if X is sequentially compact. For more general topological spaces, these are distinct notions.