Cauchy Sequences in $\mathbb{R}$

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A sequence $\{a_n\}$ of real numbers is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an N such that $|a_n - a_m| < \varepsilon$ whenever $n, m \geq N$. The goal of this note is to prove that every Cauchy sequence is convergent. This is proved in the book, but the proof we give is different, since we do not rely on the Bolzano-Weierstrass theorem.

We will call the sequence $\{a_n\}$ is *increasing* if whenever $n < m$ we have $a_n \leq a_m$. If whenever $n < m$ we have $a_n < a_m$ we will call the sequence *strictly increasing*. Similarly, the sequence $\{a_n\}$ is *decreasing* whenever $n < m$ we have $a_n \geq a_m$ and *strictly decreasing* if $n < m$ implies that $a_n > a_m$.

Proposition 1 *(i) Let $\{a_n\}$ be a bounded increasing sequence of real numbers. Then the sequence $\{a_n\}$ is convergent. The limit of the sequence coincides with its least upper bound.*

(ii) If $\{a_n\}$ is a bounded decreasing sequence, then the sequence $\{a_n\}$ converges to its greatest lower bound.

Proof The proof of (ii) is similar to (i). We will prove (i), leaving (ii) to the reader.

Since the set of a_n is bounded, it has a least upper bound L. We will prove that

$$\lim_{n \longrightarrow \infty} a_n = L.$$

Let $\varepsilon > 0$ be given. We must show that there exists an N such that if $n \geq N$ then $|a_n - L| < \varepsilon$. First, we note that there exists a positive integer N such that $a_N > L - \varepsilon$. Indeed, if no such a_N exists, then $L - \varepsilon$ is an upper bound for the set of a_n, which is a contradiction since L is the least upper bound and $L - \varepsilon < L$.

Now we may show that if $n \geq N$, then $|a_n - L| < \varepsilon$. Indeed, we have $a_n \geq a_N$ since $n \geq N$ and the sequence $\{a_n\}$ is increasing. Since $a_N > L - \varepsilon$, this means that $a_n > L - \varepsilon$. On the other hand $a_n \leq L$ since L is an upper bound. Thus $L - \varepsilon < a_N \leq L$. This implies that $|a_n - L| < \varepsilon$, and the proof is complete. □

Lemma 1 *A Cauchy sequence is bounded.*

Proof Let $\{a_n\}$ be a Cauchy sequence. By the definition of a Cauchy sequence, with $\varepsilon = 1$, there exists an N such that if $n, m \geq N$ then $|a_n - a_m| < 1$. Pick any $n_0 > N$. We will show that for any n we have $|a_n| \leq B$ where

$$B = \max\{|a_1|, |a_2|, \cdots, |a_{n_0-1}|, |a_{n_0}| + 1\}.$$

There are two cases. If $n < n_0$, then $|a_n| \leq B$ by the definition of B. On the other hand, if $n \geq n_0$ then $n, n_0 > N$ and so $|a_n - a_{n_0}| < 1$. Therefore

$$|a_n| = |a_{n_0} + (a_n - a_{n_0})| < |a_{n_0}| + |a_n - a_{n_0}| < |a_{n_0}| + 1 \leq B,$$

as required. We have proved that $\{a_n\}$ is bounded. □

Theorem 1 *Let $\{a_n\}$ be a Cauchy sequence of real numbers. Then the sequence $\{a_n\}$ is convergent.*

Proof Let us associate with the sequence $\{a_n\}$ two other sequences $\{l_n\}$ and $\{r_n\}$. We define l_n to be the greatest lower bound of the subsequence

$$\{a_n, a_{n+1}, a_{n+2}, \cdots\}. \tag{1}$$

Note that this greatest bound exists since by Lemma 1, because this is a subsequence of a bounded sequence, hence bounded. Clearly $l_n = \min(a_n, l_{n+1})$. Therefore $l_n \leq l_{n+1}$ and so the sequence l_n is increasing. It is bounded, since if $|a_n| \leq B$ for all n then $|l_n| \leq B$. By Proposition 1 the sequence $\{l_n\}$ is convergent.

Similarly, we define r_n to be the least upper bound of the subsequence (1). By the same reasoning $\{r_n\}$ is a bounded decreasing sequence, hence convergent.

Let

$$\lim_{n \longrightarrow \infty} l_n = L, \qquad \lim_{n \longrightarrow \infty} r_n = R.$$

We will show that $L = R$. Suppose that $L \neq R$. Since l_n is a lower bound of the subsequence (1) and r_n is an upper bound, we have $l_n \leq r_n$ for all n. Therefore $L \leq R$. Thus $L \neq R$ implies that $L < R$. Let $\varepsilon = \frac{1}{3}(R - L)$. Because the sequence $\{a_n\}$ is Cauchy, there exists some integer N such that if $n, m \geq N$ then $|a_n - a_m| < \varepsilon$. Because l_N is the greatest lower bound of the sequence $\{a_N, a_{N+1}, \cdots\}$, there exists $n \geq N$ such that $|a_n - l_N| < \varepsilon$. Recalling that by Proposition 1 the limit L is the least upper bound of $\{l_n\}$, we have $l_N < L$ and therefore $a_n \leq l_N + \varepsilon \leq L + \varepsilon$. Similarly, for some $m > N$, we have $a_m \geq R - \varepsilon$. Now

$$|a_m - a_n| \geq a_m - a_n > (R - \varepsilon) - (L + \varepsilon) = R - L - 2\varepsilon = \varepsilon,$$

which is a contradiction.

We have proved that $R = L$. Now we will prove that

$$\lim_{n \longrightarrow \infty} a_n = L. \tag{2}$$

Let $\varepsilon > 0$ be given. Since L is the least upper bound of $\{l_n\}$ and the greatest lower bound of $\{r_n\}$, there exists a positive integer N such that $0 \leq L - l_N < \varepsilon$ and $0 \leq r_N - L < \varepsilon$. Thus $L - \varepsilon < l_N \leq r_N \leq L + \varepsilon$. If $n \geq N$ then since l_N is the greatest lower bound of $\{a_N, a_{N+1}, a_{N+2}, \cdots\}$ and r_N is the least upper bound of this sequence, we have $l_N \leq a_n \leq r_N$. Therefore $L - \varepsilon < a_n < L + \varepsilon$ and so $|a_n - L| < \varepsilon$. This proves (2). □