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The unramified principal series of $p$-adic groups. II. The Whittaker function


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THE UNRAMIFIED PRINCIPAL SERIES OF
p-ADIC GROUPS II
THE WHITTAKER FUNCTION

W. Casselman and J. Shalika

Let $G$ be a connected reductive algebraic group defined over the non-archimedean local field $k$. We will prove in this paper an explicit formula for a certain so-called Whittaker function associated to the unramified principal series of $G(k)$, under the assumption that the group $G$ is itself unramified – that is to say, arises by base extension to $k$ from a smooth reductive group over the integers $\mathcal{O}$ of $k$. This formula has been discovered independently by Shintani [8] when $G = \text{GL}_n$ and Kato [9] for Chevalley groups, and was also in fact conjectured by Langlands several years ago (in correspondence with Godement). It plays a role in establishing the functional equations of certain $L$-functions (see [7], for example).

As a preparation for the proof of the explicit formula, we include in the paper new proofs of results due to Rodier [6] (this is done also by Bernstein and Zelevinskii in [1]). Later on, we also include a discussion of matters partly dealt with in Jacquet’s thesis [5]. Our proof of the formula itself follows along lines very similar to those pursued in [3] to prove Macdonald’s formula for the zonal spherical function. (Incidentally, the technique used by Kato and Shintani is more similar to Macdonald’s own proof of his formula.)

Notation

Throughout, algebraic groups will be written in boldface, the corresponding groups of $k$-rational points in ordinary type. Thus: $G$ and $G$.

Let

$P = \text{a minimal parabolic subgroup of } G$
A maximal split torus of G in P

M = the centralizer of A

N = the unipotent radical of P

N^- = the opposite of N

\( \Sigma \) = roots of G with respect to A

\( nd\Sigma \) = non-divisible roots of \( \Sigma \)

\( \Sigma^+ \) = positive roots determined by the choice of P

\( \Delta \) = simple roots of \( \Sigma^+ \)

W = Weyl group of \( \Sigma \)

\( \nu_\tau \) = longest element of W

For each \( \theta \subseteq \Delta \) let

\( P_{\theta} \) = the standard parabolic subgroup corresponding to \( \theta \); similarly for \( A_{\theta}, M_{\theta}, N_{\theta}, W_{\theta} \).

\( \delta_{\theta} \) = modulus character of \( P_{\theta} \), taking \( p = mn \) to \( |\det A_{\theta}(m)| \), where

\( n = \text{Lie algebra of } N \)

For each \( \alpha \in \Sigma \) let \( N_\alpha \) be the subgroup of \( N \) whose Lie algebra is \( g_\alpha + g_{2\alpha} \). Thus \( N_{\theta} \) is the product \( II N_{\alpha} \) (\( \alpha > 0 \), not divisible, not in the linear span of \( \theta \)). With a few exceptions, when we write products of expressions indexed by roots, these indices will include only elements of \( nd\Sigma \).

1. Introduction to Whittaker models

Let \( \psi \) be a smooth complex character of \( N \), and denote as \( C_\psi \) the corresponding one-dimensional \( N \)-module.

If \( (\pi, V) \) is any smooth representation of \( N \), define \( V_{\psi,N} \) to be the Jacquet space of the twisted representation \( \pi \otimes \psi^{-1} \). In other words: define \( V_{\psi}(N) \) to be the subspace of \( V \) spanned by \( \{ \pi(n)v - \psi(n)v \mid n \in N, v \in V \} \) and set \( V_{\psi,N} = V/V_{\psi}(N) \). The space \( V_{\psi}(N) \), incidentally, is also that of \( v \in V \) such that for some compact open \( N_0 \subseteq N \),

\[
\int_{N_0} \psi^{-1}(n)\pi(n)vdn = 0.
\]

The group \( N \) acts on \( V_{\psi,N} \) by \( \psi \) and \( V_{\psi,N} \) is the largest quotient of \( V \) with this property ([2] 3.2.2):
1.1. **Proposition:** If $V'$ is any space on which $N$ acts by $\psi$ then $V \rightarrow V_{\psi,N}$ induces

$$\text{Hom}_N(V, V') \cong (\text{Hom}_C(V_{\psi,N}, V')).$$

Also, by [2] 3.2.3:

1.2. **Proposition:** The functor $V \rightarrow V_{\psi,N}$ is exact.

The subgroup $\Pi N_a (\alpha > 0, \alpha \notin \Delta)$ is normal in $N$ and the quotient is isomorphic to $\Pi(N_a/N_{2a}) (\alpha \in \Delta)$. If $\psi_\alpha$ for each $\alpha \in \Delta$ is a character of $N_a/N_{2a}$ then $\Pi \psi_\alpha$ is a character of $\Pi(N_a/N_{2a})$, hence determines one of $N$ as well. A character of $N$ is said to be principal if it is of this form with no $\psi_\alpha$ trivial.

If $\psi$ is principal and $(\pi, V)$ an admissible representation of $G$, a $G$-embedding of $V$ into the smooth representation $\text{Ind}(C_\psi \mid N, G)$ of $G$ is called a Whittaker model for $V$. The space $\text{Ind}(C_\psi \mid N, G)$ (see §2.4 of [2]) is that of all $f: G \rightarrow C$ such that (1) $f(ng) = \psi(n)f(g)$ for all $n \in N, g \in G$ and (2) there exists an open subgroup $K \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K$, and $G$ acts on it by the right regular representation. (The terminology arises from the analogous theory for the real group $SL_2(\mathbb{R})$, where such functions $f$ are essentially classical Whittaker functions; see [5].) There is an intimate relationship between the space $V_{\psi,N}$ and Whittaker models. Let $\Omega$ be the map $\text{Ind}(C_\psi \mid N, G) \rightarrow C$, $f \mapsto f(1)$. It is an $N$-morphism into $C_\psi$. Frobenius reciprocity ([2] 2.4.1 (e)) and 1.1 combine to give:

1.3. **Proposition:** Let $\psi$ be any smooth character of $N$, $V$ a smooth representation of $G$. Composition with $\Omega$ induces an isomorphism

$$\text{Hom}_G(V, \text{Ind}(C_\psi \mid N, G)) \cong \text{Hom}_C(V_{\psi,N}, C).$$

Explicitly, the map $F: V \rightarrow V_{\phi,N} \rightarrow C$ corresponds to the $G$-morphism from $V$ to $\text{Ind}(C_\psi)$ which takes $v$ to the function $\Phi_v : g \mapsto F(gv)$.

When $(\pi, V)$ is induced from a proper parabolic subgroup of $G$, a result of Rodier [6] reduces the problem of determining $V_{\psi,N}$ to a similar problem for the inducing representation. We give here a proof of his result along lines used in [2] to discuss $V_N$ (which is $V_{1,N}$ in our notation).

Let $(\sigma, U)$ be an admissible representation of $M_\theta$ and let $I(\sigma) = \text{Ind}(\sigma \mid P_\theta, G)$, the corresponding induced representation of $G$. It is
admissible. Because of the disjoint decomposition

\[ G = UP_\theta \omega P \quad (w \in [W_\theta \setminus W]) \]

\( I(\sigma) \) is filtered by \( P \)-stable subspaces

\[ I_n = \{ f \in I(\sigma) \mid \text{Supp}(f) \subseteq P_\omega \omega P (\dim P_\omega \omega P \geq n) \} \]

(see [2] 6.3). The quotient \( I_n/I_{n+1} \) is isomorphic to the direct sum of spaces \( I_x \), as \( x \) ranges over a set of representatives of elements \( w \) of \([W_\theta \setminus W]\) in \( N_\theta(A) \) with \( \dim P_\theta < P_\omega \omega P = n \), and where

\[ I_x = \text{Ind}_c(x^{-1}P_\theta \omega P \cap P, P). \]

(This is [2] 6.3.2. The notation is that of §2 of [2], so that \( \text{Ind}_c \) is the non-normalized induced representation comprising functions of compact support modulo \( x^{-1}P_\theta \omega P \).)

Let \( w_{c,\theta} \) be longest element of \( W_\theta \). Then \( w_\theta = w_{c,\theta} \omega \epsilon \) is the longest element of \([W_\theta \setminus W]\) ([2], 1.1.4(b)) and \( P_\theta \omega P \omega P \) is the unique open double coset in \( P_\theta \setminus G \). Let \( d_\theta \) be its dimension. By the above remarks, we have an injection \( I_{d_\theta} \hookrightarrow I(\sigma) \).

1.4. Theorem (Rodier): If \( \psi \) is a principal character of \( N \) then the inclusion of \( I_{d_\theta} \) in \( I(\sigma) \) induces an isomorphism of \( (I_{d_\theta})_{\psi,N} \) with \( I(\sigma)_{\psi,N} \).

The proof requires a preliminary result. Let \( \Sigma_\theta^\circ \) be the intersection of \( \Sigma_\theta \) with the linear span of \( \theta \). It is, essentially, the set of positive roots of \( M_\theta \) with respect to \( A_\theta \) determined by \( P \cap M_\theta \).

1.5. Lemma: For \( w \in [W_\theta \setminus W] \), \( w \neq w_\theta \), there exists \( \alpha \in \Delta \) with \( w\alpha \in \Sigma^+ - \Sigma_\theta^\circ \).

Proof of Lemma. If \( w^{-1} \theta > 0 \) then 1.1.4(b) of [2] implies that \( l(w^{-1}_\theta) = l(w^{-1}_\omega w) + l(w^{-1}) \). Therefore by the definition of length in \( W \)

\[ \{ \alpha > 0 \mid w^{-1}_\theta \alpha < 0 \} = \{ \alpha > 0 \mid w^{-1} \alpha < 0 \} \cup w\{ \alpha > 0 \mid w^{-1}_\theta \omega \alpha < 0 \}. \]

By 1.1.4(a) of [2], the left hand side is \( \Sigma^+ - \Sigma_\theta^\circ \). Hence

\[ w^{-1}(\Sigma^+ - \Sigma_\theta^\circ) = w^{-1}\{ \alpha > 0 \mid w^{-1} \alpha < 0 \} \cup \{ \alpha > 0 \mid w^{-1}_\theta \omega \alpha < 0 \}. \]
If \( w^{-1}w \neq 1 \), the second term on the right must contain at least one element of \( \Delta \).

Conclusion of the proof of 1.4: by previous remarks and Proposition 1.2, it suffices to show that \( (I_x)_{\phi,N} = 0 \) for any \( x \) representing an element of \([W_\phi \setminus W]\) other than \( w_\phi \). By 1.1, in order to do this, it suffices to show that \( \text{Hom}_N(I_x, C_\phi) = 0 \).

An \( N \)-morphism from \( I_x \) to \( C_\phi \) may be considered as an element \( \Phi \) in the dual of \( I_x \) which is an eigenvector for \( \phi \) with eigencharacter \( \psi^{-1} \), hence as an element of the \( N \)-smooth dual of \( I_x \). Now since \( x^{-1}\theta > 0 \), \( x^{-1}(-\theta) < 0 \), so that \( x^{-1}\phi x \cap P = x^{-1}P x \cap P \), and as an \( N \)-space \( I_x \) is isomorphic to

\[
\text{Ind}_x(x^{-1}(\delta) \mid x^{-1}Nx \cap N, N).
\]

According to [2] 2.4.2, its \( N \)-smooth dual is isomorphic to

\[
\text{Ind}_x(x^{-1}(\delta) \mid x^{-1}Nx \cap N, N),
\]

where \( \delta \) is the smooth dual of \( \sigma \). Corresponding to \( \Phi \), therefore, is a function \( F: N \to \bar{U} \) (the space of \( \delta \)) such that

\[
F(n_1, n) = \sigma(xn_1x^{-1})F(n)
\]

for all \( n_1 \in x^{-1}Nx \cap N \), \( n \in N \), and

\[
F(n) = \psi^{-1}(n)F(1)
\]

for all \( n \in N \). In particular,

\[
\sigma(xn_1x^{-1})F(1) = \psi^{-1}(n_1)F(1)
\]

for all \( n_1 \in x^{-1}Nx \cap N \). Since \( \sigma \) is trivial on \( N_\phi \), in order to prove \( F(1) \) and hence \( F \) null it now suffices to find \( n_1 \in x^{-1}N_\phi x \cap N \) such that \( \psi(n_1) \neq 1 \). Since \( N_\phi = \Pi N_\alpha (\alpha \in \Sigma^+ - \Sigma^+_\phi) \) and \( \psi \) is principal, such an \( n_1 \) is guaranteed by Lemma 1.5. This concludes the proof.

Now let \( N' \) be \( M_\phi \cap N \), the unipotent radical of the minimal parabolic \( P \cap M \) in \( M_\phi \), and for \( x \in N_G(A) \) representing \( W_\phi \) let \( \psi'_s \) be the principal character of \( N' \) defined by the formula

\[
\psi'_s(n) = \psi(x^{-1}nx).
\]

This makes sense because \( w_\phi^{-1}\theta > 0 \). In fact, \( w_\phi^{-1}\theta = \bar{\theta} \) is a subset of \( \Delta \),
the conjugate of $\theta$. Because $w^{-1}_a \alpha < 0$ for every $\alpha \in \Sigma^+ - \Sigma^+$, the group $x^{-1}P_{\theta x} \cap N$ is equal to $M_\theta \cap N$. Since $N = (M_\theta \cap N)N_{\theta}$, for every $f \in I_{d_\theta}$ the function $n \mapsto f(xn)$ is a function on $N_{\theta}$ of compact support, and the integral

$$\Omega_{\sigma}(f) = \int_{N_{\theta}} f(xn)\psi^{-1}(n) \, dn$$

is well defined, and yields a map from $I(\sigma)_{d_\theta}$ to $U$.

1.6. Theorem (Rodier): The map $\Omega_{\sigma}$ induces an isomorphism of $(I_{d_\theta})_{\phi,N}$ with $U_{\phi_{\sigma},N}$.

Proof. It must first be shown that if $f$ lies in $(I_{d_\theta})_\phi(N)$ then $\Omega_{\sigma}(f)$ lies in $U_{\phi_{\sigma}}(N')$. For this, choose compact open subgroups $B_1$ of $M_\theta \cap N$ and $B_2$ of $N_{\theta}$ such that $n \mapsto f(xn)$ vanishes outside $B_2$ and $f \in (I_{d_{\theta}})_\phi(B_1 \cdot B_2)$. Then $xB_1x^{-1}$ is a compact open subgroup of $N_{\theta}$ and

$$\int_{xB_1x^{-1}} \sigma(n_1)\Omega_{\sigma}(f)\psi'_{\sigma}(n_1)^{-1} \, dn_1$$

$$= \int_{xB_1x^{-1}} \psi'_{\sigma}(n_1)^{-1} \, dn_1 \int_{B_2} f(xn_2)\psi(n_2)^{-1} \, dn_2$$

$$= \int_{B_1 \times B_2} f(xn_1n_2)\psi(n_1n_2)^{-1} \, dn_1dn_2$$

Hence $\Omega_{\sigma}$ induces a map from $(I_{d_\theta})_{\phi,N}$ to $U_{\phi_{\sigma},N'}$.

To see that this is an isomorphism, it suffices to show that the corresponding map from $\text{Hom}_N(U, C_{\phi_{\sigma}})$ to $\text{Hom}_N(I(\sigma), C_{\phi})$ is one. From the proof of 1.4, one sees that the $N$-smooth dual of $I_x$ is isomorphic to

$$\text{Ind}(x^{-1}(\hat{\sigma}) \mid M_\theta \cap N, N).$$

Following that proof a bit further one sees that the space $\text{Hom}_N(I_x, C_{\phi})$ is isomorphic to that of $\bar{u} \in \bar{U}$ such that

$$\hat{\sigma}(n)\bar{u} = \psi^{-1}(x^{-1}(x^{-1}nx)\bar{u})$$

for all $n \in M_\theta \cap N$ – i.e. to $\text{Hom}_N(U, C_{\phi_{\sigma}})$ – and if one follows details explicitly one sees that this isomorphism is the same as that induced by $\Delta$.

From 1.4 and 1.6 together:
1.7. COROLLARY: If \( \psi \) is a principal character of \( N \) then \( I(\sigma)_{\psi,N} \equiv \sigma_{\psi',N'} \), where \( x \) is any element of \( N_G(A) \) representing \( w_\theta \).

As a special case:

1.8. COROLLARY: Assume \( P_\theta = P \), \( \sigma \) one-dimensional, \( x \in N_G(A) \) representing \( w_\tau \). The functional

\[
\Omega_\sigma(f) = \int_N f(xn)\psi^{-1}(n)dn,
\]

which is defined for all \( f \in I(\sigma) \) with support on \( Pw_\tau P \), extends uniquely to a basis element of the one-dimensional space \( \text{Hom}_N(I(\sigma), C_\phi) \).

2. Holomorphicity

Our aim in this section is to show that the map \( \Omega_\sigma \) defined in §1 varies holomorphically with \( \sigma \). For convenience, we shall treat only the case we shall be concerned with later on, although it should be apparent that the argument can be generalized.

A representation of \( M \) is said to be unramified if it is trivial on the unique maximal compact subgroup \( M_0 \). Since \( M/M_0 \) is finite and free over \( Z \), the group \( X = X_{nr}(M) \) of all such characters is isomorphic to \( (C^\times)^r \) for some \( r \), and in particular has a canonical structure as a complex analytic manifold.

For \( \chi \in X \), let \( P_\chi \) be the projection from \( C_c^\infty(G) \) onto \( I(\chi) = \text{Ind}(\chi \mid P, G) \):

\[
P_\chi f(g) = \int_P \chi^{-1}\delta_{1Z}(p)f(pg)dp.
\]

Here a left Haar measure on \( P \) is assumed. Let \( \psi \) be a principal character of \( N \). Recall from §1 that \( \Omega_\chi \) is the unique \( N \)-morphism from \( I(\chi) \) to \( C_\phi \) such that for \( f \in C_c^\infty(Pw_\tau P) \) one has

\[
\Omega_\chi(P_\chi f) = \int_N P_\chi f(xn)\psi^{-1}(n)dn
\]

where \( x \) is a fixed element of \( N_G(A) \) representing \( w_\tau \). By the definition of \( P_\chi \), more explicitly:

\[
\Omega_\chi(P_\chi f) = \int_{Pw_\tau P} \Phi(y)f(y)dy
\]
where

\[ \Phi(n_1mxn_2) = \chi^{-1} \delta^{1/2}(m) \psi^{-1}(n_2) \]

on \( Pw_iP = Pw_iN \). It is clear that for a fixed \( f \in C^\infty(Pw_iP) \) this varies holomorphically with \( \chi \). In fact:

2.1. **PROPOSITION:** *For any* \( f \in C^\infty(G), \Omega_\chi(\mathcal{P}_\chi f) \) *is a holomorphic function of* \( \chi \).

Fix a compact open subgroup \( K \) with the property that \( f \) is bi-invariant under \( K \). For every compact open subgroup \( N_0 \subseteq N \) define a projection operator on \( I(\chi) \):

\[ \mathcal{P}_{\phi,N_0}\varphi(g) = (\text{meas } N_0)^{-1} \int_{N_0} \psi^{-1}(n) \varphi(gn) dn. \]

2.2. **LEMMA:** *There exists a compact open subgroup* \( N_0 \subseteq N \) *such that for every* \( \chi \in X \) *and* \( \varphi \in I(\chi)^K \) *the function* \( \mathcal{P}_{\phi,N_0}\varphi \) *has support in* \( Pw_iP \).

We first point out how the lemma implies the Proposition. Since \( f \) is bi-invariant under \( K \), \( \mathcal{P}_\chi f = \varphi_\chi \) lies in \( I(\chi)^K \). It depends holomorphically on \( \chi \) in some obvious sense, as does \( \mathcal{P}_{\phi,N_0}\varphi_\chi \). But since this latter has support in \( Pw_iP \),

\[ \Omega_\chi(\mathcal{P}_{\phi,N_0}\varphi_\chi) = \Omega_\chi(\varphi_\chi) \]

also depends holomorphically on \( \chi \).

**PROOF OF THE LEMMA.**

Step (1). Let \( G^* \) be the complement of \( Pw_iP \) in \( G \); since \( Pw_iP \) is open in \( G \), \( G^* \) is closed, and in fact it is the union of the \( PwP \) with \( w \neq w_i \). For each \( \chi \), let \( J(\chi) \) be the space of locally constant \( \varphi : G^* \rightarrow \mathbb{C} \) such that \( \varphi(nmg^*) = \chi^{1/2}(m) \varphi(g^*) \) for all \( n \in N, m \in M, g^* \in G^* \). Restriction is an \( N \)-morphism from \( I(\chi) \) to \( J(\chi) \); according to §6.1 of [2] one has an exact sequence:

\[ 0 \rightarrow I_{\delta i} \rightarrow I(\chi) \rightarrow J(\chi) \rightarrow 0. \]

Furthermore, by 1.2 and the proof of 1.4, \( J(\chi)_{\phi,N} = 0 \). In other words, for each \( \varphi \in J(\chi) \) there exists a compact open \( N_0 \subseteq N \) such that
\( \mathcal{P}_{\chi, N_0} \chi = 0. \) Since \( I(\chi)^K \) is finite dimensional, one can even choose \( N_0 \) so that this vanishing holds for all \( \varphi \in I(\chi) \) in the image of \( I(\chi)^K \); this means in turn that for all \( \varphi \in I(\chi)^K \) the function \( \mathcal{P}_{\varphi, N_0} \chi \) has support in \( P_{w_1}P \), because of the exact sequence above.

Step (2). The problem remaining is to show how one may choose this \( N_0 \) independently of \( \chi \). First of all, choose an exhaustive sequence \( N_1 \subseteq N_2 \subseteq \ldots \) of compact open subgroups of \( N \). For each \( n \geq 1 \), let \( X_n \) be the set of all \( \chi \in X \) such that for all \( \varphi \in I(\chi)^K \) the function \( \mathcal{P}_{\varphi, N_n} \chi \) has support in \( P_{w_1}P \). By the result in Step (1), \( X \) is the union of the \( X_n \). But then by Baire’s lemma, one of the \( X_n \) contains an open subset of \( X \). The condition \( \chi \in X_n \) however, is holomorphic in \( \chi \), so that in fact \( X_n \) is all of \( X \). This concludes the proof of 2.2 as well as that of 2.1.

For calculations, a refinement is useful:

2.3. COROLLARY: Given \( K \),

\[
\Omega_\chi(f) = \int_{N_\ast} \psi^{-1}(n) \varphi(xn) dn
\]

for all \( \varphi \in I(\chi)^K \) and suitably large compact open subgroups \( N_\ast \subseteq N \).

PROOF: Suppose that \( N_\ast \) is larger than the \( N_0 \) in 2.2 and also large enough so that \( \mathcal{P}_{\varphi, N_0} \chi \) has support in \( P_{w_1}N_\ast \) for all \( \varphi \in I(\chi)^K \). Then \( \mathcal{P}_{\varphi, N_\ast} \chi \) also has support in \( P_{w_1}N_\ast \) and

\[
\Omega_\chi(\varphi) = \Omega_\chi(\mathcal{P}_{\varphi, N_\ast} \chi) = \int_{N_\ast} \psi^{-1}(n) \varphi(xn) dn.
\]

3. The structure of unramified groups

From now on, through §5, we shall assume the group \( G \) to be unramified. There are two equivalent characterizations: (1) it is obtained by base extension from a smooth reductive group scheme defined over \( \text{Spec}(\mathcal{O}) \), where \( \mathcal{O} \) is the ring of integers in \( k \); (2) it is quasi-split over \( k \) and split over an unramified extension. We shall similarly assume that \( P \) and \( M \) also arise by base extension from subschemes of \( G_\mathcal{O} \). The group \( P \) is a Borel subgroup and \( M \) is an unramified torus.

For every ring \( R \) given with a homomorphism \( \mathcal{O} \to R \), let \( G(R) \) be the corresponding group of \( R \)-valued points. (We continue to write
Thus $K$ is a special maximal compact of $G$, $B$ an Iwahori subgroup, $M_0$ the maximal compact of $M$. The group $B$ has the Iwahori factorization $B = N_1 \cap P_0$, elements of $W$ may be (and we shall assume them to be) represented by elements of $K \cap N_G(A)$, and $K$ is the disjoint union of the $BwB (w \in W)$. (We do not know references for the properties of $G$ unless $G$ is split over $k$ and semi-simple, when results are in [4]. The general case may be obtained from this one by étale descent and is implicit in various announcements by Bruhat and Tits.)

If $\alpha$ is a simple root, the Levi component $M_\alpha$ of the standard parabolic $P_\alpha$ has semi-simple rank one, as does the simply connected covering $\tilde{G}_\alpha$ of its derived group. The inverse image of $P$ in $\tilde{G}_\alpha$ is a minimal parabolic of $\tilde{G}_\alpha$; the projection is an isomorphism of its unipotent radical with $N_\alpha$. There are only two possible types of simply connected, semi-simple, unramified group of rank one: (1) $SL_2(F)$, where $F$ is an unramified extension of $k$, considered as a group over $k$ by restriction of scalars; (2) the special unitary group $SU_3(F)$ as in (1) – corresponding to an unramified Hermitian form in three variables over the unramified quadratic extension $E$ of $F$. These play a special role, as we have just pointed out, in the structure of general unramified groups. We shall discuss them in detail.

Let $\mathcal{O}_F$ be the integers in $F$, etc., and let $q = [\mathcal{O}_F : \mathcal{O}_F]$. 
(1) The group $SL_2(F)$

Let

$$P = \text{upper triangular matrices}$$
$$A = \text{diagonal matrices}$$
$$K = \text{integral matrices}$$

$$B = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid c \in \mathcal{O}\right\}$$

$$w = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$
Define a special coset of $A \cap K$ in $A$:

$$a = \left( \mathcal{P}_F^+ \right)^{-1} \left( \mathcal{P}_F^+ \right)^{-1}$$

For $m \in \mathbb{Z}$, let

$$N_m = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Big| x \in \mathcal{P}^m \right\}$$

and similarly for $N_m^-$. This equation is fundamental:

$$(3.1) \quad \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

for $x \neq 0$.

(2) The group $SU_3(F)$

Let $x \mapsto \bar{x}$ be the conjugation of $E/F$, and let

$$J = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Note that $J = J^{-1}$. It is associated to the Hermitian form $x_1\bar{x}_3 + x_2\bar{x}_2 + x_3\bar{x}_1$. The group $SU_3(F)$ is that of all $X \in SL_3(E)$ with

$$'X \cdot J \cdot \bar{X} = J,$$

or

$$\bar{X} = J \cdot 'X^{-1} \cdot J.$$  

Let

- $P =$ upper triangular matrices
- $M =$ diagonal matrices
- $w = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

- $K =$ integral matrices
- $B = \{ g \in K \mid g =$ upper triangular (mod $\mathcal{P}) \}$. 

Then

\[ M = \left\{ \begin{pmatrix} y & \bar{y}/y \\ \bar{y}^{-1} & 1 \end{pmatrix} \left| y \in E^x \right. \right\} \]

\[ N = \left\{ \begin{pmatrix} 1 & x & y \\ 1 & -\bar{x} & 1 \end{pmatrix} \left| \text{Trace}(y) = -\text{Norm}(x) \right. \right\} \]

and \( w \) represents the non-trivial element of the Weyl group.

Define the class

\[ a = \begin{pmatrix} \mathcal{P}_F^* \\ 1 \end{pmatrix}. \]

Define subgroups \( N_m, m \in \mathbb{Z} \):

\[ N_{2m} = \left\{ \begin{pmatrix} 1 & x & y \\ 1 & -\bar{x} & 1 \end{pmatrix} \in N \left| x \equiv 0(\mathcal{P}^m), y \equiv 0(\mathcal{P}^{2m}) \right. \right\} \]

\[ N_{2m} = \left\{ \begin{pmatrix} 1 & x & y \\ 1 & -\bar{x} & 1 \end{pmatrix} \in N \left| x \equiv 0(\mathcal{P}^{m+1}), y \equiv 0(\mathcal{P}^{2m+1}) \right. \right\}. \]

Here the fundamental equation is:

\[
\begin{pmatrix}
1 & 1 & 1 \\
\bar{x}/y & 1/y & 1/y \\
y^{-1} & 1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1/x/y & 1/x/y \\
1 & x/y & 1 \\
y & 1 & 1
\end{pmatrix}
\]

(3.2)

for \( y \neq 0 \).

In either case, we have this fact:

\[ N_m^{-} - N_{m+1}^{-} \subseteq N_m a^{-m} w (N_m - N_{m+1}) \]

and if \( n \in N_m^{-} - N_{m+1}^{-} \) is expressed accordingly as

\[ n - n_0 a^{-m} w n \]
the map \( n \mapsto n^*, N_m - N_{m+1} \mapsto N_{-m} - N_{-m+1} \) is a bijection transforming the restriction of a Haar measure to the restriction of a Haar measure. If one chooses measures on \( N, N^- \) such that \( \text{meas}(N_0) = \text{meas}(N_0^-) = 1 \) then, more precisely,

\[
\frac{dn^*}{dn} = \begin{cases} 
q^{2m} & (SL_2) \\
q^{4m} & (SU_3)
\end{cases}
\]

We also need to remark that, in the case of \( SL_2 \) one can find characters \( \psi : N \to \mathbb{C} \) trivial on \( N_0 \) but not on \( N_{-1} \); in the case of \( SU_3 \), one can find characters \( \psi : N \to \mathbb{C} \) trivial on \( N_{-1} \) but not on \( N_{-2} \). Such characters we call \textit{unramified}.

Note that if \( \delta \) is the modulus character of \( P \) then

\[
\delta(a) = \begin{cases} 
q^{-2} & (SL_2) \\
q^{-4} & (SU_3)
\end{cases}
\]

since \( aN_m a^{-1} = N_{m+2} \) in both cases.

If \( \alpha \) is the simple root corresponding to the choice of \( P \), define

\[
q_\alpha = q \\
q_{\alpha/2} = \begin{cases} 
1 & (SL_2) \\
q^2 & (SU_3)
\end{cases}
\]

This agrees with the notation of \cite{3}.

To return to the more general case: if \( \alpha \) is any simple root, we shall always assume an identification of \( \tilde{G}_\alpha \) with one of the groups above in such a way that the inverse image of \( P \cap \tilde{G}_\alpha \) is the parabolic above, the image of the above \( K \) is contained in \( G(\mathfrak{o}) \), etc. We shall similarly refer to the groups \( N_{\alpha,m}, N_{\alpha,m}^- \), the coset \( a_\alpha \) of \( M_0 \) (the image of the element \( a \) in \( \tilde{G}_\alpha \)), and the numbers \( q_\alpha, q_{\alpha/2} \).

A remark is in order concerning the validity of our paper when \( G \) is not unramified. It seems likely that all results still hold as long as one chooses for \( K \) what might be called an absolutely special maximal compact—i.e. one descending from a special compact over any field extension. Already in \( SU_3 \), for example, there is a second type of maximal compact which is \textit{not} absolutely special and for which our eventual formula does not hold.

We ought also to mention that globally the local groups, representations, and characters of \( N \) are unramified at almost all primes.
4. Whittaker models and intertwining operators

All characters of $M$ will be unramified in §§4, 5.

Fix also for these sections a principal character $\psi = \Pi \psi_\alpha$ of $N$, which we assume to be unramified.

Assume on $PwP$ the restriction of a Haar measure with $\text{meas}(P_0w_1N_0) = 1$. Fix a representative $x_l$ of $w_l$. Let $\Omega_\chi : I(\chi) \to C_\psi$ be the unique $N$-morphism which for $f \in C^\infty(PwP)$ satisfies

$$\Omega_\chi(Pxf) = \int_{PwP} \Phi(y)f(y)dy$$

where $\Phi(n_1mhn_2) = \chi^{-1}(m)\psi^{-1}(n_2)$. As we have shown in §2, $\Omega_\chi$ varies holomorphically with $\chi$.

For each $\chi$ and each $q \in W$, let $\varphi_{w,q} = \mathcal{P}_\chi(\text{char } BwB)$. These form a basis of $I(\chi)^q$. The function $\varphi_{k_0,q} = \mathcal{P}_\chi(\text{char } K)$ spans $I(\chi)^K$. (We shall often drop reference to $\chi$ in subscripts.)

For each $\alpha \in \Sigma$, let

$$\xi_\alpha(\chi) = \begin{cases} (1 - q^{-1/2}q_\alpha^{-1}(a_\alpha))(1 + q^{-1/2}q_\alpha^{-1}(a_\alpha)) & (q_{\alpha\overline{\alpha}} \neq 1) \\ 1 - q^{-1/2}q_\alpha^{-1}(a_\alpha) & (q_{\alpha\overline{\alpha}} = 1) \end{cases}$$

$$\eta_\alpha(\chi) = \begin{cases} 1 - \chi(a_\alpha) & (q_{\alpha\overline{\alpha}} \neq 1) \\ 1 - \chi(a_\alpha) & (q_{\alpha\overline{\alpha}} = 1) \end{cases}$$

and for each $w \in W$ let

$$\xi_w(\chi) = \Pi \xi_\alpha(\chi)$$

$$\eta_w(\chi) = \Pi \eta_\alpha(\chi) \quad (\alpha > 0, w\alpha < 0).$$

Further let

$$c_\alpha(\chi) = \xi_\alpha(\chi)/\eta_\alpha(\chi)$$

$$c_w(\chi) = \xi_w(\chi)/\eta_w(\chi)$$

In [3] it is shown that if $x$ is a regular character – i.e. $w_x = \chi$ only for $w = 1$ – then there exists a unique $G$-morphism $T_w : I(\chi) \to I(w\chi)$ such that $T_w(\varphi_k) = c_w(\chi)\varphi_k$. The operator $T_w$ depends holomorphically on $\chi$; $T_{w_1w_2} = T_{w_1}T_{w_2}$ if $l(w_1w_2) = l(w_1) + l(w_2)$; and if $\alpha$ is a simple root, $w \in W$ such that $l(w_\alpha w) > l(w)$, then

$$T_{w_\alpha}(\varphi_{w_\alpha w} + \varphi_w) = c_\alpha(\chi)(\varphi_{w_\alpha w} + \varphi_w).$$

Let $T_w^*$ be the transpose of $T_w$ from the linear dual of $I(w\chi)$ to that
of \( I(\chi) \). Since the space of \( N \)-morphisms from \( I(\chi) \) to \( C_\phi \) has dimension one, \( T^*_w(\omega_w) \) is a scalar multiple of \( \Omega_\chi \). The scalar will depend holomorphically on \( \chi \). Since every \( w \) is a reduced product of elementary reflections, it suffices to calculate it for \( w = w_\alpha, \alpha \in \Delta \).

4.1. **Lemma:** (a) For any \( \alpha \in \Delta \),

\[
\Omega_\chi(\varphi_{w_\alpha}) = \begin{cases} 
- q^{-1}_\alpha \chi(a) & (q_{a_2} = 1) \\
q^{1/2}_a \chi(a) - q^{-1/2}_a q^{-1}_a \chi(a) - q^{-1/2}_a q^{-1}_a \chi(a) & (q_{a_2} \neq 1);
\end{cases}
\]

(b) \( \Omega_\phi(\varphi_w) = 1 \).

**Proof:** Claim (b) is trivial since \( Bw_1B = P_0w_1N_0 \). For claim (a): first apply 2.3 to see that we must evaluate

\[
\int_{P_0w_1N \cap Bw_1B} \Phi(y)dy
\]

for \( N_+ \) suitably large, where \( \Phi(n,m_\alpha,n_2) = \chi^{-1}\delta^{1/2}(m)\psi^{-1}(n_2) \) as above.

Recall that \( Bw_1B = P_0N^{-1}_{\alpha,1}w_1w_0 \), and express \( N^{-1}_{\alpha,1} \) as \( (N^{-1}_{\alpha,1} - N^{-1}_{\alpha,2}) \cup (N^{-1}_{\alpha,2} - N^{-1}_{\alpha,3}) \cup \cdots \). Recall also (from 3) that every \( n \in (N^{-1}_{\alpha,m} - N^{-1}_{\alpha,m+1}) \) may be expressed

\[
n = n_1a^{-m}_\alpha m_0w_1n_2
\]

with \( n_1 \in N_{\alpha,-m}, m_0 \in M_0, n_2 \in (N_{\alpha,-m} - N_{\alpha,-m+1}) \) Therefore

\[
nw_1w = n_1a^{-m}_\alpha m_0w_1 \cdot n \ast \cdot n_2
\]

where here

\[
n \ast \in (N_{\alpha,-m} - N_{\alpha,-m+1}) = (w_1w)^{-1}(N_{\alpha,-m} - N_{\alpha,-m+1})(w_1w).
\]

(The element \( \bar{\alpha} \in \Delta \) is the conjugate of \( \alpha \).

The above integral becomes

\[
\int_{P_0w_1N \cap P_0N^{-1}_{\alpha,1}w_1w_0} \Phi(y)dy = \sum_{m=1} \int_{P_0w_1N \cap P_0(N_{\alpha,m} - N_{\alpha,m+1})w_1w_0} \Phi(y)dy
\]

\[
= \sum_{m=1} \int_{P_0(N_{\alpha,m} - N_{\alpha,m+1})w_1w_0} \Phi(y)dy
\]
which for large \( n \)
\[
\sum_{m=1}^{\infty} \int_{P_0} dp_0 \int_{N_0} dn_0 \int_{N_{\alpha-m} - N_{\alpha,m+1}} \Phi(p_0 n_1 a_{\alpha}^{-m} x, n \ast n_0) dn \\
= \sum_{m=1}^{\infty} \chi^{-1} \delta^{1/2}(a_{\alpha}^{-m}) \int_{N_{\alpha,m} - N_{\alpha,m+1}} \psi^{-1}(n_*) dn_*
\]
where we have assumed measures on \( P_0, N_0 \) with total measure 1. We must in fact be careful about measures: we have assumed \( \text{meas}(Bw_B) = 1 \), so that since \( Bw_B B \cdot Bw_B = Bw_B \) is the Hecke algebra \( \mathcal{H}(G, B) \), \( \text{meas}(Bw_B) = 1/\alpha \), and in the above integral, with the stated assumptions on \( P_0, N_0 \) we assume on \( N_\alpha \) the measure with \( \text{meas}(N_{\alpha,\delta}) = 1 \). Assuming as well the measure on \( N_\alpha \) to be such that \( \text{meas}(N_{\alpha,\delta}) = 1 \) (consistent with the assumption on \( N \)), one can check that the map from \( N_{\alpha,m} - N_{\alpha,m+1} \) to \( N_{\delta,-m} - N_{\delta,-m+1} \) taking \( n \) to \( n_* \) is such that

\[
\frac{dn_*}{dn} = \begin{cases} q_{\alpha}^{2m} & (\alpha = 1) \\ q_{\alpha}^{4m} & (\alpha \neq 1). \end{cases}
\]

The above becomes (since \( \psi = \psi_{\delta} \) on \( N_{\delta} \))
\[
\sum_{m=1}^{\infty} \chi^{-1} \delta^{1/2}(a_{\alpha}^{-m}) \left( \frac{dn}{dn_*} \right)_{N_{\delta,-m} - N_{\delta,-m+1}} \psi^{-1}(n_*) dn_*
\]
At this point the two cases have to be treated differently:

1. \( \tilde{G}_\alpha = SL_2 \). Here \( \psi_{\delta} \) is trivial on \( N_{\delta,0} \) but not on \( N_{\delta,-1} \), so that
\[
\int_{N_{\delta,-m}} \psi_{\delta}(n) dn = \begin{cases} 1 & (m = 0) \\ 0 & (m \geq 1) \end{cases}
\]
and
\[
\int_{N_{\delta,-m} - N_{\delta,-m+1}} \psi^{-1}(n_*) dn_* = \begin{cases} -1 & (m = 1) \\ 0 & (m \geq 2) \end{cases}
\]
Our sum only has one term and is equal to
\[
-\chi^{-1} \delta^{1/2}(a_{\alpha}^{-1}) q_{\alpha}^{-2} = -q_{\alpha}^{-1}(a_{\alpha}).
\]

2. \( \tilde{G}_\alpha = SU_3 \). Here \( \psi_{\delta} \) is trivial on \( N_{\delta,-1} \) but not on \( N_{\delta,-2} \).
Therefore
\[ \int_{N_{d,m} - N_{d,m+1}} \psi^{-1}(n) dn = \begin{cases} q_a - 1 & (m = 1) \\ -q_a & (m = 2) \\ 0 & (m \geq 3) \end{cases} \]

and our sum, with two terms only, equals
\[
\begin{align*}
\chi^{-1} \delta(a^{-2}_a) q^{-8}_a (-q_a) &+ \chi^{-1} \delta^{1/2}(a^{-1}_a) q^{-4}_a (q - 1) \\
&= -q^{-3}_a \chi(a_a)^2 + q^{-1}_a \chi(a_a) - q^{-2}_a \chi(a_a)
\end{align*}
\]

which agrees with the Lemma since \(q_{a_2} = q^2\).

**4.2. COROLLARY:** One has
\[
\Omega_\chi(\varphi_{w,\omega} + \varphi_{\omega}) = \zeta'_\alpha(\chi).
\]

Now define
\[
\begin{align*}
\lambda_\alpha(\chi) &= \zeta'_\alpha(\chi^{-1}) / \eta_\alpha(\chi) \\
\lambda_w(\chi) &= \Pi \lambda_\alpha(\chi) & (\alpha > 0 \mid w\alpha < 0).
\end{align*}
\]

**4.3. PROPOSITION:** One has
\[
T^*_w(\Omega_{w\chi}) = \lambda_w(\chi) \Omega_\chi.
\]

This follows from a comparison of 4.2 with Equation (4.1). (Note that \(\lambda_w(\chi) = c_w(\chi) \zeta_w(\chi^{-1}) / \zeta_w(\chi)\).)

**5. The unramified Whittaker function**

Corresponding to the \(N\)-morphism \(\Omega_\chi: I(\chi) \to C_\varphi\) we have a \(G\)-morphism:\n\(I(\chi) \to \text{Ind}(C_\varphi \mid N, G)\). Define the unramified Whittaker function corresponding to \(\chi\) to be the image of \(\varphi_\chi\) under this map. It is thus defined by the formula
\[
W_\chi(g) = \Omega_\chi(R_g \varphi_\chi)
\]

and satisfies the conditions
\[
W_\chi(ngk) = \psi(n) W_\chi(g)
\]
for $k \in K$, $n \in N$, $g \in G$. Because $G = NAK$ the function $w_\chi$ is determined by its restriction to $A$. We shall obtain an explicit formula for this restriction.

5.1. **Lemma:** For $a \in A$, $a \not\in A^-$, $W_\chi(a) = 0$.

**Proof:** For $n \in N_0$, $a \in A$,

$$W_\chi(an) = W_\chi(a)$$

$$= W_\chi(ana^{-1} \cdot a)$$

$$= \psi(ana^{-1})W_\chi(a).$$

For $a \not\in A^-$, $|\alpha(a)| > 1$ for some $\alpha \in \Delta$. Then for some $n \in N_{a,0}$ one has $\psi_n(ana^{-1}) \neq 1$, so that $W_\chi(a) = 0$.

The idea we use from this point on is extremely close to that used in [3] to find an explicit formula for the spherical function. We use notation and results from [3] freely. (The common point of both problems is simply that in each case one has for each $\chi$ a certain unique functional on $I(\chi)$ invariant under $N_0$; we shall explain this more precisely later.) Because $W_\chi$ is holomorphic in $\chi$, we may assume $\chi$ to be a regular character.

Let $\{f_w\}$ be the basis of $I(\chi)^B$ which is as a linear space canonically isomorphic to $I(\chi)_N$, dual to the maps $f \rightarrow T_w(f)(1)$. By definition, then,

$$\varphi_K = \sum c_w(\chi)f_w.$$

For $a \in A^-$, [3] 2.5 implies

$$\mathcal{P}_{N_0}(R_\sigma \varphi_K) = \sum c_w(\chi)(w\chi)\delta^{\text{inv}}(a)f_w$$

Hence

$$W_\chi(a) = \Omega_\chi(R_\sigma \varphi_K)$$

$$= \Omega_\chi(\mathcal{P}_{N_0}(R_\sigma \varphi_K))$$

$$= \sum c_w(\chi)\Omega_\chi(f_w)(w\chi)\delta^{\text{inv}}(a).$$

Because $\varphi_{w_i} = f_{w_i}$ ([3] 3.7),

$$\Omega_\chi(f_{w_i}) = 1,$$
but the other values of $\Omega_\chi(f_w)$ must be obtained by using a functional equation for $W_x$.

5.2. LEMMA: For any $w \in W, \chi$ such that $\zeta_w(\chi) \neq 0$, 

$$W_{wx} = (\zeta_w(\chi^{-1})/\zeta_w(\chi))W_x.$$  

PROOF: Let $F_\chi: \text{Ind}(\chi) \rightarrow \text{Ind}(C_\phi | N, G)$ be the $G$-morphism corresponding to $\Omega_\chi$. Thus 

$$F_\chi(\phi)(g) = \Omega_\chi(R_g\phi)$$

for all $g \in G, \phi \in I(\chi)$. According to 4.3, 

$$F_{wx}(T_w\phi) = (\zeta_w(\chi^{-1})/\eta_w(\chi))F_\chi(\phi).$$

Now $W_x = F_\chi(\phi_K)$; therefore 

$$F_{wx}(T_w\phi_K) = c_w(\chi)W_{wx} = (\zeta_w(\chi)/\eta_w(\chi))W_{wx}$$

on the one hand and 

$$= (\zeta_w(\chi^{-1})/\eta_w(\chi))W_x$$

on the other. For generic $\chi$, therefore, 

$$W_{wx} = (\zeta_w(\chi^{-1})/\zeta_w(\chi))W_x.$$  

This is Jacquet’s functional equation for the $\mathcal{P}$-adic Whittaker function in [5].

Let $\zeta(\chi)$ be $\zeta_w(\chi)$.

5.3. COROLLARY: The function $W_x/\zeta(\chi)$ is $W$-invariant as a function of $\chi$.

PROOF: It must be shown, by 5.2, that 

$$\frac{\zeta(wx)}{\zeta(\chi)} = \frac{\zeta_w(\chi^{-1})}{\zeta_w(\chi)}.$$
The left hand side is

\[ \prod_{\alpha > 0} \xi_\alpha(w\chi) = \frac{\prod_{\alpha > 0} \xi_\alpha^{-1}(\chi)}{\prod_{\alpha > 0} \xi_\alpha(\chi)} = \frac{\prod_{\omega \alpha > 0} \xi_\alpha(\chi)}{\prod_{\alpha > 0} \xi_\alpha(\chi)} = \frac{\prod_{\alpha < 0 \omega \alpha > 0} \xi_\alpha(\chi)}{\prod_{\alpha > 0} \xi_\alpha(\chi)} = \frac{\xi_w(\chi^{-1})}{\xi_w(\chi)}. \]

We now have

\[ \frac{W_\chi}{\zeta(\chi)} = \sum_{w \in W} (c_w(\chi)/\xi(\chi)) \Omega_\chi(f_w)(w\chi)\delta^{1/2}. \]

The coefficient for \( w = w_l \) is

\[ \frac{c_w(\chi)}{\xi(\chi)} = \frac{1}{\eta_{w_l}(\chi)} = \prod_{\alpha > 0} \frac{1}{1 - \chi(a_\alpha) d_\alpha}, \]

where

\[ d_\alpha = \begin{cases} 1 & (q_{a/2} = 1) \\ 2 & (q_{a/2} \neq 1) \end{cases}. \]

This coefficient may also be expressed as

\[ \prod_{\alpha > 0} \frac{1}{1 - (w_l \omega_l \chi)(a_\alpha)^{d_\alpha}}. \]

Because of the linear independence of characters on \( A^- \) and the \( W \)-invariance of \( W_\chi/\zeta(\chi) \), one must therefore have

\[ \frac{W_\chi}{\zeta(\chi)} = \sum_{w \in W} \prod_{\alpha > 0} \frac{1}{1 - (w)(a_\alpha)^{-d_\alpha}} (w\chi)\delta^{1/2}. \]

This expression can be reduced further. Let

\[ \epsilon(\chi) = \prod_{\alpha > 0} \frac{1}{1 - \chi(a_\alpha)^{-d_\alpha}}. \]
Then for $\beta \in \Delta$

$$\epsilon(w_\beta \chi) = \prod_{\alpha > 0} \frac{1}{1 - \chi(a_{\alpha \beta})^{-d_\alpha}}$$

$$= \prod_{\alpha > 0} \frac{1}{1 - \chi(a_\alpha)^{-d_\alpha}} \cdot \frac{1}{1 - \chi(a_\beta)^{d_\beta}}$$

$$= -\chi(a_\beta)^{d_\beta} \epsilon(\chi).$$

Hence in general

$$\epsilon(w \chi) = (\text{sgn } w) \cdot \prod_{\substack{\alpha > 0 \\text{wa} < 0}} \chi(a_\alpha)^{-d_\alpha} \cdot \epsilon(\chi)$$

and

$$W_x = \zeta(\chi) \cdot \epsilon(\chi) \cdot \sum_{w \in W} (\text{sgn } w) \cdot \prod_{\substack{\alpha > 0 \\text{wa} < 0}} \chi(a_\alpha)^{-d_\alpha} \cdot (w \chi) \delta^{1/2}.$$ 

When $a = 1$:

$$W_x(a) = \zeta(\chi) \cdot \prod_{\alpha > 0} \frac{1}{1 - \chi(a_\alpha)^{-d_\alpha}} \cdot \sum_{w \in W} (\text{sgn } w) \prod_{\alpha > 0} \chi(a_\alpha)^{-d_\alpha} (w \chi) \delta^{1/2}(a).$$

$$= \zeta(\chi).$$

All in all we have

5.4. THEOREM: For $a \in A^-$,

$$W_x(a) = \zeta(\chi) \cdot \prod_{\alpha > 0} \frac{1}{1 - \chi(a_\alpha)^{-d_\alpha}} \cdot \sum_{w \in W} (\text{sgn } w) \prod_{\alpha > 0} \chi(a_\alpha)^{-d_\alpha} (w \chi) \delta^{1/2}(a).$$

In particular, $W_x(1) = \zeta(\chi)$.

To repeat,

$$\zeta(\chi) = \prod_{\alpha > 0} \left\{ \frac{1}{1 - q^{-1} \chi(a_\alpha)} \right\} \left( 1 + \frac{q^{-1/2}}{1} \chi(a_\alpha) \right) \left( 1 - \frac{q^{-1/2}}{1} \chi(a_\alpha) \right) \quad (q_{a/2} = 1)$$

$$\zeta(\chi) = \prod_{\alpha > 0} \left\{ \frac{1}{1 - q^{-1} \chi(a_\alpha)} \right\} \left( 1 + \frac{q^{-1/2}}{1} \chi(a_\alpha) \right) \left( 1 - \frac{q^{-1/2}}{1} \chi(a_\alpha) \right) \quad (q_{a/2} \neq 1).$$
5.5. Remark: It may be helpful if we summarize our argument, and especially to bring out the points which it has in common with the argument in [3].

(1) In each case, one has a functional $F_\chi : I(\chi) \to \mathbb{C}$ which (a) depends holomorphically on $\chi$, (b) is unique, in some sense, up to scalar multiplication, (c) is $N_0$-invariant. Here the functional is $\Omega_\chi$, in [3] it is the map $f \to \mathcal{P}_\chi(f)(1)$.

(2) Because of the uniqueness property, whenever $\chi$ is regular one has

$$T_w^*(F_{w_\chi}) = \gamma_w(\chi)F_\chi$$

for some scalar $\gamma_w(\chi)$. This scalar is multiplicative in $w$ since $T_w$ is, and can be calculated explicitly for an elementary reflection.

(3) One is looking for a formula for the function $\Phi_\chi(a) = F_\chi(R_a \varphi_K)$, $a \in A^-$. Because of holomorphicity, it suffices to find the formula when $\chi$ is regular; in this case, the result in (2) gives some relation between $CP_x$ and $\gamma_{wx}$.

(4) The value of $F_\chi(\varphi_w) = F_\chi(f_w)$ can be calculated explicitly.

(5) One expresses $\varphi_K$ in terms of the basis $\{f_w\}$:

$$\varphi_K = \sum_w c_w(\chi)f_w$$

then continues

$$\mathcal{P}_{N_0}(R_a \varphi_K) = \sum_w c_w(\chi)(w\chi)\delta(a)f_w$$

$$\Phi_\chi(a) = \sum_w c_w(\chi)(w\chi)\delta(a)F_\chi(f_w)$$

The term $F_\chi(f_w)$ is known, and one uses the functional equation from (3) to find the other coefficients.

6. Whittaker functions and Jacquet modules

The explicit formula of §5 suggests properties of the functions in any Whittaker model of an admissible representation. The properties of $W_\chi$ that we wish to generalize here are (1) it vanishes off $A^-$; (2) on $A^-$ it is the restriction of an $A$-finite function.

We again allow $G$ to be an arbitrary reductive group. Let $\psi = \Pi \psi_a$ be any character of $N$ (not necessarily principal). Let $(\pi, V)$ be an admissible representation of $G$. 

6.1. PROPOSITION: Let \( \psi \) be a principal character of \( N \), \( \Omega : V \rightarrow C_\psi \) an \( N \)-morphism. For any \( v \in V \) there exists \( \epsilon > 0 \) such that whenever \( |\alpha(a)| > \epsilon^{-1} \) for some \( \alpha \in \Delta \), \( \Omega(\pi(a)v) = 0 \).

PROOF: Choose a compact open subgroup \( N_0 \) of \( N \) fixing \( V \). For any \( n \in N_0 \), \( a \in A \):

\[
\Omega(\pi(a)n) = \Omega(\pi(\pi(n)v)) = \Omega(\pi(an^{-1})v) = \psi(an^{-1})\Omega(\pi(a)v).
\]

Since \( \psi \) is principal, there exists \( \epsilon > 0 \) such that for whenever \( |\alpha(a)| > \epsilon^{-1} \) for some \( \alpha \in \Delta \), \( an_0a^{-1} \not\subseteq \ker(\psi) \). For such an \( a \), \( \Omega(\pi(a)v) = 0 \).

Another way of saying this that the support on \( A \) of any Whittaker function lies in a translate of \( A^- \). Our next result says that there is some other translate of \( A^- \) on which the function is \( A \)-finite.

Let \( u \) be a vector of the Jacquet module \( V_N \). Choose a compact open subgroup \( K_0 \) of \( G \) with the Iwahori factorization \( K_0 = N_0M_0N_0 \) such that \( u \) is fixed by \( M_0 \) and \( \psi \) is trivial on \( N_0 \). Let \( v \) be the canonical lifting of \( u \) in \( V^{K_0} \) (refer to §4 of [2]).

6.2. LEMMA: If \( K'_0 \subseteq K_0 \) is any smaller group with Iwahori factorization and \( v' \in V^{K'_0} \) the corresponding canonical lifting of \( u \), then \( v' \) and \( v \) have the same image in \( V_{\psi,N} \).

Here it is not necessary to assume \( \psi \) principal.

PROOF. By Proposition 4.1.8 of [2],

\[ v = \mathcal{P}_{M_0}(v'). \]

But then \( v \) and \( v' \) clearly have the same image in \( V_{\psi,N} \) since \( \psi = 1 \) on \( N_0 \).

The correspondence \( u \mapsto v \mapsto v \mod V_\psi(N) \) thus defines a canonical linear map \( \Phi \) from \( V_N \) to \( V_{\psi,N} \).

6.3. PROPOSITION: Let \( \Omega : V \rightarrow C_\psi \) be an \( N \)-morphism, \( v \) an element of \( V \) and \( u \) its image in \( V_N \). There exists \( \epsilon > 0 \) such that whenever \( |\alpha(a)| < \epsilon \) for all \( \alpha \in \Delta \),

\[ \Omega(\pi(a)v) = (\Omega \circ \Phi)(\pi_N(a)u). \]

PROOF: Choose \( K_0 \) for \( v \) and \( u \) as above. By definition of the
canonical lifting in [2] there exists $\epsilon > 0$ such that $\mathcal{P}_{\kappa}(\pi(a)v) = \mathcal{P}_{\kappa}(\pi(a)v)$ is the canonical lifting of $\pi_N(a)u$ whenever $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta$. But then one has only to apply the definition of $\Phi$.

It happens that the two translates of $A^-$ used in 6.1 and 6.3 coincide in the case of the Whittaker function $W_\chi$, and indeed for the Whittaker function attached to any element of $I(\chi)^B$. One can see this directly from the proofs of 6.1 and 6.3.

What we have proven so far about the support of a Whittaker function $W$ can be summarized in a picture, representing $A$ rather figuratively as a plane, $A^-$ as a cone. In fact one can also say something about the behaviour all over $A$; we have left it to last because it is not so simple to formulate as the above.

Let $\theta$ be a subset of $\Delta$, $\psi_\theta$ the character of $M_\theta \cap N$ defined as $\Pi_\psi_\theta(\alpha \in \theta)$. Just as above, one can define a canonical linear map $\Phi_\theta$ from $(V_{N_\theta})_{\psi_\theta M\cap N_\theta}$ to $V_{\psi,N}$, and almost exactly the same proof yields:

6.4. PROPOSITION: Let $\Omega : V \rightarrow C_\phi$ be an $N$-morphism, $v \in V$ with image $u \in V_{N_\theta}$. There exists $\epsilon > 0$ such that whenever $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta - \theta$,

$$\Omega(\pi(a)v) = (\Omega \circ \Phi_\theta)(\pi_N(a)u).$$

In other words, in the direction of $A^-$ any function for $V$ eventually becomes a Whittaker function for $V_{N_\theta}$.

One well known consequence:

6.5. COROLLARY: If $(\pi, V)$ is absolutely cuspidal then any Whittaker function associated to $V$ has compact support on $G$ modulo $N$.

One might note that in general the canonical maps $\Phi_\theta$ are neither injective nor surjective.

One might also note that the above argument will show that whenever $\Omega : V \rightarrow C$ is a functional invariant under some compact
open subgroup $N_0 \subseteq N$ then one can prove $\Omega(\pi(a)v)$ is $A$-finite on some translate of $A^-$. 

REFERENCES


(Oblatum 13–XI–1978)