Lecture 9: More about vertex algebras

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Fields

Today’s lecture will continue discussing the Heisenberg vertex operator following Frenkel and Ben-Zvi, Chapter 2.

We fix a vector space $V$, which will later be given some additional structure. A field is a formal power series

$$A(z) = \sum_{j \in \mathbb{Z}} A_j z^{-j}$$

with $A_j \in \text{End}(V)$ such that for any $v \in V$ we have $A_j v = 0$ for sufficiently large $j$. 
Locality

Two fields $A(z)$ and $B(w)$ are considered \textit{(mutually) local} if for every $\Phi \in V$ and $\Psi \in V^*$ the matrix coefficients

$$\langle \Psi | A(z)B(w) | \Phi \rangle, \quad \langle \Psi | B(w)A(z) | \Phi \rangle$$

defined the same element of $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$ expanded in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively.

Less precisely, for sufficiently large $N$

$$(z - w)^N[A(z), B(w)] = 0.$$
Vertex algebra axioms

A vertex algebra is a purely algebraic structure in which the state-field correspondence associates a field

\[ Y(A, z) = \sum_{n} A_{(n)} z^{-n-1}. \]

We also need a vacuum vector \(|0\rangle\) and a translation operator \(T : V \to V\) subject to the following axioms.

**Vacuum Axiom.** We have \(Y(|0\rangle, z) = I_v\). Moreover if \(A \in V\) then \(Y(A, z)|0\rangle \in V[[z]]\) so the limit \(Y(A, z)|0\rangle\) as \(z \to 0\) exists and is \(A\).

**Translation Axiom.** \([T, Y(A, z)] = (d/dz)Y(A, z)\) and \(T|0\rangle = 0\).

**Locality Axiom.** The fields \(Y(A, z)\) are mutually local.
Origins in CFT

In 2-dimensional Euclidean conformal field theory, the state-field correspondence associates to every point \( z \) in \( \mathbb{R}^2 \) or its conformal completion \( \mathbb{P}^1(\mathbb{C}) \) and to every vector \( A \) in the Hilbert space \( \mathcal{H} \) a field; the correspondence is such that \( v \) is the limiting value of the field at \( v \).

Also in conformal field theory, there are Virasoro operators \( L_n \) corresponding to local conformal transformations. (We have not lectured on this yet.)

The translation operator is the Virasoro generator \( L_{-1} \), an infinitesimal translation along the one-parameter subgroup of special conformal transformations

\[
\begin{align*}
  z &\mapsto \frac{z - |z|^2 b}{1 - 2\langle z, b \rangle + |z|^2|b|^2}
\end{align*}
\]
Normal ordering

In a field $A(z) = \sum A_n z^{-n-1}$ we may often think of $A_n$ as a creation operator if $n > 0$ and an annihilation operator if $n < 0$. As we saw last time there is a difficulty interpreting

$$A(z)B(w) = \sum_{n,m} A_n B_m z^{-n-1} w^{-m-1}$$

due to the fact that infinitely many times, a creation operator is done before the annihilation operator.

To avoid this problem, we define the normal ordered product

$$:A(z)B(w): = \sum_m \left( \sum_{n<0} A_n B_m z^{-n-1} + \sum_{n>0} B_m A_n z^{-n-1} \right) w^{-m-1}.$$

Often this makes sense when $A(z)B(w)$ is problematic, particularly when $z = w$. 
Normal order as a residue

Recall that we defined the delta distribution to equal
\[
\delta_-(z - w) = \frac{1}{z} \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n
\]

so
\[
\delta(z - w) = \delta_-(z - w) + \delta_+(z - w)
\]

where
\[
\delta_-(z - w) = \frac{1}{z} \sum_{n \geq 0} \left( \frac{w}{z} \right)^n, \\
\delta_+(z - w) = \frac{1}{z} \sum_{n > 0} \left( \frac{z}{w} \right)^n.
\]

\[
\delta_-(z - w) = \frac{1}{z - w}, \\
\delta_+(z - w) = -\frac{1}{z - w}.
\]

We will show
\[
: A(w)B(w) : \text{Res}_{z=0}(\delta_-(z - w)A(z)B(w) + \delta_+(z - w)B(w)A(z))
\]
Proof

To see this, first consider

\[ \text{Res}_{z=0} \delta_-(z-w)A(z)B(w). \]

This is the \( z^{-1} \) coefficient in

\[ \frac{1}{z} \sum_{k \geq 0} \left( \frac{w}{z} \right)^k \sum_{n,m} A_n B_m z^{-n-1} w^{-m-1}. \]

We must have \( -k - n - 1 = 0 \) so \( -n - 1 = k \geq 0 \) and thus \( n < 0 \). The residue is

\[ \sum_{n < 0} A_n B_m w^{-n-m-1}. \]

Similarly

\[ \text{Res}_{z=0} \delta_+(z-w)B(w)A(z) = \sum_{n > 0} B_m w^{-n-m-1} A_n. \]
Review: the Heisenberg Lie Algebra

The Heisenberg Lie algebra $\mathfrak{h}$ has a central basis element $1$ and other basis elements $b_n$ such that

$$[b_m, b_n] = m\delta_{m,-n}1.$$ 

Note that both $b_0$ and $1$ are central. A version of the Stone-von-Neumann theorem asserts $\mathfrak{h}$ has a unique irreducible module in which $b_0$ and $1$ act by scalars $0$ and $1$.

A model of this representation is the **bosonic Fock space** $B = \mathbb{C}[b_{-1}, b_{-2}, \cdots] \subset U(\mathfrak{h})$ where $b_{-n}$ with $n > 0$ act by multiplication and $b_n$ with $n > 0$ acts by $n\partial/\partial b_{-n}$. 
Review: The Bosonic Fock Space

The Fock space $B$ is a graded module in which $b_{-k}$ has degree $k$. Secretly $B$ may be identified with the ring of symmetric functions, with $b_{-k}$ corresponding to the power-sum symmetric function $p_k$. (See Kac-Raina, equation (6.5)).

The vacuum element is $1 \in B = \mathbb{C}[b_{-1}, b_{-2}, \cdots]$ (not to be confused with $\mathbb{1}$).

The translation operator $T$ is defined to satisfy $T(1) = 0$, $[T, b_m] = -mb_{m-1}$ from which its effect may be deduced recursively using

$$T \cdot b_k f = b_k T(f) + [T, b_k] \cdot f.$$
We now have to define $Y(b_k, z)$. First we define $Y(b_{-1}, z)$ to be

$$b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-1}.$$ 

More generally we define

$$Y(b_{-k}, z) = \frac{1}{(k-1)!} \left( \frac{d}{dz} \right)^{k-1} b(z).$$

However the $b_k$ are not a basis of $B$, since we may have elements like $b_{-1}^2$. We cannot define this to be

$$b(z)^2 = \sum_n \left( \sum_{k+l=n} b_k b_l \right) z^{-n-2}$$

since the sum in parentheses is problematic if $n = -2$. Applied to $f \in B$ there are infinitely many terms. See [FBZ] p. 30.
Locality

We have defined $Y(b_{-k}, z)$ but what about $Y(b_{-i_1} \cdots b_{-i_k}, z)$? This is the normal ordered product of $Y(b_{-i_1}, z)$, in other words

$$Y(b_{-i_1} \cdots b_{-i_k}, z) = \frac{1}{(i_1 - 1)!} \cdots \frac{1}{(i_k - 1)!} : (\partial_{\bar{z}}^{i_1-1} b(z)) \cdots (\partial_{\bar{z}}^{i_k-1} b(z)) : .$$

So far we have reviewed what we did in Lecture 8. Now let us show that $b(z)$ is local with respect to itself. This depends on the formula

$$b(z)b(w) = \frac{1}{(z - w)^2} + : b(z)b(w) : ,$$

which we will now discuss.
We are pondering

\[ b(z)b(w) = \sum_{m,n} b_n b_m z^{-n-1} w^{-m-1}. \]

Note that \( b_m b_n \) and \( :b_m b_n : \) are equal unless \( n = m \) and \( m < 0 \). So

\[ b(z)b(w) =: b(z)b(w) : + \sum_{n>0} [b_n, b_{-n}] z^{-n-1} w^{n-1}. \]

The correction term is

\[ \sum_{n>0} n z^{-n-1} w^{n-1} \]

which is the expansion in \( \mathbb{C}((z))((w)) \) of \( (z - w)^{-2} \).
Locality (continued)

Note that since the operators $b_n$ and $b_m$ commute except in the diagonal case $b_n = b_{-n}$ we have $b_nb_m := b_mb_n$ and so $b(z)b(w) := b(w)b(z)$. Therefore the expression

$$b(z)b(w) := b(z)b(w) + \frac{1}{(z-w)^2}$$

is symmetric under the interchange. This means that $b(z)$ and $b(w)$ are mutually local.
Dong’s Lemma

**Dong’s Lemma** If $A(z)$, $B(z)$ and $C(z)$ are three fields that are pairwise mutually local, then $:A(z)B(z):$ and $C(z)$ are mutually local.

This is proved using the interpretation

$$:A(w)B(w): \text{Res}_{z=0} (\delta_-(z-w)A(z)B(w) + \delta_+(z-w)B(w)A(z)).$$

See [FBZ] Section 2.3.5 for details.
We have checked that $Y(b_{-1}, z)$ and $Y(b_{-1}, w)$ are local. Next let us check that $Y(b_{-n}, z)$ and $Y(b_{-m}, w)$ are local. Since $Y(b_n, z) = ((n - 1)!)^{-1} \partial^{n-1} b(z)$ it is enough to show that if $A(z)$ and $B(w)$ are local, so are $\partial_z A(z)$ and $B(w)$. Indeed $(z - w)^N [A(z), B(w)] = 0$ for sufficiently large $N$, so differentiating the identity $(z - w)^{N+1} [A(z), B(w)] = 0$ gives $(z - w)^N [\partial_z A(z), B(w)] = 0$. 

**Locality of the fields** $Y(b, z)$