Lecture 9: More about vertex algebras

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Fields

Today's lecture will continue discussing the Heisenberg vertex operator following Frenkel and Ben-Zvi, Chapter 2.

We fix a vector space V, which will later be given some additional structure. A field is a formal power series

$$A(z) = \sum_{j \in \mathbb{Z}} A_j z^{-j}$$

with $A_j \in \mathrm{End}(V)$ such that for any $v \in V$ we have $A_j v = 0$ for sufficiently large j.

Locality

Two fields $A(z)$ and $B(w)$ are considered (mutually) local if for every $\Phi \in V$ and $\Psi \in V^*$ the matrix coefficients

$$\langle \Psi | A(z) B(w) | \Phi \rangle, \qquad \langle \Psi | B(w) A(z) | \Phi \rangle$$

defined the same element of $\mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ expanded in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively.

Less precisely, for sufficiently large N

$$(z-w)^N [A(z), B(w)] = 0.$$

Vertex algebra axioms

A vertex algebra is a purely algebraic structure in which the state-field correspondence associates a field

$$Y(A,z) = \sum_n A_{(n)} z^{-n-1}.$$

We also need a vacuum vector $|0\rangle$ and a translation operator $T : V \to V$ subject to the following axioms.

Vacuum Axiom. We have $Y(|0\rangle, z) = I_v$. Moreover if $A \in V$ then $Y(A,z)|0\rangle \in V[[z]]$ so the limit $Y(A,z)|0\rangle$ as $z \to 0$ exists and is A.

Translation Axiom. $[T, Y(A,z)] = (d/dz)Y(A,z)$ and $T|0\rangle = 0$.

Locality Axiom. The fields $Y(A,z)$ are mutually local.

Origins in CFT

In 2-dimensional Euclidean conformal field theory, the state-field correspondence associates to every point z in $\mathbb{R}^{(2)}$ or its conformal completion $\mathbb{P}^1(\mathbb{C})$ and to every vector A in the Hilbert space $\mathcal{H}$ a field; the correspondence is such that v is the limiting value of the field at v.

Also in conformal field theory, there are Virasoro operators L_n corresponding to local conformal transformations. (We have not lectured on this yet.)

The translation operator is the Virasoro generator L_{-1}, an infinitesimal translation along the one-parameter subgroup of special conformal transformations

$$z \mapsto \frac{z - |z|^2 b}{1 - 2\langle z, b\rangle + |z|^2|b|^2}$$

Normal ordering

In a field $A(z) = \sum A_n z^{-n-1}$ we may often think of A_n as a creation operator if $n > 0$ and an annihilation operator if $n < 0$. As we saw last time there is a difficulty interpreting

$$A(z)B(w) = \sum_{n,m} A_n B_m z^{-n-1} w^{-m-1}$$

due to the fact that infinitely many times, a creation operator is done before the annihilation operator.

To avoid this problem, we define the normal ordered product

$$:A(z)B(w) := \sum_m \left(\sum_{n<0} A_{(n)} B_m z^{-n-1} + \sum_{n>0} B_m A_{(n)} z^{-n-1} \right) w^{-m-1}.$$

Often this makes sense when $A(z)B(w)$ is problematic, particularly when $z = w$.

Normal order as a residue

Recall that we defined the delta distribution to equal

$$\delta_-(z-w) = \frac{1}{z}\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n$$

so

$$\delta(z-w) = \delta_-(z-w) + \delta_+(z-w)$$

where

$$\delta_-(z-w) = \frac{1}{z}\sum_{n\geqslant 0}\left(\frac{w}{z}\right)^n, \qquad \delta_+(z-w) = \frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n.$$

$$\delta_-(z-w) = \frac{1}{z-w}, \qquad \delta_+(z-w) = -\frac{1}{z-w}.$$

We will show

$$:A(w)B(w)\;:\mathrm{Res}_{z=0}(\delta_-(z-w)A(z)B(w) + \delta_+(z-w)B(w)A(z)).$$

Proof

To see this, first consider

$$\mathrm{Res}_{z=0}\,\delta_-(z-w)A(z)B(w).$$

This is the z^{-1} coefficient in

$$\frac{1}{z}\sum_{k\geqslant 0}\left(\frac{w}{z}\right)^k\sum_{n,m}A_nB_mz^{-n-1}w^{-m-1}.$$

We must have $-k-n-1=0$ so $-n-1=k\geqslant 0$ and thus $n<0$. The residue is

$$\sum_{n<0}A_nB_mw^{-n-m-1}.$$

Similarly

$$\mathrm{Res}_{z=0}\,\delta_+(z-w)B(w)A(z)=\sum_{n>0}B_mw^{-n-m-1}A_n.$$

Review: the Heisenberg Lie Algebra

The Heisenberg Lie algebra $\mathfrak{H}$ has a central basis element $\mathbb{1}$ and other basis elements b_n such that

$$[b_m, b_n] = m\delta_{m,-n}\mathbb{1}.$$

Note that both b_0 and $\mathbb{1}$ are central. A version of the Stone-von-Neumann theorem asserts $\mathfrak{H}$ has a unique irreducible module in which b_0 and $\mathbb{1}$ act by scalars 0 and 1.

A model of this representation is the bosonic Fock space $B = \mathbb{C}[b_{-1}, b_{-2}, \cdots] \subset U(\mathfrak{H})$ where b_{-n} with $n > 0$ act by multiplication and b_n with $n > 0$ acts by $n\partial/\partial b_{-n}$.

Review: The Bosonic Fock Space

The Fock space B is a graded module in which b_{-k} has degree k. Secretly B may be identified with the ring of symmetric functions, with b_{-k} corresponding to the power-sum symmetric function p_k. (See Kac-Raina, equation (6.5)).

The vacuum element is $1 \in B = \mathbb{C}[b_{-1}, b_{-2}, \cdots]$ (not to be confused with $\mathbb{1}$).
The translation operator T is defined to satisfy $T(1) = 0$, $[T, b_m] = -mb_{m-1}$ from which its effect may be deduced recursively using

$$T \cdot b_k f = b_k T(f) + [T, b_k] \cdot f.$$

Review: Vertex operators

We now have to define $Y(b_k, z)$. First we define $Y(b_{-1}, z)$ to be

$$b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-1}.$$

More generally we define

$$Y(b_{-k}, z) = \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} b(z).$$

However the b_k are not a basis of B, since we may have elements like b_{-1}^2. We cannot define this to be

$$b(z)^2 = \sum_n \left(\sum_{k+l=n} b_k b_l \right) z^{-n-2}$$

since the sum in parentheses is problematic if $n = -2$. Applied to $f \in B$ there are infinitely many terms. See [FBZ] p. 30.

Locality

We have defined $Y(b_{-k}, z)$ but what about $Y(b_{-i_1} \cdots b_{-i_k}, z)$? This is the normal ordered product of $Y(b_{-i_1}, z)$, in other words

$$Y(b_{-i_1} \cdots b_{-i_k}, z) = \frac{1}{(i_1 - 1)!} \cdots \frac{1}{(i_k - 1)!} : (\partial_z^{i_1 - 1} b(z)) \cdots (\partial_z^{i_k - 1} b(z)) : \ .$$

So far we have reviewed what we did in Lecture 8. Now let us show that $b(z)$ is local with respect to itself. This depends on the formula

$$b(z)b(w) = \frac{1}{(z-w)^2} + : b(z)b(w) : ,$$

which we will now discuss.

Locality (continued)

We are pondering

$$b(z)b(w) = \sum_{m,n} b_n b_m z^{-n-1} w^{-m-1}.$$

Note that $b_m b_n$ and $:b_m b_n$ are equal unless $n = m$ and $m < 0$. So

$$b(z)b(w) =: b(z)b(w) : + \sum_{n>0} [b_n, b_{-n}] z^{-n-1} w^{n-1}.$$

The correction term is

$$\sum_{n>0} n z^{-n-1} w^{n-1}$$

which is the expansion in $\mathbb{C}((z))((w))$ of $(z-w)^{-2}$.

Locality (continued)

Note that since the operators b_n and b_m commute except in the diagonal case $b_n = b_{-n}$ we have $: b_n b_m :=: b_m b_n :$ and so $: b(z)b(w) :=: b(w)b(z)$. Therefore the expression

$$b(z)b(w) =: b(z)b(w) : + \frac{1}{(z-w)^2}$$

is symmetric under the interchange. This means that $b(z)$ and $b(w)$ are mutually local.

Dong's Lemma

Dong's Lemma If $A(z)$, $B(z)$ and $C(z)$ are three fields that are pairwise mutually local, then $:A(z)B(z):$ and $C(z)$ are mutually local.

This is proved using the interpretation

$$:A(w)B(w): \mathrm{Res}_{z=0}(\delta_-(z-w)A(z)B(w) + \delta_+(z-w)B(w)A(z))\,.$$

See [FBZ] Section 2.3.5 for details

Locality of the fields $Y(b,z)$

We have checked that $Y(b_{-1},z)$ and $Y(b_{-1},w)$ are local. Next let us check that $Y(b_{-n},z)$ and $Y(b_{-m},w)$ are local. Since $Y(b_n,z) = ((n-1)!)^{-1}\partial^{n-1}b(z)$ it is enough to show that if $A(z)$ and $B(w)$ are local, so are $\partial_z A(z)$ and $B(w)$. Indeed $(z-w)^N[A(z),B(w)] = 0$ for sufficiently large N, so differentiating the identity $(z-w)^{N+1}[A(z),B(w)] = 0$ gives $(z-w)^N[\partial_z A(z),B(w)] = 0$.