Lecture 8: Vertex Algebras

Daniel Bump

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Background

Vertex operators appeared in string theory, called "dual resonance models" in the early days. An early application was to the basic representations of affine Lie algebras by Lepowski and Wilson, and by Kac and I. Frenkel.

Borcherds abstracted the purely algebraic structure underlying these constructions as Vertex Operator Algebras.

Subsequently the foundations underwent some simplification. The original vertex operator algebras are equivalent to conformal vertex algebra in modern texts. We recommend E. Frenkel and Ben-Zvi, *Vertex Algebras and Algebraic Curves*, available on-line through the library. See also Kac, *Vertex Algebras for Beginners* and Chapter 10 of Schottenloher.

Delta Function

In a vertex algebra one often works with formal power series, which might be infinite in both directions. For example we define

$$\delta(z-w)=\sum_{-\infty}^{\infty} z^m w^{-m-1}.$$

This may be multiplied by a power series in one variable:

$$A(w)\delta(z-w)=\left(\sum_{k\in\mathbb{Z}} A_k w^k\right)\delta(z-w)=\sum_{m,n} A_{m+n+1} z^m w_n.$$

This, together with $\delta(z-w)=\delta(w-z)$ implies that

$$A(z)\delta(z-w)=A(w)\delta(z-w),$$

justifying the term “delta function.” As a special case

$$\boxed{(z-w)\delta(z-w)=0.}$$

Locality in QFT

In Lorentzian QFT locality refers to the fact that two operators $\phi(x)$ and $\psi(y)$ commute if x and y are spacelike separated. (Actually $\phi(x)$ and $\phi(y)$ are distributions so this explanation is by abuse of language.)

In Euclidean QFT, after Wick rotation all points are spacelike separated, so $\phi(x)$ and $\phi(y)$ always commute except on the diagonal. This means that the operator-valued distribution $\phi(x)\phi(y) - \phi(y)\phi(x)$ on $\mathbb{R}^d \times \mathbb{R}^d$ is singular, and supported on the diagonal. We haven't proved this in class yet but see Schottenloher, Property S2 in Theorem 8.25.

Thus $\phi(x)\phi(y) - \phi(y)\phi(x)$ is a distribution like $\delta(x-y)$ in a Euclidean theory.

Rings of formal power series

The ring $\mathbb{C}[[z]]$ is the ring of formal power series $\sum_n A(n)z^{-n}$ where $A(n) = 0$ when $n > 0$. So it is an expansion in positive powers of z. The use of z^{-n} is motivated by the application to vertex algebras. This contains the polynomial ring $\mathbb{C}[z]$ as a subring. $\mathbb{C}[[z]]$ is an integral domain whose quotient is the field of formal power series $\mathbb{C}((z))$ of expressions $\sum_n A(n)z^{-n}$ where $A(n) = 0$ for n sufficiently large.

As an exercise, remember

$$\delta(z-w) = \sum_{-\infty}^{\infty} z^m w^{-m-1}.$$

Delta function again

$\delta(z-w)$ may be written $\delta_-(z-w)+\delta_+(z-w)$:

$$\delta_-(z-w) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n, \qquad \delta_+(z-w) = \frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{z}{w}\right)^n.$$

Note that $\delta_-(z-w)$ is the image of the rational function $1/(z-w)$ in $\mathbb{C}((z))((w))$, while $\delta_+(z-w)$ is the image of $-1/(z-w)$ in in $\mathbb{C}((w))((z))$. It is wrong to think that $\delta(z-w)$ is zero because these rational functions sum to zero. The correct statement is that $\delta_-(z-w)$ and $-\delta_+(z-w)$ are expansions of the same element of $\mathbb{C}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ embedded in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively.

Fields and Matrix Coefficients

Let V be a vector space, usually infinite-dimensional, that will play the role of the Hilbert space $\mathcal{H}$ in QFT. A field is a formal power series

$$A(z) = \sum_{j \in \mathbb{Z}} A_j z^{-j}$$

with $A_j \in \mathrm{End}(V)$ such that for any $v \in V$ we have $A_j v = 0$ for sufficiently large j.

If we write a "matrix coefficient"

$$\langle \Phi | A(z) | \Psi \rangle$$

we are thinking of the Hilbert space case. However we have not imposed an inner product on V, so now $\Psi \in V$ and $\Phi \in V^*$, the algebraic dual space.

Locality in Vertex Algebras

We wish to define a notion of locality that is analogous to the notion of locality in a Euclidean QFT. Several equivalent notions are described in Kac's book *Vertex algebras for beginners*. For us we will just say that A and B are (mutually) local if for sufficiently large N

$$\boxed{(z-w)^N[A(z), B(w)] = 0.}$$

As with the Delta distribution, this definition conceals a nuance. Both Frenkel and Ben-Zvi have a good discussion of locality, and Kac gives a number of equivalent conditions that may be more useful in particular situations.

A nuance

This definition conceals a subtlety. Consider the matrix coefficient

$$\langle\Psi|A(z)B(w)|\Phi\rangle.$$

This is a formal power series in $\mathbb{C}((z))((w))$ meaning it has an expansion

$$\sum_n \left(\sum_m C_{m,n}w^{-m}\right) z^{-n}$$

where $C_{m,n}$ vanishes for sufficiently large m, and $\sum_m C_{m,n}w^{-m}$ vanishes for sufficiently large n. On the other hand

$$\langle\Psi|B(w)A(z)|\Phi\rangle$$

lives in $\mathbb{C}((w))((z))$.

A nuance (continued)

So it is most correct to define locality as the condition that for some N

$$(z-w)^N \langle \Psi | A(z) B(w) | \Phi \rangle$$

$$(z-w)^N \langle \Psi | B(w) A(z) | \Phi \rangle$$

define the same element of $\mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ embedded in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively.

The state-field correspondence

A feature of two-dimensional conformal field theories is the state-field correspondence, a bijection between states (vectors in $\mathcal{H}$) and fields. We will give an impressionistic explanation referring for more details to:

- David Tong, Lectures in String Theory (See page 99)
- Remark by Pavel Etingof

Imagine that the world-sheet is the Riemann sphere, and z is a point. After Wick rotation switch to the Lorentzian viewpoint. Looking at a field near the point z may be thought of as extrapolating the state backwards in time to $t = -\infty$. This associates a state to the field.

From CFT to VA

Although Kac's book *Vertex operators for beginners* is not available on-line, the first chapter may be viewed:

- Kac's book at the AMS webpage

In the first chapter (more cryptically written than the rest of the book, unfortunately) he constructs a vertex algebra from a two-dimensional conformal field theory.

The production of a vertex operator from a 2d CFT is also discussed in Schottenloher Chapter 10.

Translation Operator

One feature is a translation operator T on the Hilbert space of states having nice algebraic properties. Let $\mathbf{P} = P_0 + P_1$ be the Hermitian energy-momentum operator. We switch to the light-cone coordinates and write

$$P = P_0 + iP_1, \qquad \overline{P} = P_0 - iP_0.$$

Now let Q be the conjugate of P by inversion $x \to -\bar{x}/|x|^2$ which is a conformal map. Then $T = \frac{1}{2}(P + [P, Q] - Q)$.

Question/Exercise What is this in the Euclidean model?

Vertex Algebras

The definition here is used in both Kac and Frenkel-Ben-Zvi. We require a vector space V with an element $|0\rangle$, to be called the vacuum, a translation operator $T : V \to V$ and a linear operation that associates to a vector $A \in V$ a field

$$Y(A,z) = \sum_n A_{(n)} z^{-n-1}.$$

Vacuum Axiom. We have $Y(|0\rangle, z) = I_v$. Moreover if $A \in V$ then $Y(A,z)|0\rangle \in V[[z]]$ so the limit $Y(A,z)|0\rangle$ as $z \to 0$ makes sense, and this limit is A.

Translation Axiom. $[T, Y(A,z)] = (d/dz)Y(A,z)$ and $T|0\rangle = 0$.

Locality Axiom. The fields $Y(A,z)$ are mutually local.

A first example

Let A be a commutative associative ring with a derivation T. Then we may define $Y(A, z) = e^{zT}A$ and obtain a derivation in which the fields are in $A[[z]]$, and locality reduces to simple commutativity.

Thus a vertex algebra is a generalization of a commutative ring.

The Heisenberg Lie algebra and its bosonic module

References: Kac and Raina, Chapters, 2,4,5,6, and Frenkel-Ben-Zvi Chapter 2.

Let $\mathfrak{H}$ be the Heisenberg (oscillator) Lie algebras with a central basis element $\mathbb{1}$ and other basis elements b_n such that

$$[b_m, b_n] = m\delta_{m,-n}\mathbb{1}.$$

Note that both b_0 and $\mathbb{1}$ are central. A version of the Stone-von-Neumann theorem asserts $\mathfrak{H}$ has a unique irreducible module in which b_0 and $\mathbb{1}$ act by scalars 0 and 1.

A model of this representation is the bosonic Fock space $B = \mathbb{C}[b_{-1}, b_{-2}, \cdots] \subset U(\mathfrak{H})$ where b_{-n} with $n > 0$ act by multiplication and b_n with $n > 0$ acts by $n\partial/\partial b_{-n}$.

Vacuum, Translation, Grading

The Fock space B is a graded module in which b_{-k} has degree k. Secretly B may be identified with the ring of symmetric functions, with b_{-k} corresponding to the power-sum symmetric function p_k. (See Kac-Raina, equation (6.5)).

The vacuum element is $1 \in B = \mathbb{C}[b_{-1}, b_{-2}, \cdots]$ (not to be confused with $\mathbb{1}$).
The translation operator T is defined to satisfy $T(1) = 0$, $[T, b_m] = -mb_{m-1}$ from which its effect may be deduced recursively using

$$T \cdot b_k f = b_k T(f) + [T, b_k] \cdot f.$$

Vertex operators

We now have to define $Y(b_k, z)$. First we define $Y(b_{-1}, z)$ to be

$$b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-1}.$$

More generally we define

$$Y(b_{-k}, z) = \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1} b(z).$$

However the b_k are not a basis of B, since we may have elements like b_{-1}^2. We cannot define this to be

$$b(z)^2 = \sum_n \left(\sum_{k+l=n} b_k b_l\right) z^{-n-2}$$

since the sum in parentheses is problematic if $n = -2$. Applied to $f \in B$ there are infinitely many terms. See [FBZ] p. 30.

Normal ordering

The solution to this dilemma involves normal ordering which generally is an ordering that tries to apply annihilation operators (b_n with $n > 0$ in this case) before creation operators (b_n with $n < 0$). Note that $b_k b_l = b_l b_k$ if $k \neq l$. We define

$$:b_k b_l := \begin{cases} b_l b_k & \text{if } l = -k, k > 0, \\ b_k b_l & \text{otherwise} \end{cases}$$

and

$$:b(z)b(w) := \sum_n \in \mathbb{Z} \left(\sum_{k+l=n} :b_k b_l: \right) z^{-n-2}.$$

Now applying this as an operator to an element of B resolves the problem with the $n = -2$ term by doing the annihilations before the creations.

Normal ordering and vertex operators

The normal product of any two fields is defined the same way:

$$:A(z)B(w) := \sum_n \left(\sum_{m<0} A_m B_n z^{-m-1} + \sum_{m\geqslant 0} B_n A_m z^{-m-1} \right) w^{-n-1}.$$

Note that normal ordering is commutative provided $A_m B_n$ commute when m, n have the same sign.

Now we may define $Y(b_{-1}^2, z) =: b(z)^2$: and more generally

$$Y(b_{j_1} \cdots b_{j_k}, z) =: Y(b_{j_1}, z) \cdots Y(b_{j_k}, z):$$

This concludes the definition of the vertex algebra structure on the bosonic Fock space. The proof of locality is quite instructive: see FBZ Section 2.3, where two proofs are given.

Other vertex algebras

- An irreducible Verma module of **Vir** has a vertex algebra structure. See FBZ Section 2.5
- The basic representation (and other integrable representations) of affine Lie algebras have vertex algebra structures. See FBZ Section 2.4.
- A vertex operator may be associated with any lattice. Applied to the root lattice of an affine Lie algebra, this gives the last example. Applied to the Leech lattice, it leads to the Monster vertex algebra, after Frenkel, Lepowski, Meurman and Borcherds.