Lecture 7: Correlation Functions and Wick Rotation

Daniel Bump

January 1, 2020

Symmetric spaces

References:

- Helgason, Differential Geometry, Lie Groups and Symmetric Spaces
- Bump, Lie Groups, Chapter 28
- Piatetski-Shapiro, Automorphic Functions and the Geometry of Classical Domains

A symmetric space is a homogeneous space G/K where G is a semisimple Lie group and K its maximal compact subgroup. A list of these is tabulated in Helgason, reproduced in Bump's reference above.

Hermitian Symmetric spaces

Studying the lists of symmetric spaces, you will find examples where the maximal compact subgroup has a central $U(1)$ factor and is of the form $K = K_1 \otimes U(1)$. In this case, G/K has a complex structure and is called a Hermitian symmetric space.

In the series $G = SO(p,q)$, $K = S(O(p) \times O(q))$, the quotient G/K is Hermitian if $q = 2$, since $SO(2) \cong U(1)$.

There are also cases where K has a central $SU(2)$ factor. In this case G/K is a quaternionic space.

Symmetric domains

Hermitian symmetric spaces can be realized as symmetric domains in $\mathbb{C}^n$. These will have two realizations, one bounded, one unbounded, related by a Cayley transform. For example, the quotient $SL(2,\mathbb{R})/SO(2)$ can be realized as the upper half plane or the unit disk; the map $z \to (z-i)/(z+i)$ is a conformal map from the upper half plane to the disk.

The Hermitian symmetric spaces are of different types, but the type we will be concerned with are called tube domains.

Self-dual cones

Let C be a convex cone in $\mathbb{R}^n$. The dual cone

$$C' = \{x \in \mathbb{R}^n | \langle x, y\rangle \geqslant 0 \text{ for all } y \in C\}.$$

A cone such that $C = C'$ is a product of a symmetric space by $\mathbb{R}_+^\times$. The classification of self-dual cones is accomplished through the theory of Jordan algebras.

Examples of self-dual cones are symmetric matrices in $\mathrm{Mat}_n(\mathbb{R})$, realized as $GL(n,\mathbb{R})/SO(n)$, or the positive vectors in $\mathbb{R}^{(d-1,1)}$. This cone is the union of hyperboloids, each realized as $SO(d-1,1)/S(O(d-1)\times O(1))$. The hyperboloids are symmetric spaces of dimension $d-1$.

Tube domains

If $\mathcal{C} \subset \mathbb{R}^n$ is a self-dual cone, then we may build another symmetric space in $\mathbb{C}^n$. This is the tube domain

$$\mathcal{T}_C = \{z = x + iy \in \mathbb{C}^n | y \in C\}.$$

These may be thought of as generalizations of the Poincaré upper half plane. They have interesting theories of modular and automorphic forms. For example the tube domain over the cone of symmetric matrices is the Siegel upper half plane $\mathcal{H}_n = Sp(2n, \mathbb{R})/U(n)$, and its arithmetic quotient $Sp(2n, \mathbb{Z})\backslash\mathcal{H}_n$ is the moduli space of polarized abelian varieties, important in algebraic geometry and number theory.

References

We are following Schottenloher, Chapters 8 and 9. Chapter 9 does not logically depend on Chapter 8 but the material in Chapter 8 is important foundational material for Axiomatic Quantum Field Theory, a field whose goal is the mathematically rigorous development of QFT.

In addition to Schottenloher, see:

- Streater and Wightman, PCT, Spin and Statistics and All That
- Barry Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory

Both texts are available on-line through the Stanford Libraries.

Correlation functions

Let ϕ be a field in a It follows from a theorem of L. Schwartz (the nuclear or kernel theorem) that

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$$

is a distribution, the map on $\mathcal{S}(R^{(1,d-1)})^n$ that sends $(f_1,\cdots,f_n)$ to

$$\langle 0|\phi(f_1)\cdots\phi(f_n)|0\rangle$$

is multilinear and continuous on each component, and the Schwartz kernel theorem then implies that this is a distribution on $\mathbb{R}^{nd}$.

Following the notational convention of writing distributions as if they are functions, we denote this discribution as $w_n(z_1,\cdots,z_n)$.

Correlation functions

In the context of axiomatic QFT these are called Wightman functions. Streater and Wightman call them vacuum expectation values. In most of the QFT literature they are called n-point functions or correlation functions.

Apart from their significance in Axiomatic QFT, the correlation functions are basic objects of study since they are the amplitudes of events with concrete physical meaning. Thus if the field ϕ is associated with a type of particle, $\langle 0|\phi(x)|0\rangle$ is the amplitude that a particle will be created from the vacuum by quantum tunnelling. (This does not happen in the conformal field theories we will be concerned with.)

Two-point functions

Similarly consider $\langle 0|\phi(x_1)\phi(x_2)|0\rangle$. By locality, this is zero unless x_1 and x_2 are timelight separated (or lightlike separated if the particle has zero mass. If x_2 is earlier than x_1, this is the amplitude that a particle will travel from x_2 to x_1.

Higher correlation functions express amplitudes of scattering processes. For example the 4-point functions are related to two-particle scattering.

Reconstruction and Analytic Continuation

Schottenloher (following Streater and Wightman) prove that there is enough information in the functions w_N to reconstruct the QFT. In particular, this result requires reconstructing the Hilbert space.

Another important issue is the analytic continuation of the fields. If we can analytically continue to imaginary time, we can change a Lorentzian theory in $\mathbb{R}^{(1,d-1)}$ to a Euclidean one in $\mathbb{R}^{(d,0)}$. This process is called Wick rotation; the correspondence between Lorentzian and Euclidean field theories was observed by Schwinger. It is somewhat analogous to the relationship between the finite-dimensional irreducible representations of $SL(2,\mathbb{R})$ and its compact form $SU(2)$.

Two-point functions

We consider the analytic continuation of the 2-point functions; the general case is similar. Thus consider

$$f(z,w) = \langle 0|\phi(z)\phi(w)|0\rangle.$$

This is zero if z and w are spacelike separated; we assume timelike separation and that w is earlier than z. Our goal is to obtain the analytic continuation of $f(z,w)$.

If $a \in \mathbb{R}^{(1,d-1)}$, let T_a denote translation by a, an element of the Poincaré group. Since $U(T_a)$ fixes the vacuum,

$$f(z,w) = \langle 0|U(T_a)\phi(z)U(T_a)^{-1}U(T_a)\phi(w)U(T_a)^{-1}|0\rangle =$$

$$\langle 0|\phi(z+a)\phi(w+a)|0\rangle = f(z+a,w+a).$$

Hence $f(z,w)$ only depends on $z-w$ and we write $f(z,w) = F(z-w)$.

Support of a Fourier transform

Let Φ and Ψ be elements of the domain $D \subset \mathcal{H}$ of the functions operators $\phi(f), f \in \mathcal{S}(\mathbb{R}^{(1,d-1)})$ as in the Wightman axioms.

We will need the following.

Proposition

$$\int e^{-ip\cdot x}(\Phi, U(T_a)\Psi)\, dx = 0$$

unless p lies in the joint spectrum of the energy-momentum operators, i.e. unless p lies in the positive cone.

Streater and Wightman deduce this from a theorem of Stone, Naimark, Ambrose and Godement (SNAG).

Proof

It is enough to show that if f is a test function

$$f(a) = \frac{1}{(2\pi)^d} \int \hat{f}(p) e^{-ipa}\, da$$

whose Fourier transform $\hat{f}$ has support outside the positive cone then

$$\int f(a)(\Phi, U(T_a)\Psi) dx = 0.$$

The SNAG theorem asserts the existence of a projection-valued measure μ such that

$$U(T_a) = \int e^{ip\cdot a} d\mu(p).$$

If E is a measurable subset of $\mathbb{R}^{1,d-1}$, then $\mu(E)$ is a projection operator on $\mathcal{H}$. The spectral assumption in Wightman Axiom 0 implies that the measure μ is supported in the positive cone.

Proof (concluded)

We are proving

$$\int f(a)(\Phi, U(T_a)\Psi)dx = 0 \tag{1}$$

where f is a Schwartz function whose Fourier transform has support outside the positive cone. By SNAG we have

$$\int f(a)U(T_a)\, da = \int \hat{f}(p)\, d\mu(p) = 0.$$

This implies (1).

Wick rotation

As an application, we consider how to analytically continue the two-point function

$$f(z,w) = \langle 0|\phi(z)\phi(w)|0\rangle$$

to "imaginary time." This procedure relates the Lorentzian field theory on $\mathbb{R}^{(1,d-1)}$ to a Euclidean one on $\mathbb{R}^d$.
By translation invariance, $f(z,w)$ is really only a function of $z-w$. Also, it vanishes if $z-w$ has spacelike separation, by locality. Define

$$F(a) = f(w-a,w).$$

We have $\phi(z) = U(T_{z-w})\phi(w)U(T_{z-w})^{-1}$. Taking $a = w-z$ and remembering that the vacuum is fixed by $U(T_a)$ we see that

$$F(a) = \langle 0|\phi(w)U(T_a)\phi(w)|0\rangle.$$

Wick rotation (continued)

This is defined for $a \in \mathbb{R}^{1,d-1}$. We wish to analytically continue this to a in the tube domain over C, with the distribution F as a limiting value. As we have already shown

$$\int_C e^{-ip\cdot x} F(x)\, dx = 0$$

unless p lies in the positive cone (in momentum space). Therefore

$$F(x) = \frac{1}{(2\pi)^d} \int_C e^{ip\cdot x} \hat{F}(p) dp$$

where $\hat{F}(p)$ is a tempered distribution with support in the positive cone C. This expression gives analytic continuation of $F(x)$ to the tube domain $\mathcal{T}_C$ over C.